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THE ZERO-DIVISOR GRAPH UNDER A GROUP ACTION IN A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity, X the set of all nonzero, nonunits of R and G the group of all units of R. We will investigate some ring theoretic properties of R by considering $\Gamma(R)$, the zero-divisor graph of R, under the regular action on X by G as follows: (1) If R is a ring such that X is a union of a finite number of orbits under the regular action on X by G, then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if R is a local ring or $R \simeq \mathbb{Z}_2 \times F$ where F is a field; (2) If R is a local ring such that X is a union of n distinct orbits under the regular action of G on X, then all ideals of R consist of $\{\{0\}, J, J^2, \ldots, J^n, R\}$ where J is the Jacobson radical of R; (3) If R is a ring such that X is a union of a finite number of orbits under the regular action on X by G, then the number of all ideals is finite and is greater than equal to the number of orbits.

1. Introduction and basic definitions

The zero-divisor graph of a commutative ring has been studied extensitively by Akbari, Anderson, Frazier, Lauve, Livinston and Maimani in [1, 2, 3] since its concept had been introduced by Beck in [4]. Recently, zero-divisor graph of a noncommutative ring (resp. a semigroup) has studied by Redmond and Wu (resp. F. DeMeyer and L. Demeyer) in [9, 10, 11] (resp [5]). Zero-divisor graph is very useful to find the algebraic structures and properties of rings. In this paper, the zero-divisor graph of a commutative ring is also studied by considering some group action.

Throughout this paper all rings are assumed to be rings with identity $1 \neq 0$. For a commutative ring R, let $Z(R)$ be the set of all zero-divisors of R, and $\Gamma(R)$ be the zero-divisor graph of R consisting of all vertices in $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero zero-divisors of R, and edges $x \longleftrightarrow y$, which means that $xy = 0$ for $x, y \in Z(R)^*$. In this paper, a loop (i.e., an edge from some vertex to

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itself) can be considered an edge in a zero-divisor graph $\Gamma(R)$. Recall that $\Gamma(R)$ is *connected* if there is a path between any two distinct vertices. For vertices x and y of $\Gamma(R)$, if there exists a path between x and y, we will denote $d(x, y)$ by the length of the shortest path between x and y, otherwise, $d(x, y) = \infty$. The diameter of $\Gamma(R)$ (denoted by diam($\Gamma(R)$)) is defined by the supremum of $d(x, y)$ for all distinct vertices x and y in $\Gamma(R)$. In particular, if $x = y$ and $d(x, x) = k \geq 3$, then the path is called the cycle of length k. If $\Gamma(R)$ contains a cycle, then the girth of $\Gamma(R)$ (denoted by $g(\Gamma(R))$) is defined by the length of the shortest cycle in $\Gamma(R)$, otherwise, $g(\Gamma(R)) = \infty$. In [6, Proposition 1.3.2], if $\Gamma(R)$ contains a cycle, then $1 + 2\text{diam}(\Gamma(R)) \geq g(\Gamma(R))$. We say that $\Gamma(R)$ is *complete* if $xy = 0$ for any distinct vertices x, y in $\Gamma(R)$. In [3], Anderson and Livingston have shown that for a commutative ring R, (1) $\Gamma(R)$ is connected and $3 \geq \text{diam}(\Gamma(R))$; (2) there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if $R \simeq \mathbb{Z}_2 \times A$ (A is an integral domain) or $Z(R)$ is an annihilator ideal.

Let R be a ring, $X(R)$ (simply, denoted by X) the set of all nonzero, nonunits of R, $G(R)$ (simply, denoted by G) the group of all units of R and J, the Jacobson radical of R. In this paper, we will consider a group action of G on X given by $((g, x) \longrightarrow gx)$ from $G \times X$ to X, called the regular action. If $\phi : G \times X \longrightarrow X$ is the regular action, then for each $x \in X$, we define the *orbit* of x by $o(x) = \{\phi(g, x) : \forall g \in G\}$. Recall that G is *transitive* on X (or G acts transitively on X) if there is an $x \in X$ with $o(x) = X$ and the group action on X by G is *trivial* if $o(x) = \{x\}$ for all $x \in X$. In [7], it has been shown that if X is a union of a finite n number of orbits under the regular action of G on X, then (1) $x^{n+1} = 0$ for all $x \in J$, and X is the set of all nonzero left zero-divisors of R; (2) R is a local ring, $J^n \neq (0)$ and $J^{n+1} = (0)$ if and only if there exists $x \in J$ such that $x^n \neq (0)$ if and only if $J > J^2 > \cdots > J^{n-1} > J^n \neq (0)$.

For a subset S of $Z(R)^*$, we will denote the induced subgraph of $\Gamma(R)$ with vertices in S by $\Gamma_S(R)$, that is, $x, y \in S$ are adjacent in $\Gamma_S(R)$ if and only if x and y are adjacent in $\Gamma(R)$. In particular, if R is a commutative ring such that X is a union of a finite number of orbits under the regular action of G on X, then X is the set of all nonzero zero-divisors of R, i.e., $X = Z(R)^*$, and so $\Gamma(R) = \Gamma_X(R)$. In Section 2, for a commutative ring R such that X is a union of a n orbits under the regular action on X by G, we will investigate some ring theoretic properties of R by considering $\Gamma(R)$, the zero-divisor graph of R, as follows: (1) if $n = 1$, then $\Gamma(R)$ is complete; (2) there is an element $x \in X$ such that x is adjacent to every other vertex in $\Gamma(R)$ if and only if R is a local ring or $R \simeq \mathbb{Z}_2 \times F$ (F is a field); (3) if R is a local ring, then every ideal of R is an annihilator of some element $x \in X$ (denoted by $ann(x)$); (3) the number of all ideals in R is equal to the number of all annihilators in R and is greater than or equal to n, the number of orbits.

Recall that a ring R is called von Neumann regular (simply, regular) (resp. unit-regular) if for every $x \in R$ there exists $y \in R$ (resp. $g \in G$) such that $xyx = x$ (resp. $xgx = x$). Note that for a commutative ring R, R is regular if and only if R is unit-regular. In Section 3, we will investigate some properties of a commutative regular ring R as follows: (1) $\Gamma_X(R)$ is complete if and only if the set of all idempotents in R is orthogonal and the regular action of G on X is trivial; (2) if $2 = 2{\cdot}1$ is a unit in R, then there exists a cycle of length 4 in $\Gamma(R)$.

2. Zero-divisor graph under the regular action

For each $x \in X$, we will denote the set of every element which is adjacent to x by $av(x)$. In fact, $av(x) = ann(x)^* = ann(x) \setminus \{0\}$.

Proposition 2.1. *Let R be a commutative ring. If the regular action of G on X is transitive, then $\Gamma_X(R)$ is complete.*

Proof. It follows from [7, Theorem 2.2]. □

Example 1 (See Example 2.1 in [3])**.** Let $R_1 = \mathbb{Z}_9$ and $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Even though R_1 is not isomorphic to R_2, $\Gamma(R_1) = \Gamma(R_2)$. On the other hand, we can note that (1) all the vertices of $\Gamma(R_1)$ are nilpotent but all the vertices of $\Gamma(R_2)$ are not nilpotent; (2) the adjacency matrix of R_1 is not also equal to the one of R_2; (3) the regular action in R_1 is transitive but the regular action in R_2 is trivial.

Example 2 (See Example 2.1 in [3])**.** Let $R_1 = \mathbb{Z}_2[x, y]/\langle x^2, xy, y^2\rangle$ and $R_2 = F_4[x]/\langle x^2\rangle$. Even though R_1 is not isomorphic to R_2, but $\Gamma(R_1) = \Gamma(R_2)$. On the other hand, we can note that (1) all the vertices of $\Gamma(R_1)$ (resp. $\Gamma(R_2)$) are nilpotent; (2) the adjacency matrix of R_1 is equal to the one of R_2; (3) the regular action in R_1 is trivial but the regular action in R_2 is transitive.

Proposition 2.2. *Let R be a commutative ring such that X is a union of 2 orbits $o(x)$ and $o(y)$ under the regular action on X by G. If $\Gamma_{o(x)}(R)$ and $\Gamma_{o(y)}(R)$ are complete, then $\Gamma_X(R)$ is complete.*

Proof. Note that the set of all the vertices of $\Gamma(R)$ is $X = o(x) \cup o(y)$. Since $\Gamma_X(R)$ is connected by [3, Theorem 2.3], there exists $a \in o(x)$ and $b \in o(y)$ such that $ab = 0$. Let $x_1 \in o(x)$ (resp. $y_1 \in o(y)$) be arbitrary. Then $x_1 = ga$ and $y_1 = hb$ for some $g, h \in G$, and then $x_1y_1 = (gh)(ab) = 0$. Hence X is complete. □

Lemma 2.3. *Let R be a commutative ring such that X is a union of finite number of orbits under the regular action on X by G. Then for each $x \in X$, $av(x)$ is a union of finite number of orbits.*

Proof. It follows from the observation that $av(x) = \bigcup_{y \in av(x)} o(y)$ for each $x \in X$. □

Theorem 2.4. *Let R be a commutative ring such that X is a union of a finite number of orbits under the regular action on X by G. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if R is a local ring or $R \simeq \mathbb{Z}_2 \times F$ where F is a field.*

Proof. (⇒) Suppose that there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. Then $R \simeq \mathbb{Z}_2 \times A$ (A is an integral domain) or $Z(R)$ is an annihilator ideal by [1, Theorem 2.5]. Let X be a union of n distinct orbits under the regular action on X by G. Then $Z(R)^* = X$ by [7, Lemma 2.1]. Since $Z(R)^* = X$, in case that $R \simeq \mathbb{Z}_2 \times A$, A must be a field; in case that $Z(R)$ is an annihilator ideal, $Z(R) = X \cup \{0\}$ is an ideal, which means that R is a local ring.

(⇐) Suppose that R is a local ring. Then there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$ and $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ by [7, Lemma 2.3]. Thus $av(x^n) = X$ and so there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. Suppose that $R \simeq \mathbb{Z}_2 \times F$ where F is a field. Without loss of generality, we can let $R = \mathbb{Z}_2 \times F$. Then there exists $(1,0) \in R$ such that $av((1,0)) = X$, and so there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. □

Corollary 2.5. *Let R be a finite commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if R is a local ring or $R \simeq \mathbb{Z}_2 \times F$ where F is a finite field.*

Proof. Since R is a finite commutative ring, clearly X is a union of finite number of orbits under the regular action of G on X. Hence it follows from Theorem 2.4. □

Proposition 2.6. *Let R be a commutative ring with $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ under the regular action on X by G for some positive integer n. If $n = 1$ and $|X| \geq 3$, or $n = 2$ and $o(x^2) \neq \{x^2\}$, or $n = 3$ and $o(x^i) \neq \{x^i\}$ for some $i = 2$ or 3, or $n \geq 4$, then there exists a cycle of length 3 in $\Gamma(R)$.*

Proof. If $n = 1$, i.e., the regular action is transitive, then $\Gamma(R)$ is complete by Proposition 2.1. Since $|X| \geq 3$, there exists a cycle of length 3 in $\Gamma(R)$. If $n = 2$ and $o(x^2) \neq \{x^2\}$, then there exists $g \in G$ such that $gx^2 \neq x^2$. Since $X = o(x) \cup o(x^2)$ and $x^2g \in X$, $gx^2 = hx$ or hx^2 for some $h \in G$. Thus $x^2 \longrightarrow x \longrightarrow gx^2 \longrightarrow x^2$ is a cycle of length 3. If $n = 3$ and $o_r(x^i) \neq \{x^i\}$ for some $i = 2$ or 3, then there exists $g \in G$ such that $gx^i \neq x^i$. Since $X = o(x) \cup o(x^2) \cup o(x^3)$ and $gx^i \in X$, $gx^i = hx$ or hx^2 or hx^3 for some $h \in G$. Thus $x^3 \longrightarrow x^2 \longrightarrow gx^i \longrightarrow x^3$ is a cycle of length 3. Finally, if $n \geq 4$, then clearly $x^{n-2} \longrightarrow x^{n-1} \longrightarrow x^n \longrightarrow x^{n-2}$ is a cycle of length 3. □

Theorem 2.7. *Let R be a local commutative ring such that X is a union of n distinct orbits under the regular action of G on X. Then the set of all the distinct nonzero proper ideals of R consists of $\{\{0\}, J, J^2, \ldots, J^n, R\}$.*

Proof. Since R is a local ring with identity such that X is a union of n distinct orbits under the regular action of G on X, there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$, $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ by [7, Lemma 2.3] and also $J^n \neq \{0\} = J^{n+1}$ by [7, Lemma 2.9]. Thus $J \supset J^2 \supset \cdots \supset J^n$ and $J^i \neq J^j$ for all $i, j = 1, \ldots, n\,(i \neq j)$. Consider $av(x), av(x^2), \ldots, av(x^n)$.

Since $x^{n-i+1}, x^{n-i+2}, \ldots, x^n \in av(x^i)$ for all $i = 1, 2, \ldots, n$, we can have that $av(x^i) = o(x^{n-i+1}) \cup o(x^{n-i+2}) \cup \cdots \cup o(x^n)$. Also we can note that $av(x^j) \neq av(x^k)$ and $av(x^j) \supset av(x^k)$ for all j, k $(n \geq j > k \geq 1)$. Next, we will show that $J^k = ann(x^{n-k+1})(= av(x^{n-k+1}) \cup \{0\})$ for all $k = 1, \ldots, n$ by using induction on n. When $n = 1$, $J = ann(x)$ since $X = av(x)$. Assume that $J^k = ann(x^{n-k+1})$ holds. To show that $J^{k+1} = ann(x^{n-k})$, let $y(\neq 0) \in J^{k+1}$ be arbitrary. Then $y = y_1 y_2$ for some $y_1 \in J^k, y_2 \in J$. By assumption, $y_1 \in ann(x^{n-k+1})$ and $y_2 \in ann(x^n)$. Since $ann(x^{n-k+1}) \setminus \{0\} = av(x^{n-k+1}) = o(x^k) \cup o(x^{k+1}) \cup \cdots \cup o(x^n)$ and $ann(x^n) \setminus \{0\} = av(x^n) = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$, $y_1 = ax^k, y_2 = bx$ for some $a, b \in R \setminus \{0\}$. Thus $yx^{n-k} = y_1 y_2 x^{n-k} = abx^{k+1}x^{n-k} = abx^{n+1} = 0$, which implies $y \in av(x^{n-k})$. Hence $J^{k+1} \subset ann(x^{n-k})$. To show the convere inclusion holds, let $z \in ann(x^{n-k})$ be arbitrary. Then $zx^{n-k} = 0$. Since $ann(x^{n-k}) \setminus \{0\} = av(x^{n-k}) = o(x^{k+1}) \cup o(x^{k+2}) \cup \cdots \cup o(x^n)$, $z \in o(x^i)$ for some i $(n \geq i \geq k+1)$, and so $z = gx^i$ for some $g \in G$. Thus $z = gx^i = (gx)(x^{i-1}) \in J^{k+1}$ since $gx \in J$ and $x^{i-1} \in J^k$. Thus $J^{k+1} \supset ann(x^{n-k})$. Hence we have $J^{k+1} = ann(x^{n-k})$. Let $A = \{J, J^2, \ldots, J^n\}$. Therefore, $J^k = ann(x^{n-k+1})(= av(x^{n-k+1}) \cup \{0\})$ for all $k = 1, \ldots, n$. Finally, we will show that for any nonzero proper ideal I of R, $I \in A$. Since I is a nonzero ideal of R, there exists $y \in X$. Since $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$, $y \in o(x^i)$ for some i, and then $o(x^i) \subset I$. Since $x^i \in I$ and I is an ideal of R, $x^{i+1}, \ldots, x^n \in I$, and so $o(x^i), \ldots, o(x^n) \subset I$, which implies that $J^i = o(x^i) \cup \cdots \cup o(x^n) \cup \{0\} \subseteq I$. If $I \neq J^i$, then there exists $z \in I \setminus J^i$. Then $z \in o(x^j)$ for some j $(i > j \geq 1)$. By the same argument given as above, $J^j \subseteq I$ $(i > j)$. If $I \neq J^j$, then we will continue in this way. Since $A = \{J, J^2, \ldots, J^n\}$ is a finite set of ideals in R, I must be J^k for some k $(n \geq k \geq 1)$. Hence the set of all ideals of R consists of $\{\{0\}, J, J^2, \ldots, J^n, R\}$. □

For any set S, we denote the cardinality of S by $|S|$.

Corollary 2.8. *Let R be a local commutative ring such that X is a union of n orbits under the regular action of G on X. If $S = \{av(a) : \forall a \in X\}$, then $S = \{J^i \setminus \{0\} : i = 1, \ldots, n\}$, and so $|S| = n$.*

Proof. Let $I_a^* = av(a)$ for all $a \in X$. Then I_a^* is a union of some orbits by Lemma 2.3. Since $I_a = I_a^* \cup \{0\} = ann(a)$ is an ideal of R, $I_a = J^k$ for some k $(n \geq k \geq 1)$ by Theorem 2.7. In the proof in Theorem 2.7, $J^k = ann(x^{n-k+1})$. Hence we have the result from Theorem 2.7. □

Corollary 2.9. *Let R be a finite local commutative ring such that X is a union of n orbits under the regular action of G on X and let m be the number of all ideals of R. Then*

$$m - 2 = n = \frac{1}{|G|} \sum_{g \in G} |X_g|,$$

where $X_g = \{x \in X : gx = x\}$.

Proof. It follows from the Theorem 2.7 and the Burnside's formula. □

Lemma 2.10. *Let $R = R_1 \times R_2 \times \cdots \times R_t$ be the direct product of commutative rings $R_1, R_2, \ldots, R_t$ and let $B = \{ann(x) : \forall x \in X\} \cup \{\{0\}, R\}$ and $B_i = \{ann(x_i) : \forall x \in X\} \cup \{\{0_i\}, R_i\}$ for all $i = 1, \ldots, t$ where each X_i is the set of all nonzero, nonunits of R_i and 0_i is the additive identity of R_i. Then $B_1 \times B_2 \times \cdots \times B_t \subseteq B$.*

Proof. Let $b_1 \times b_2 \times \cdots \times b_t \in B_1 \times B_2 \times \cdots \times B_t$ be arbitrary.

Case 1. $b_i \neq \{0_i\}, R_i$ for all i, i.e., $b_i = ann(x_i)$ for some $x_i \in X_i$.

Thus $b_1 \times b_2 \times \cdots \times b_t = ann(x_1) \times ann(x_2) \times \cdots \times ann(x_t)$. Then clearly, $ann(x_1) \times ann(x_2) \times \cdots \times ann(x_t) = ann((x_1, x_2, \ldots, x_t)) \in B$.

Case 2. $b_i = \{0_i\}$ for some i.

Thus $b_1 \times b_2 \times \cdots \times b_t = ann(x_1) \times \cdots \times \{0_i\} \times \cdots \times ann(x_t)$. Then $ann(x_1) \times \cdots \times \{0_i\} \times \cdots \times ann(x_t) \subseteq ann((x_1, \ldots, 1_i, \ldots, x_t)) \in B$, where 1_i is the unity of R_i.

Case 3. $b_i = R_i$ for some i.

Thus $b_1 \times b_2 \times \cdots \times b_t = ann(x_1) \times \cdots \times R_i \times \cdots \times ann(x_t)$. Then $ann(x_1) \times \cdots \times R_i \times \cdots \times ann(x_t) \subseteq ann((x_1, \ldots, 0_i, \ldots, x_t)) \in B$.

Case 4. $b_i = \{0_i\}$ for some i and $b_j = R_j$ for some j $(i \neq j)$.

Thus by Case 2 and Case 3, $b_1 \times \cdots \times b_i \times \cdots \times b_i \times \cdots \times b_t = ann(x_1) \times \cdots \times \{0_i\} \times \cdots \times R_j \times \cdots \times ann(x_t)$. Then $ann(x_1) \times \cdots \times \{0_i\} \times \cdots \times R_j \times \cdots \times ann(x_t)) \subseteq ann((x_1, \ldots, 1_i, \ldots, 0_j, \ldots, x_t)) \in B$.

Case 5. $b_i = \{0_i\}$ or $b_i = R_i$ for all i.

Thus $b_1 \times \cdots \times b_i \times \cdots \times b_t = ann((a_1, \ldots, \ldots, a_i, \ldots, a_t)) \in B$, where $a_i = 1_i$ or $a_i = 0_i$ for all i. □

Lemma 2.11. *Let $R = R_1 \times R_2 \times \cdots \times R_t$ be the direct product of commutative rings $R_1, R_2, \ldots, R_t$ and let $C = \{o(x) : \forall x \in X\} \cup \{\{0\}, R\}$ and $C_i = \{o(x_i) : \forall x \in X\} \cup \{\{0_i\}, R_i\}$ for all $i = 1, \ldots, t$ where each X_i is the set of all nonzero, nonunits of R_i and 0_i is the additive identity of R_i. Then $C \subseteq C_1 \times C_2 \times \cdots \times C_t$.*

Proof. Let $c \in C$ be arbitrary

Case 1. $c = \{0\}$ or $c = R$.

Then clearly, $c \in C_1 \times C_2 \times \cdots \times C_t$.

Case 2. $c = o(x)$ for some $x = (x_1, \ldots, x_t) \in X$.

Subcase 1. $x_i \in X_i$ for all i.

Subcase 2. $x_i = 0_i$ for some i.

Then $c = o(x) = o((x_1, \ldots, 0_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times \{0_i\} \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_t$.

Subcase 3. $x_i = 1_i$ for some i.

Then $c = o(x) = o((x_1, \ldots, 1_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times R_i \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_t$.

Subcase 4. $x_i = 0_i$ for some i and $x_j = 1_j$ for some j $(i \neq j)$.

Thus by Subcase 2 and Subcase 3, $c = o(x) = o((x_1, \ldots, x_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times \{0_i\} \times \cdots \times R_j \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_j \times \cdots \times C_t$. □

Remark 1. Let R be a commutative ring such that X is a union of a finite number of orbits under the regular action of G on X. Then R is an Artinian ring since $I \setminus \{0\}$ is a union of some orbits for every ideal I of R by Lemma 2.3. Therefore, R is a finite direct product of Artinian local rings, say $R = R_1 \times R_2 \times \cdots \times R_t$ with each R_i Artinian local ring $(i = 1, \ldots, n)$.

Theorem 2.12. *Let R be a commutative ring such that X is a union of a finite number of orbits under the regular action of G on X and let $R = R_1 \times R_2 \times \cdots \times R_t$ where each R_i is Artinian local ring $(i = 1, \ldots, n)$ as mentioned in Remark 1. Then*

(1) *for all ideal I of R, $I = I_1 \times I_2 \times \cdots \times I_t$ where $I_i \in \{\{0_i\}, J_i, J_i^2, \ldots, J_i^{n_i}, R_i\}$ ($\{0_i\}$ is the zero ideal of R_i and J_i is the Jacobson radical of R_i with $J_i^{n_i} \neq \{0_i\} = J_i^{n_{i}+1}$) for all $i = 1, \ldots, t$.*

(2) *the number of all nonzero proper ideals of R is $(n_1 + 2) \cdots (n_t + 2) - 2$, is equal to $|\{av(x) : \forall x \in X\}|$ and greater than or equal to $|\{o(x) : \forall x \in X\}|$.*

Proof. (1) Note that any ideal I of R is of the form $I_1 \times I_2 \times \cdots \times I_t$ where I_i is an ideal of R_i for all $i = 1, \ldots, n$. Since R_i is a local commutative ring for all $i = 1, \ldots, n$, $I_i \in \{\{0_i\}, J_i, J_i^2, \ldots, J_i^{n_i}, R_i\}$ by Theorem 2.7 and so we have the result.

(2) Let A (resp. A_i) be the set of all ideals of R (resp. the set of all ideals of R_i) for all $i1, \ldots, t$, $B = \{ann(x) : \forall x \in X\} \cup \{\{0\}, R\}$ and $C = \{o(x) : \forall x \in X\} \cup \{\{0\}, R\}$. By (1), $A = A_1 \times \cdots \times A_t$ and so $|A| = \prod_{i=1}^{t} |A_i| = (n_1 + 2) \cdots (n_t + 2)$. In the proof of Theorem 2.7, we have that

$$(*) \qquad A_i = \{\{0_i\}, J_i, \ldots, J_i^{n_i}, R_i\}$$

with $J_i^{n_i+1} = \{o_i\}$ and $J_i^{k_i} = ann(x_i^{n_i-k_i+1})$ for some $x_i \in X_i$, the set of all nonzero, nonunits of R_i for all $i = 1, \ldots, t$ where $n_i \geq k_i \geq 1$. Since for all $x \in X$, $ann(x)$ is a nonzero proper ideal of R, $B \subseteq A$, and so $(|A| - 2) \geq |\{ann(x) : \forall x \in X\}|$. Let $B_i = \{ann(x_i) : \forall x_i \in X\} \cup \{\{0_i\}, R_i\}$ for all $i = 1, \ldots, t$. Clearly, $A_i \subseteq B_i$ for all $i = 1, \ldots, t$. By above $(*)$, we have $B_i \subseteq A_i$ for all $i = 1, \ldots, t$. Therefore, $A_i = B_i$ for all $i = 1, \ldots, t$. By Lemma 2.10, we have $B_1 \times \cdots \times B_t \subseteq B$. Hence $B \subseteq A = A_1 \times \cdots \times A_t = B_1 \times \cdots \times B_t = B$, and so $A = B$. Therefore, $|A| - 2 = |\{ann(x) : \forall x \in X\}| = |\{av(x) : \forall x \in X\}|$. On the other hand, let $C_i = \{o(y_i) : \forall y_i \in X\} \cup \{\{0_i\}, R_i\}$ for all $i = 1, \ldots, t$. By Lemma 2.11, we also have $C \subseteq C_1 \times \cdots \times C_t$, and so $|C| \leq |C_1| \times \cdots \times |C_t|$. Since $|B_i| = |\{av(y_i) : \forall y_i \in X\} \cup \{\{0_i\}, R_i\}| = |C_i|$ for all $i = 1, \ldots, t$ by Corollary 2.8, $|A| = |B| = |B_1| \times \cdots \times |B_t| = |C_1| \times \cdots \times |C_t| \geq |C|$. □

We can have the following question:

Question 1. Let R be a commutative ring with identity such that X is a union of n orbits under the regular action of G on X. Is $|\{av(x) : \forall x \in X\}| = |\{o(x) : \forall x \in X\}|$?

Example 3. Let $R = \mathbb{Z}_{36}$. Then R has 7 nonzero proper ideals. We can compute that $av(x)$ and $o(x)$ for all $x \in X$ as follows: $av(2) = 18R = \{18\}$, $av(3) = 12R = \{12, 24\}$, $av(4) = 9R = \{9, 18, 27\}$, $av(6) = 6R = \{6, 12, 18, 24, 30\}$, $av(9) = 4R = \{4, 8, \ldots, 32\}$, $av(12) = 3R = \{3, 6, \ldots, 33\}$, $av(2) = 2R = \{2, 4, \ldots, 34\}$; $o(18) = \{18\}$, $o(6) = \{6, 30\}$, $o(9) = \{9, 27\}$, $o(12) = \{12, 24\}$, $o(3) = \{3, 15, 21, 33\}$, $o(2) = \{2, 10, 14, 22, 26, 34\}$ and $o(4) = \{4, 8, 16, 20, 28, 32\}$. Note that the number of $av(x)$s' is 7 and is equal to the number of $o(x)$s'.

Example 4. Let $R = \mathbb{Z}_3[x]/\langle x^3 \rangle$ and for simple notation, denote $f(x) = f(x) + \langle x^3 \rangle \in R$ for all $f(x) \in \mathbb{Z}_3[x]$. Then $X = \{x, 2x, x^2, 2x^2, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\}$ and R has 2 nonzero proper ideals xR and x^2R. We can also compute that $av(y)$ and $o(y)$ for all $y \in X$ as follows: $av(x) = \{x^2, 2x^2\}$, $av(x^2) = \{x, 2x, x^2, 2x^2, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\}$; $o(x^2) = \{x^2, 2x^2\}$, $o(x) = \{x, 2x, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\}$. Note that the number of $av(y)$s' is 2 and is also equal to the number of $o(y)$s'.

3. Zero-divisor graph of regular rings

In [4], it has been shown that if R is a unit-regular ring, then for every orbit $o(x)$ ($x \in X$) under the regular action of G on X, there exists some idempotent $e \in X$ such that $o(x) = o(e)$. Note that for a commutative ring R with identity, R is regular if and only R is unit-regular.

Proposition 3.1. *Let R be a commutative regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in R is orthogonal and the regular action of G on X is trivial, i.e., $o(x) = \{x\}$ for all $x \in X$.*

Proof. ($\Rightarrow$) Suppose that $\Gamma_X(R)$ is complete. Clearly, the set of all idempotents in R is orthogonal. Assume that the regular action of G on X is not trivial. Then there exists $y \in X$ such that $o(y) \neq \{y\}$. By [8, Lemma 2.3], there exists idempotent $e(\neq y) \in X$ such that $y = ge$ for some $g \in G$. Since $\Gamma_X(R)$ is complete and $y, e \in X$, $0 = ye = (ge)e = ge = y$, a contradiction. Hence the regular action of G on X is trivial.

($\Leftarrow$) Suppose that the set of all idempotents in R is orthogonal and the regular action of G on X is trivial. Let $x, y(x \neq y) \in X$ be arbitrary. By [8, Lemma 2.3], there exist idempotents $e_1, e_2 \in X$ such that $o(x) = o(e_1)$ and $o(y) = o(e_2)$. Since the regular action of G on X is trivial, $\{x\} = o(x) = o(e_1) = \{e_1\}$ and $\{y\} = o(y) = o(e_2) = \{e_2\}$, and so $x = e_1, y = e_2$. Since $x \neq y$, $e_1 \neq e_2$ and so $xy = e_1e_2 = 0$ by assumption. Thus $\Gamma_X(R)$ is complete. □

Lemma 3.2. *Let R be a commutative regular ring. Then the following are equivalent*:

(1) *$x^2 = x$ for all $x \in X$;*
(2) *the regular action of G on X is trivial*;
(3) $G = \{1\}$.

Proof. (1) $\Rightarrow$ (2). Suppose that $x^2 = x$ for all $x \in X$. Let $y \in o(x)$ be arbitrary. Then $y = gx$ for some $g \in G$. Since $y \in X$, $y^2 = y$ by assumption, and then $y^2 = (gx)^2 = g^2x = y = gx$, which implies $y = gx = x$, and so $o(x) = \{x\}$. Thus the regular action of G on X is trivial.

(2) $\Rightarrow$ (3). Suppose that the regular action of G on X is trivial and let $e \in X$ be an idempotent. Then $o(e) = \{e\}$ and $o(1-e) = \{1-e\}$, and so $ge = e$ and $g(1-e) = 1-e$ for all $g \in G$. Thus $g - e = g(1-e) = 1-e$, which implies $g = 1$. Thus $G = \{1\}$.

(3) $\Rightarrow$ (1). Suppose that $G = \{1\}$. Let $x \in X$ be arbitrary. Since $G = \{1\}$, $o(x) = \{x\}$, and so $o(x) = \{x\} = e$ for some idempotent $e \in X$ by [8, Lemma 2.3]. Hence $x^2 = x$ for all $x \in X$. □

Corollary 3.3. *Let R be a commutative regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in R is orthogonal and one of the statements in Lemma 3.2 is satisfied.*

Proof. It follows from Proposition 3.1 and Lemma 3.2. □

Remark 2. Let R be a ring. If the regular action of G on X is transitive, then there exists no idempotent in X. Indeed, assume that there exists an idempotent $e \in X$. Since the regular action of G on X is transitive, $X = o(1-e)$, and then $e = g(1-e)$ for some $g \in G$. Thus $0 = e(1-e) = g(1-e)^2 = g(1-e)$, and so $1 = e$, a contradiction. Therefore for a unit-regular (commutative regular) ring R with identity, there is no transitive regular action of G on X by the above argument and [8, Lemma 2.3].

Proposition 3.4. *Let R be a commutative regular ring with $X \neq \emptyset$. Then for each $x \in X$, there exists an idempotent $e \in X$ such that $av(x) = av(e)$.*

Proof. By [8, Lemma 2.3], for each $x \in X$ there exists an idempotent $e \in X$ such that $o(x) = 0(e)$. Thus $e = gx$ for some $g \in G$, and then $av(e) = av(x)$. □

Proposition 3.5. *Let R be a commutative regular ring such that $2 = 2{\cdot}1$ is a unit in R. Then there exists a cycle of length 4 in $\Gamma(R)$.*

Proof. Let $e \in X$ be an idempotent. Since $2 = 2 \cdot 1 \in G$, $e \neq 1-e, -e$. Thus $e \longleftrightarrow 1-e \longleftrightarrow -e \longleftrightarrow e-1 \longleftrightarrow e$ is a cycle of length 4 in $\Gamma(R)$. □

We note that for any idempotent $e(\neq 0,1)$ in a commutative regular ring R, under the regular action of G on X, $o(1-e) \subseteq av(e)$. In particular, if $R = F_1 \times F_2$ (F_1, F_2 : fields), then $o(1-e) = av(e)$ for all idempotent $e(\neq 0,1) \in R$.

We raise the following question:

Question 2. For any idempotent $e(\neq 0,1)$ in a commutative regular ring R with identity, when is $o(1-e) = av(e)$?

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