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KD-(k_0, k_1)-HOMOTOPY EQUIVALENCE AND ITS APPLICATIONS

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Abstract. Let $\mathbb{Z}^n$ be the Cartesian product of the set of integers $\mathbb{Z}$ and let $(\mathbb{Z}, T)$ and $(\mathbb{Z}^n, T^n)$ be the Khalimsky line topology on $\mathbb{Z}$ and the Khalimsky product topology on $\mathbb{Z}^n$, respectively. Then for a set $X \subset \mathbb{Z}^n$, consider the subspace (X, T^n_X) induced from $(\mathbb{Z}^n, T^n)$. Considering a k-adjacency on (X, T^n_X), we call it a (computer topological) space with k-adjacency and use the notation $(X, k, T^n_X) := X_{n,k}$. In this paper we introduce the notions of KD-(k_0, k_1)-homotopy equivalence and KD-k-deformation retract and investigate a classification of (computer topological) spaces $X_{n,k}$ in terms of a KD-(k_0, k_1)-homotopy equivalence.

1. Introduction

Let $\mathbb{N}$ represent the set of natural numbers. Let $\mathbb{Z}$ represent the set of integers. In relation to the study of some topological properties of a set in $\mathbb{Z}^n$ we have often used Scott, Lawson, Khalimsky topologies, and so forth [7]. In algebraic topology, it is well known that *two any simple closed circles in Euclidean topological space* (E^n, U) *are homotopic to each other.*

Unlike this property, in Khalimsky topology this kind of approach cannot be available. Indeed, in computer (Khalimsky, or digital) topology two any closed circles need not be homotopic to each other (see Theorems 5.5 and 5.9). Thus we need to study this property from the viewpoint of computer topology.

As usual, for a set $X \subset \mathbb{Z}^n$ a pair (X, k) is assumed to be a (*binary*) *digital space* with k-adjacency (or a *digital space* if not confused) in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$ [31], where $k \neq \bar{k}$ except $X \subset \mathbb{Z}$ [34], the pair $(k, \bar{k})$ depends on

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the situation, and k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ [34] (see also [31]). For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, $[a, b]_{\mathbb{Z}} = \{a \leq n \leq b \,|\, n \in \mathbb{Z}\}$ is considered in $(\mathbb{Z}, 2, 2, [a, b]_{\mathbb{Z}})$ [3]. But in this paper we will not concern with the $\bar{k}$-adjacency of $\mathbb{Z}^n - X$.

Motivated by both the digital continuity in [34] and the (k_0, k_1)-continuity in [3], for two digital spaces (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, the notion of (k_0, k_1)-continuity in a different form was established [13] (see also [19]). For a map $f : (X, k_0) \to (Y, k_1)$, since the preservation of the k_0-connectivity into the k_1-one is strongly required in discrete geometry, the (k_0, k_1)-continuity has been used in studying digital spaces in digital topology [4, 9, 13, 19, 31, 34]. Using the (k_0, k_1)-continuity, we obtain the *digital topological category*, briefly DTC, consisting of two things [13] (see also [20]):

(1) A class of objects (X, k) in $\mathbb{Z}^n$;
(2) For every ordered pair of objects (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, all (k_0, k_1)-continuous maps $f : (X, k_0) \to (Y, k_1)$ as morphisms.

In DTC both a (k_0, k_1)-homotopy and a k-homotopy have been used in classifying digital spaces in terms of the digital fundamental group and the digital k-(or (k_0, k_1))-homotopy equivalence [2, 4, 5, 8, 13, 15, 16, 17, 30, 32].

By computer topology is now meant the mathematical recognition of a set in $\mathbb{Z}^n$ with some reasonable topological structure and one of the k-adjacency relations of $\mathbb{Z}^n$, *e.g.,* a development of tools implementing topological concepts for use in science and engineering. Computer topology can play a significant role in computer graphics, image synthesis, image analysis, and so forth. It grew out of discrete geometry expanded into applications where significant topological issues arise. It may be of interest both for computer scientist who try to apply topological knowledge for investigating discrete spaces and for mathematicians who want to use computers to solve complicated topological problems.

Motivated by an Alexandroff space [1], Khalimsky topology for $\mathbb{Z}^n, n \in \mathbb{N}$, was often studied [6, 10, 20, 33, 35]. Thus, in order to study $X \subset \mathbb{Z}^n$ with Khalimsky topology, we consider the subspace (X, T^n_X) induced from the Khalimsky n-dimensional space $(\mathbb{Z}^n, T^n)$ in [1, 6, 20]. Moreover, for a space (X, T^n_X), we strongly need to take into account a reasonable k-adjacency relations of $\mathbb{Z}^n$ in (2.1) because we apply our results to discrete geometry, digital topology, and computer science and further, a Khalimsky continuous map need not preserve a k-connectivity. After considering (X, T^n_X) with k-adjacency, we call it a (*computer topological*) *space with k-adjacency* or a *space* in this paper if not confused, and use the notation $(X, k, T^n_X) := X_{n,k}$.

Up to now, several approaches of the study of (X, T^n_X) have been proposed:

- Khalimsky topological approach in relation with Jordan theorem in the digital case and a computer topological function space [1, 6, 20, 25, 33, 35].

- Combinatorial topological approach using several continuities and homeomorphisms in computer topology [20].
- Extension problem of a Khalimsky topological space with some condition [24, 33].
- Connected ordered topological space (briefly, COTS) approach [27].

Recently, the notion of k-(or (k_0, k_1))-homotopy equivalence in DTC was firstly introduced in [8] (see also [5, 11, 14, 26, 28]) and has often used in classifying digital images from the viewpoint of digital homotopy. The recent paper [5] discussed some properties of a digital homotopy equivalence (Although, it should be noted those earlier papers or information including [8, 11, 26]).

In computer topology the preservation of the connectivity is also important in studying a computer topological space. However, if a computer topological spaces X_{n_0,k_0} and Y_{n_1,k_1} are not connected from the viewpoint of Khalimsky topology, a Khalimsky continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ need not preserve the k_0-connectivity into the k_1-one [20] (see Remark 3.2 of the current paper). This is one of the reasons why we study a set $X \subset \mathbb{Z}^n$ or a Khalimsky topological subspace with (k_0, k_1)-continuity related to the preservation of the digital connectivity as well as Khalimsky continuity. Besides, the study of several computer topological continuities and homeomorphisms was partially done [20]. In this paper in relation to the study of a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$, we use KD-$(k_0, k_1)$-*continuity* (see Definition 4). By the same method as the establishment of DTC, we obtain a Khalimsky digital topological category (briefly, KDTC) consisting of a collection $Ob(C)$ of $X_{n,k}$ and a class $Mor(X, Y)$ of KD-(k_0, k_1)-continuous maps for each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(C)$. Our study in this paper will be in what comes to be called a KD-(k_0, k_1)-*homotopy equivalence*. Thus, in relation with the study of a computer topological space $X_{n,k}$, we need various mathematical tools, such as KD-(k_0, k_1)-homeomorphism, KD-k-deformation retract, KD-(k_0, k_1)-homotopy equivalence, and so forth.

This paper is organized as follows. In Section 2, we provide some basic notions. In Section 3, we establish some properties of KD-(k_0, k_1)-continuity. In Section 4, we introduce the notion of KD-(k_0, k_1)-homotopy equivalence and its various properties. In Section 5, we study a classification of closed k-curves in KDTC. In Section 6, we conclude the paper with a summary. Finally, we give an appendix for characterizing the *general k-adjacency relations* of $\mathbb{Z}^n$.

2. Preliminaries

In order to study a set in $\mathbb{Z}^n$, we essentially need both some continuity under a useful topology and reasonable k-adjacency relations of $\mathbb{Z}^n$. Thus, generalizing the commonly used 4- and 8-adjacency of $\mathbb{Z}^2$ and further, 6-, 18- and 26-adjacency of $\mathbb{Z}^3$ in [31], we have used the following general k-adjacency relations of $\mathbb{Z}^n, n \geq 1$ [9, 13] (for more details, see Appendix):

For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \dots, p_n),\ q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n,$$

are $k(m,n)$-(briefly, k- or k_m-)adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In this operator, $k := k(m,n)$ is the number of points q which are k-adjacent to a given point p according to the numbers m and n in $\mathbb{N}$, where " :=" means *equal* by definition. Indeed, this $k(m,n)$-adjacency is another presentation of the k-adjacency of [9] (see also [13]). Consequently, this operator establishes the k-adjacency relations of $\mathbb{Z}^n$:

$$(2.1)\quad k \in \{2n(n \geq 1), 3^n - 1(n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(2 \leq r \leq n-1, n \geq 3)\}.$$

The recent paper [22] represented the k-adjacency relations of $\mathbb{Z}^n$ in (2.1) in a simpler form as follows.

$$(2.2)\qquad k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n,$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.

For instance, $(n,k) \in \{(1,2); (2,4), (2,8); (3,6), (3,18), (3,26); (4,8), (4,32),$ (4, 64), (4, 80); (5, 10), (5, 50), (5, 130), (5, 210), (5, 242); (6, 12), (6, 72), (6, 232), (6, 472), (6, 664), (6, 728)}.

Hereafter, we are not concerned with the $\bar{k}$-adjacency of $(\mathbb{Z}^n, k, \bar{k}, X)$.

We say that a set of lattice points is *k-connected* if it is not a union of two disjoint non-empty sets not k-adjacent to each other [31]. For a digital space (X,k) in $\mathbb{Z}^n$, two points $x, y \in X$ with $x \neq y$ are called *k-connected* if there is a sequence $(x_0 = x, x_1, \ldots, x_m = y) \subset X$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbb{Z}}$ [31]. Then we call it a *k-path* [31]. The *length* of the k-path is called the number m [31]. The digital interval $[a,b]_{\mathbb{Z}} = \{a \leq n \leq b \,|\, n \in \mathbb{Z}\}$ with 2-adjacency can be considered as a set if it is not related to some topology [3].

A *simple k-curve* is considered to be a k-path(or a sequence) $(x_0, x_1, \ldots, x_m)$ such that x_i and x_j are k-adjacent if and only if $j = i \pm 1$ [31]. Furthermore, a *simple closed k-curve* with l elements in $\mathbb{Z}^n$ is a k-path $(w_0, w_1, \ldots, w_{l-1})$ derived from a simple k-curve $(w_0, w_1, \ldots, w_{l-1}, w_l)$ with $w_0 = w_l$, where w_i and w_j are k-adjacent if and only if $j = i+1 \pmod{l}$ or $i = j+1 \pmod{l}$ [31]. Furthermore, a *closed k-curve* with l elements in $\mathbb{Z}^n$ is denoted by a k-path $(c_0, c_1, \ldots, c_{l-1})$ derived from a k-curve $(c_0, c_1, \ldots, c_{l-1}, c_l)$ with $c_0 = c_l$(briefly k-loop), where c_i and c_j are k-adjacent if $j = i+1 \pmod{l}$ or $i = j+1 \pmod{l}$ [31].

Let us now recall *Khalimsky line topology* on $\mathbb{Z}$ is induced from the following subbasis $\{[2n-1, 2n+1]_{\mathbb{Z}} : n \in \mathbb{Z}\}$ [27] (see also [6, 10, 33, 35]), and is denoted by $(\mathbb{Z}, T)$ and called the *Khalimsky line*. Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbb{Z}} : m, n \in \mathbb{Z}\}$, which induces open sets for $(\mathbb{Z}, T)$, is a basis of the Khalimsky line topology $(\mathbb{Z}, T)$. Furthermore, the product topology on $\mathbb{Z}^n$ derived from $(\mathbb{Z}, T)$ is the *Khalimsky product topology* on $\mathbb{Z}^n, n \geq 2$, and is denoted by $(\mathbb{Z}^n, T^n)$. Indeed, in the Khalimsky line topological space $(\mathbb{Z}, T)$, since the singletons $\{2n : n \in \mathbb{Z}\}$ and $\{2n+1 : n \in \mathbb{Z}\}$ are closed and open, respectively, we can observe that the union of any subsets of the closed sets is also closed.

Let us examine the structure of the Khalimsky n-space. A point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is *open* if all coordinates are odd, and *closed* if each of the coordinates is even [27] (see also [6, 20, 33]). These points are called *pure* and the other points in $\mathbb{Z}^n$ is called *mixed.* In all subspaces in $(\mathbb{Z}^n, T^n), n \geq 2$, of Figures 1, 2, 3, 4, 5, and 6 the symbols ■, black big circle, and • mean a pure closed point, a pure open point, and a mixed point, respectively.

3. Some properties of KD-(k_0, k_1)-continuity

For the n-dimensional Khalimsky space $(\mathbb{Z}^n, T^n)$ and a subset $X \subset \mathbb{Z}^n$, we obtain the subspace (X, T^n_X) induced from $(\mathbb{Z}^n, T^n)$, where $T^n_X = \{O \cap X \,|\, O \in T^n\}$. For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, $[a, b]_{\mathbb{Z}} = \{a \leq n \leq b\}$ with 2-adjacency can be assumed to be a subspace of $(\mathbb{Z}, T)$ if it is related to Khalimsky topology $(\mathbb{Z}, T)$.

Definition 1 ([20])**.** We say that a space (X, T^n_X) with k-adjacency is a (*computer topological*) *space* and use the notation $(X, k, T^n_X) := X_{n,k}$.

In order to establish the notions of KD-(k_0, k_1)-continuity and KD-(k_0, k_1)-homotopy *rel.* A (see Definitions 4 and 6), let us now recall the *digital* (*topological*) k-*neighborhood* in [10] (see also [13]).

Definition 2 ([10], see also [13])**.** Let (X, k) be a space in DTC, $X \subset \mathbb{Z}^n$, $x, y \in X$, and $\varepsilon \in \mathbb{N}$. By $N_k(x, \varepsilon)$ we denote the set

$$\{y \in X : l_k(x, y) \leq \varepsilon\} \cup \{x\},\ \varepsilon \in \mathbb{N},$$

where $l_k(x, y)$ is the length of a shortest simple k-path x to y in X. Besides, we assume that $l_k(x, y) = \infty$ if there is no k-path from x to y. Thus, if the k-component of x is the singleton $\{x\}$, then we assume that $N_k(x, \varepsilon) = \{x\}$ for any $\varepsilon \in \mathbb{N}$.

Consider the spaces in Figure 1. Precisely, in $SC^{2,12,*}_4$, we observe that $N_4(a_0, 1) = \{a_{11}, a_0, a_1\}$; in $SC^{3,6,*}_{18}$, $N_{18}(b_0, 1) = \{b_0, b_1, b_5\}$ and $N_{18}(b_0, 2) = \{b_0, b_1, b_2, b_4, b_5\}$.

For a map $f : (X, k_0) \to (Y, k_1)$, we observe that the preservation of k_0-connectedness into k_1-connectedness at a point $x_0 \in X$ in [3, 34] is exactly presented by the transformation from $N_{k_0}(x_0, 1)$ into $N_{k_1}(f(x_0), 1)$ [12] (see also [20, 23]).

Let us now consider a computer topological k-neighborhood which is different from $N_k(x, \varepsilon)$ of Definition 2.

Definition 3 ([9], see also [20])**.** Consider $X_{n,k}$, $x, y \in X$, and $\varepsilon \in \mathbb{N}$.

(1) A subset V of X is called a *neighborhood* of x if there exists $O_x \in T_X^n$ such that $x \in O_x \subseteq V$.

(2) If a digital k-neighborhood $N_k(x,\varepsilon)$ is a (Khalimsky product topological) neighborhood of x in (X, T_X^n), then this set is called a computer topological k-neighborhood of x with radius ε and we use the notation $N_k^*(x,\varepsilon)$.

Hereafter, let $SC_k^{n,l}$ be a simple closed k-curve with l elements as a subspace of the Khalimsky n-dimensional space $(\mathbb{Z}^n, T^n)$ whose every element x has an $N_k^*(x,1)$ (see the space $SC_4^{2,12}$ in Figure 1). Let $SC_k^{n,l,*}$ be a simple closed k-curve with l elements as a subspace of $(\mathbb{Z}^n, T^n)$ whose some point x does not have an $N_k^*(x,1)$ (see the spaces $SC_4^{2,12,*}$ and $SC_{18}^{3,6,*}$ in Figure 1).

Unlike the digital k-neighborhood $N_k(x,\varepsilon)$ in Definition 2, we can observe that the current computer topological neighborhood $N_k^*(x,\varepsilon)$ absolutely depends on the topological structure $X_{n,k}$. Precisely, consider the spaces $SC_4^{2,12,*} := (a_i)_{i\in[0,11]_{\mathbb{Z}}}$ and $SC_{18}^{3,6,*} := (b_i)_{[0,5]_{\mathbb{Z}}}$ in Figure 1. In $SC_4^{2,12,*}$ we observe that $N_4^*(a_0,\varepsilon) = \emptyset$ if $\varepsilon \in [1,5]_{\mathbb{Z}}$ and $N_4^*(a_0,6) = SC_4^{2,12,*}$ because the smallest open set containing the point a_0 is the set $\{a_{10}, a_{11}, a_0, a_1, a_2, a_6\}$ in $SC_4^{2,12,*}$. Besides, in $SC_{18}^{3,6,*}$ we obtain that $N_{18}^*(b_0,1) = \emptyset$ because the smallest open set containing the point b_0 is the set $\{b_2, b_1, b_0, b_4, b_5\}$ and $N_{18}^*(b_0,2) = SC_{18}^{3,6,*} - \{b_3\}$.

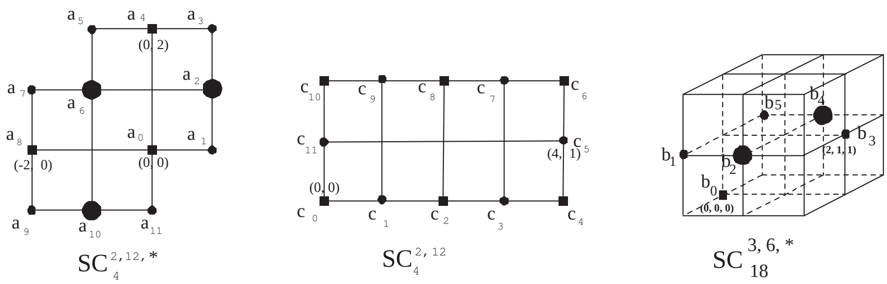

Figure 1. Simple closed 4- and 18-curves

Furthermore, in $X_{n,k}$ we observe that
if $k = 3^n - 1$, then $N_k^*(x,1) = N_k(x,1)$; and
if $k \neq 3^n - 1$, then $N_k^*(x,1)$ need not be equal to $N_k(x,1)$.

In Khalimsky topology, let us recall that $f : (X, T_X^{n_0}) \to (Y, T_Y^{n_1})$ is *continuous* at a point $x_0 \in X$ if for any $O_{f(x_0)} \in T_Y^{n_1}$ there is $O_{x_0} \in T_X^{n_0}$ satisfying $f(O_{x_0}) \subset O_{f(x_0)}$ as usual.

Definition 4 ([20]). For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, we say that a function $f : X \to Y$ is KD-(k_0,k_1)-continuous at a point $x_0 \in X$ if

(1) f is continuous at the point x_0; and
(2) for any $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $N_{k_0}(x_0, \delta) \subset X$ such that

$$f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon), \text{ where } \varepsilon, \delta \in \mathbb{N}.$$

Furthermore, we say that a map $f : X \to Y$ is KD-(k_0, k_1)-continuous if the map f is KD-(k_0, k_1)-continuous at every point $x \in X$.

We observe that there is a big difference between the (k_0, k_1)-continuity in DTC and the current KD-(k_0, k_1)-continuity owing to the condition (1) of Definition 4. Besides, the above condition (2) of Definition 4 is exactly the digital (k_0, k_1)-continuity in [34] (see also [13]). Thus it can be represented as the following simpler form [21] because every digital space (X, k) has an $N_k(x, 1), x \in X$.

$$f(N_{k_0}(x_0, 1)) \subset N_{k_1}(f(x_0), 1). \tag{3.1}$$

Unlike the pasting property of classical continuity in topology, (k_0, k_1)-continuity has some intrinsic features: (k_0, k_1)-continuity has *the almost pasting property* [29] instead of *the pasting property* of classical continuity.

Furthermore, we observe that neither of the conditions (1) and (2) of Definition 4 implies to the other. To be specific, consider the space X with $X_{2,4}$ in Figure 2 and the map $f : X \to \mathbb{Z}$ for which $f(X_1) = \{1\}$, $f(\{x_4, x_5\}) = \{3\}$, where $X = \{x_i\}_{i \in [0,5]_{\mathbb{Z}}}$ and $X_1 = \{x_i\}_{i \in [0,3]_{\mathbb{Z}}}$. Then, while the map f is a continuous map because the sets X_1 and $\{x_4, x_5\}$ are open in T_X^2, f cannot satisfy the property (2) of Definition 4, especially at the points x_3 and x_4, which means that the condition (1) of Definition 4 does not imply the condition (2) of Definition 4 [20].

Meanwhile, in general, since the digital k-neighborhood $N_k(x, \varepsilon)$ need not be a topological k-neighborhood in $X_{n,k}$, the condition (2) may not imply the condition (1) of Definition 4.

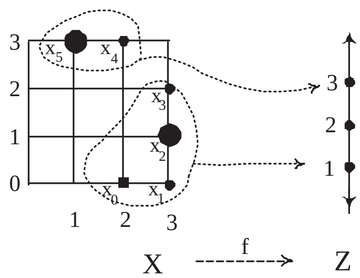

FIGURE 2. Comparison between Khalimsky continuity and KD-(k_0, k_1)-continuity

Furthermore, the current KD-(k_0, k_1)-continuity has strong merits to study each space $X_{n,k}$ as well as the space whose some point x does not have an

$N_k^*(x,1)$. Let us consider the *KD-topological category*, denoted by KDTC, consisting of two things:

• A class of objects $X_{n,k}$;

• KDTC has KD-(k_0,k_1)-continuous maps as morphisms.

In this paper we work in this category KDTC. We now remind again that a point $x=(x_1,x_2,\ldots,x_n)\in\mathbb{Z}^n$ is (*pure*) *open* if all coordinates are odd, and (*pure*) *closed* if each of the coordinates is even [6, 20, 33] and the other points in $\mathbb{Z}^n$ is called *mixed*.

Theorem 3.1. *Let $f:(X,T_X^{n_0})\to(Y,T_Y^{n_1})$ be a map. Consider a point $x_0\in X\subset\mathbb{Z}^{n_0}$ and its image $f(x_0)=y_0\in Y\subset\mathbb{Z}^{n_1}$, we obtain the following.*

(1) *If the map f is continuous at a closed point $x_0\in X$ with the image $f(x_0)=y_0\in Y$ closed, then f can be used in examining KD-$(3^{n_0}-1,3^{n_1}-1)$-continuity at the point x_0.*

(2) *If the map f is continuous at a closed point $x_0\in X$ with the image $f(x_0)=y_0\in Y$ open, then f can be used in examining KD-$(3^{n_0}-1,2n_1)$-continuity at the point x_0.*

(3) *If the map f is continuous at an open point $x_0\in X$ with the image $f(x_0)=y_0\in Y$ closed, then f can be used in examining KD-$(2n_0,3^{n_1}-1)$-continuity at the point x_0.*

(4) *If the map f is continuous at an open point $x_0\in X$ with the image $f(x_0)=y_0\in Y$ open, then f can be used in examining KD-$(2n_0,2n_1)$-continuity at the point x_0.*

Proof. (1) If the two points $x_0\in X$ and $y_0\in Y$ are closed, then the smallest open sets containing the points x_0 and y_0 are $N_{3^{n_0}-1}(x_0,1)$ and $N_{3^{n_1}-1}(y_0,1)$, respectively. Thus the Khalimsky continuity of $f:(X,T_X^{n_0})\to(Y,T_Y^{n_1})$ at the point x_0 with the image y_0 implies that $f(N_{3^{n_0}-1}(x_0,1))\subset N_{3^{n_1}-1}(y_0,1)$. Consequently, we observe that f is a KD-$(3^{n_0}-1,3^{n_1}-1)$-continuous map at the point $x_0\in X$.

(2) If a point $x_0\in X$ is closed, then the smallest open set containing the point x_0 is $N_{3^{n_0}-1}(x_0,1)$. In $(Y,T_Y^{n_1})$, if $y_0\in Y$ is open, then the smallest open set containing the point y_0 is $\{y_0\}$.

Thus the continuity of $f:(X,T_X^{n_0})\to(Y,T_Y^{n_1})$ at the point x_0 with the image y_0 implies that $f(N_{3^{n_0}-1}(x_0,1))\subset\{y_0\}\subset N_{2n_1}(y_0,1)$ if y_0 is open, where x_0 is closed.

Consequently, it turns out that f is a KD-$(3^{n_0}-1,2n_1)$-continuous map at the point $x_0\in X$.

The proofs of (3) and (4) are proceeded by the same methods as those of (1) and (2) by using the following fact: If the point $x_0\in X$ is open or closed, then the smallest open set containing the point x_0 is $\{x_0\}$, $N_{2n_0}(x_0,1)$, or $N_{3^{n_0}-1}(x_0,1)$, respectively. □

Remark 3.2. (1) Without the four cases in Theorem 3.1 Khalimsky continuity need not imply the KD-(k_0,k_1)-continuity. Precisely, consider the map f :

$X := SC_{18}^{3,6,*} \to Y := SC_{26}^{3,4}$ in Figure 3 for which

$$f(x_0) = y_0, f(x_1) = f(x_2) = y_1, f(x_3) = y_2, f(x_4) = f(x_5) = y_3. \tag{3.2}$$

Then we observe that the map f is continuous at the point x_0. While the map f is KD-$(18, 26)$- and KD-$(26, 26)$-continuous at the point x_0, it can be neither KD-$(18, 18)$-continuous nor KD-$(26, 18)$-continuous at the point x_0.

(2) Since each singleton $\{x\} \subset [a, b]_{\mathbb{Z}}$ is an open set or a closed set, we observe that the smallest open sets containing the points $2n + 1$ and $2n \in [a, b]_{\mathbb{Z}}$ are $\{2n + 1\}$ and $N_2(2n, 1) = N_2^*(2n, 1)$, respectively. Thus, as a special case of Theorem 3.1, a Khalimsky continuous map $f : [a, b]_{\mathbb{Z}} \to (X, T_X^n)$ can be served to examine KD-$(2, 3^n - 1)$-continuity of f if $f(t)$ is closed and further, the map f can be used in examining KD-$(2, 2n)$-continuity of f if $f(t)$ is open in (X, T_X^n), where $t \in [a, b]_{\mathbb{Z}}$.

Proof. (1) First, we prove that the map $f : X := SC_{18}^{3,6,*} \to Y := SC_{26}^{3,4}$ of (3.2) is KD-$(18, 26)$-continuous at the point x_0. Precisely, we can observe that the smallest open set O_{y_0} containing the point y_0 is the set $\{y_0, y_1, y_3\} \subset Y$ in Figure 3, and there is an pen set containing the point x_0, denoted by $O_{x_0} = N_{18}(x_0, 2)$, such that $f(O_{x_0}) \subset O_{y_0}$. Thus, by (3.1), we observe that $f(N_{18}(x_0, 1)) \subset N_{26}(y_0, 1) = O_{y_0}$, where $N_{18}(x_0, 1) = \{x_0, x_1, x_5\}$. Thus the map f is KD-$(18, 26)$- and KD-$(26, 26)$-continuous at the point x_0.

But we observe that the map f can be neither KD-$(18, 18)$- nor KD-$(26, 18)$-continuous at the point x_0. Even though the map f is continuous at the point x_0, since $\{y_0\}$ is the 18-component of the point y_0, we observe that $N_{18}(y_0, \varepsilon)$ is the singleton $\{y_0\}$ for any $\varepsilon \in \mathbb{N}$. Meanwhile, for $N_{18}(x_0, 1)$ and $N_{26}(x_0, 1) = \{x_0, x_1, x_2, x_4, x_5\}$, each of both $f(N_{18}(x_0, 1))$ and $f(N_{26}(x_0, 1))$ cannot be a subset of $\{y_0\} = N_{18}(y_0, \varepsilon)$ for any $\varepsilon \in \mathbb{N}$, by (3.1), which implies that the map f is neither a KD-$(18, 18)$-continuous map nor a KD-$(26, 18)$-continuous map at the point x_0. □

Consequently, Remark 3.2 justifies the current KD-(k_0, k_1)-continuity for the study of spaces in $(\mathbb{Z}^n, T^n)$ with some adjacency relations of $\mathbb{Z}^n$.

As discussed in Remark 3.2, since a Khalimsky continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ need not preserve the k_0-connectivity into the k_1-connectivity, by the use of the k-neighborhood $N_k(x, 1)$ we need to establish another continuity which is Khalimsky continuous and preserves the k_0-connectivity into the k_1-connectivity, as follows.

Definition 5 ([20])**.** For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a function $f : X \to Y$ is said to be a KD-(k_0, k_1)-homeomorphism if

(1) the map f is bijective, and

(2) the map f is a KD-(k_0, k_1)-continuous map and further, f^{-1} is a KD-(k_1, k_0)-continuous map.

Then we say that the space X is KD-(k_0, k_1)-homeomorphic to Y.

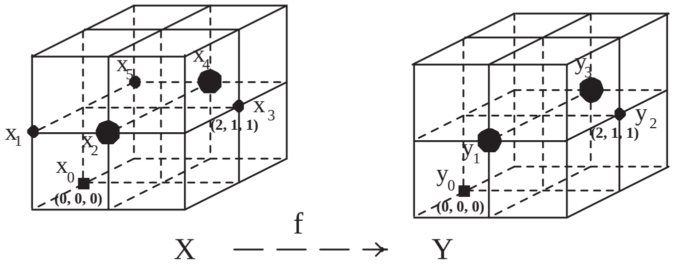

FIGURE 3. Non-existence of KD-$(18, 18)$-continuity

We can observe that neither of the conditions (1) and (2) of Definition 5 implies the other, as follows.

Example 3.3. Consider the two spaces $SC_4^{2,12,*}$ and $SC_4^{2,12}$ in Figure 1. Even if they have the same cardinality, they cannot be KD-$(4,4)$-homeomorphic to each other contrary to the property (2) of Definition 5 because there is no KD-$(4,4)$-continuous bijection from $SC_4^{2,12,*}$ onto $SC_4^{2,12}$.

4. KD-(k_0, k_1)-homotopy equivalence

In this section we establish the notion of KD-(k_0, k_1)-homotopy equivalence and classify closed k_i-curves in KDTC up to the KD-(k_0, k_1)- or KD-k-homotopy equivalence (see Definition 7). Furthermore, we study some KD-k-homotopic properties of $X_{n,k}$ in relation with the classification of computer topological spaces $X_{n,k}$. Precisely, in $(\mathbb{Z}^n, T^n)$ let $C_k^{n,l,*}$ be a closed k-curve with l elements whose some point $x \in C_k^{n,l,*}$ does not have an $N_k^*(x,1)$ in $C_k^{n,l,*}$. Let $C_k^{n,l}$ be a closed k-curve with l elements whose every point $x \in C_k^{n,l}$ has an $N_k^*(x,1)$ in $C_k^{n,l}$. Then we show that all kinds of closed k-curves including $C_k^{n,l,*}$ as well as $C_k^{n,l}$ can be classified in terms of a KD-k-homotopy equivalence (see Theorems 5.9 and 5.11).

In order to study some KD-k-homotopic properties of spaces $X_{n,k}$ in KDTC, we introduce the notions of KD-k-deformation retract and KD-k-homotopic thinning, which can be used in topology and computer science. Especially, we study a relation between a KD-k-homeomorphism and a KD-k-homotopy equivalence in the category of closed k-curves in KDTC including $C_k^{n,l,*}$ and $SC_k^{n,l,*}$. For the purpose of studying these problems, we first establish the new notion of KD-(k_0, k_1)-homotopy which is the computer topological analog of the (k_0, k_1)-*homotopy rel.* A in DTC [19]. Now we need some basic terminology as follows. For a space $X_{n,k}$ and its subspace $A_{n,k}$, we call (X, A) a *space pair* with k-adjacency and use the notation $(X, A)_{n,k}$. For two space pairs

$(X, A)_{n_0,k_0}$ and $(Y, B)_{n_1,k_1}$, we say that $f : (X, A)_{n_0,k_0} \to (Y, B)_{n_1,k_1}$ is KD-(k_0, k_1)-continuous if $f : (X, A)_{n_0,k_0} \to (Y, B)_{n_1,k_1}$ is KD-(k_0, k_1)-continuous and $f(A) \subset B$. Let us now establish the notion of KD-(k_0, k_1)-*homotopy rel.* A in terms of the KD-(k_0, k_1)-continuity as follows.

Definition 6. For three spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, and a subspace $A_{n_0,k_0} := A \subset X_{n_0,k_0}$, let $f, g : X \to Y$ be KD-(k_0, k_1)-continuous functions. Let $[0, m]_{\mathbb{Z}}$ be considered as a subspace of the Khalimsky line. Suppose there exist $m \in \mathbb{N}$ and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is KD-$(2, k_1)$-continuous;
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ for all $x \in X$ is KD-(k_0, k_1)-continuous.

Then we say that F is a KD-(k_0, k_1)-homotopy between f and g, and f and g are KD-(k_0, k_1)-homotopic in Y. And we use the notation $f \simeq_{KD\cdot(k_0,k_1)} g$.

- If, further, for all $t \in [0, m]_{\mathbb{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbb{Z}}$. Then, we say that the homotopy is a *KD-*(k_0, k_1)*-homotopy rel.* A and denote it $f \simeq_{KD\cdot(k_0,k_1)rel.A}\, g$. In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed KD-(k_0, k_1)-homotopy.

If $X = [0, m_X]_{\mathbb{Z}}$, for all $t \in [0, m]_{\mathbb{Z}}$, we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, then we say that F holds the endpoints fixed.

As the computer topological analogs of the notions of *pointed* k*-contractibility* and k*-nullhomotopic* in [4], we say that the space $X_{n,k} := X$ is *pointed KD-*k*-contractible* if $1_X \simeq_{KD\cdot k} c_{\{x_0\}}$, where $c_{\{x_0\}}$ is a constant map for some point $x_0 \in X$. We say that a KD-(k_0, k_1)-continuous function $f : X_{n_0,k_0} \to Y_{n_1,k_1} := Y$ is *KD-*k_1*-nullhomotopic* in Y if f is KD-k_1-homotopic in Y to a constant function $c_{\{y_0\}}$ for some $y_0 \in Y$.

Obviously, we observe that the space with 8-adjacency in KDTC

$$X \approx_{KD\cdot 8} \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$$

is pointed KD-8-contractible by the same method as the digital 8-contractibility of the space X in [5, 13] because each singleton $\{x\} \subset X$ is an open set in T^2_X. Unlike the KD-8-contractibility of X, we strongly need to show the KD-8-contractibility of the spaces Y and Z in Figure 4 as follows.

Example 4.1. Consider the two spaces in KDTC
$Y :\approx_{KD\cdot 8} \{(0, 0), (1, 1), (0, 2), (-1, 1)\} := (y_i)_{i\in[0,3]_{\mathbb{Z}}}$ and
$Z :\approx_{KD\cdot 8} \{(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), (0, 1)\} := (z_i)_{i\in[0,7]_{\mathbb{Z}}}$.
Then the two spaces Y and Z are also pointed KD-8-contractible.

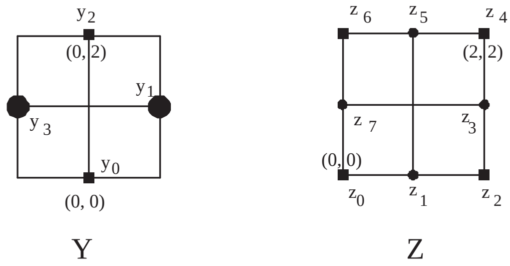

Figure 4. KD-8-contractibility

Proof. (1) We show the KD-8-contractibility of the space Y by the following KD-$(8, 8)$-homotopy rel. $\{y_0\}$:

$$H : Y \times [0, 2]_{\mathbb{Z}} \to Y$$

such that

$$H(y_i, 0) = y_i \text{ for each } y_i \in Y;$$
$$H(y_1, 1) = H(y_2, 1) = H(y_3, 1) = y_3, H(y_0, 1) = y_0; \text{ and}$$
$$H(y_i, 2) = y_0 \text{ for each } y_i \in Y.$$

(2) We show the KD-8-contractibility of the space Z by the following KD-$(8, 8)$-homotopy rel. $\{z_0\}$:

$$H : Z \times [0, 2]_{\mathbb{Z}} \to Z$$

such that

$$H(z_i, 0) = z_i \text{ for each } z_i \in Z;$$
$$H(z_5, 1) = H(z_6, 1) = H(z_7, 1) = z_7,$$
$$H(z_0, 1) = z_0,$$
$$H(z_i, 1) = z_1, \ i \in \{1, 2, 3, 4\};$$
$$H(z_i, 2) = z_0 \text{ for each } z_i \in Z.$$

□

The notion of KD-(k_0, k_1)-homotopy equivalence is now introduced in order to classify computer topological spaces $X_{n,k}$ up to a KD-k-homotopy equivalence.

Definition 7. For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, if there are both a KD-(k_0, k_1)-continuous map $h : X \to Y$ and a KD-(k_1, k_0)-continuous map $l : Y \to X$ such that $l \circ h$ is KD-k_0-homotopic to 1_X and $h \circ l$ is KD-k_1-homotopic to 1_Y, then the maps h and l are called KD-(k_0, k_1)-homotopy equivalences. Then we use the notation, $X \simeq_{KD\cdot(k_0,k_1)\cdot h\cdot e} Y$. Furthermore, if $k_0 = k_1$, we call h a KD-k_0-homotopy equivalence and we use the notation $X \simeq_{KD\cdot k_0\cdot h\cdot e} Y$.

Example 4.2. For $SC_4^{2,12,*}$ in Figure 1, consider the subspaces $SC_4^{2,12,*} - \{a_5, a_9\} := X'$ and $X := X' - \{a_3, a_7\}$. Then we observe that $X'_{2,8}$ is KD-8-homotopy equivalent to $X_{2,8}$ in terms of $a_7 \to a_6$, $a_3 \to a_2$, and the other points are remained.

Remark 4.3. By Example 4.2, we observe that there is no need to have the same cardinality of between X and Y in Definition 7.

Theorem 4.4 (transitivity)**.** *The composition also preserves a homotopy equivalence in KDTC. Namely, if*

$$X_{n_0,k_0} \simeq_{KD\cdot(k_0,k_1)\cdot h\cdot e} Y_{n_1,k_1} \textit{ and } Y_{n_1,k_1} \simeq_{KD\cdot(k_1,k_2)\cdot h\cdot e} Z_{n_2,k_2},$$

then $X_{n_0,k_0} \simeq_{KD\cdot(k_0,k_2)\cdot h\cdot e} Z_{n_2,k_2}$.

Remark 4.5. There is an equivalence relation on the family of subspaces in $(\mathbb{Z}^n, T^n)$ in terms of a KD-k-homotopy equivalence. Indeed, there are various types of $SC_k^{n,l,*}$ up to a KD-k-homotopy equivalence depending on its topological structure and the cardinality of the set $\{x | \nexists N_k^*(x,1) \subset SC_k^{n,l,*}\}$.

Definition 8 ([20])**.** For two spaces $X_{n,k} := X$ and $X'_{n,k} := Y$, we say that a KD-(k,k)-continuous map $r : X' \to X$ is a KD-k-retraction if
(1) $X \subset X'$,
(2) $r(x) = x$ for all $x \in X$.
Then we say that X is a KD-k-retract of X'.

Definition 9. For a space pair $(X, A)_{n,k}$, $A_{n,k} := A$ is said to be a strong KD-k-deformation retract of $X_{n,k} := X$ if there is a KD-k-retraction r of X to A such that $F : i \circ r \simeq_{KD\cdot k\cdot rel.A} 1_X$. Then the KD-$k$-homotopy F is called a strong KD-k-deformation of X to A. A point $x \in X - A$ is called strong KD-k-deformable.

By the notion of strong KD-k-deformation we introduce the notion of KD-k-homotopic thinning as follows.

Definition 10. We call a strong KD-k-deformable point a *KD-k-homotopic thinning point*. For a space $X_{n,k}$, we can delete all KD-k-homotopic thinning points from $X_{n,k}$ in terms of a strong KD-k-deformation retract. Then this processing is called a KD-k-homotopic thinning.

Example 4.6. Consider the space $X := \{x_i | i \in [0,9]_{\mathbb{Z}}\}$ in KDTC (see Figure 5(a)). Then consider the map $j : X \to X_1 := X - \{x_3, x_7\}$ for which $j(x_3) = x_2, j(x_7) = x_8$, and the other points in X_1 remain by j. Furthermore, consider the inclusion $i : X_1 \to X$. Then we observe that $j \circ i$ is KD-$(4,4)$-homotopic to 1_{X_1} relative to X_1 and the composition $i \circ j$ is KD-$(4,4)$-homotopic to 1_X relative to X_1, as required.

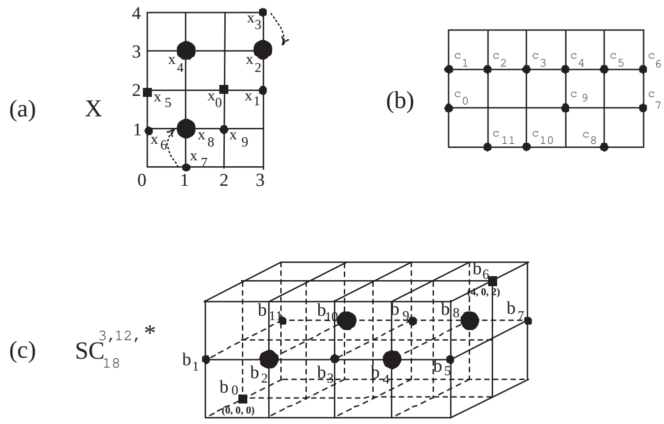

FIGURE 5

5. Classification of closed k-curves up to the KD-k-homotopy equivalence

In this section we study a method of classifying two closed k_i-curves in KDTC, $i \in \{1, 2\}$, up to a KD-(k_0, k_1)-homotopy equivalence. In this section we assume that both $SC_k^{n,l}$ and $SC_k^{n,l,*}$ are not KD-k-contractible because they are trivial in KD-k-homotopy theory. As a generalization of a closed spline in $\mathbb{Z}^3$, in this paper we consider $C_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ to be a sequence such that c_i and c_j are k-adjacent if $j = i \pm 1 \pmod{l}$ as usual and with the further condition that each of the index sets

$$I_k(i) = \{j | c_j \in N_k(c_i, 1)\} \text{ is consecutive modulo } l. \tag{5.1}$$

For instance, consider the closed 8-curve $(c_i)_{i\in[0,11]_{\mathbb{Z}}}$ in Figure 5(b). Let us examine the two points c_4 and c_9. Then we observe that both $I_8(4) = \{3, 4, 5, 9\}$ and $I_8(9) = \{3, 4, 5, 8, 9, 10\}$ are not consecutive modulo 12. Thus in this paper we will not consider such a kind of closed k-curve in relation with Theorems 5.9 and 5.11 and Corollary 5.7, and Examples 5.8 and 5.10. After observing the two spaces $SC_4^{2,12,*}$ and $SC_4^{2,12}$ in Figure 1, we obtain the following.

Theorem 5.1. *$SC_{k_0}^{n_0,l,*}$ cannot be KD-(k_0, k_1)-homotopy equivalent to $SC_{k_1}^{n_1,l}$.*

Proof. Suppose that there is a KD-(k_0, k_1)-homotopy equivalence $f : SC_{k_0}^{n_0,l,*} \to SC_{k_1}^{n_1,l}$. In order for the two spaces $SC_{k_0}^{n_0,l,*}$ and $SC_{k_1}^{n_1,l}$ to be KD-(k_0, k_1)-homotopy equivalent to each other, we observe that the KD-(k_0, k_1)-homotopy equivalence $f : SC_{k_0}^{n_0,l,*} \to SC_{k_1}^{n_1,l}$ should be bijective owing to the same cardinality of them. If not, the two spaces cannot be KD-(k_0, k_1)-homotopy equivalent owing to the non-preservation of both the k_0-connectivity into the k_1-connectivity by the map f.

Meanwhile, any bijection $f : SC_{k_0}^{n_0,l,*} \to SC_{k_1}^{n_1,l}$ cannot satisfy KD-(k_0, k_1)-continuity at the point $x_0 \in SC_{k_0}^{n_0,l,*}$ which does not have $N_{k_0}^*(x_0, 1)$ in $SC_{k_0}^{n_0,l,*}$. To be specific, for a point x_0 which does not have $N_{k_0}^*(x_0, 1)$ in $SC_{k_0}^{n_0,l,*}$, let $O_{f(x_0)}$ be the smallest open set containing the point $f(x_0)$. Then we can obtain

$$O_{f(x_0)} \subset N_{k_1}(f(x_0), 1) \subset SC_{k_1}^{n_1,l}. \tag{5.2}$$

Next, let O_{x_0} be the smallest open set containing the point x_0 in $SC_{k_1}^{n_1,l,*}$. Then, owing to the non-existence of $N_{k_0}^*(x_0, 1)$ in $SC_{k_0}^{n_0,l,*}$, we observe

$$O_{x_0} \not\subseteq N_{k_0}(x_0, 1). \tag{5.3}$$

Thus, by (5.2) and (5.3) $f(O_{x_0}) \not\subseteq O_{f(x_0)}$ because $f(O_{x_0}) \not\subseteq N_{k_1}(f(x_0), 1)$ and f is a bijection. This implies that f cannot be KD-(k_0, k_1)-continuous at the point $x_0 \in SC_{k_0}^{n_0,l,*}$. □

Example 5.2. $SC_{18}^{3,6,*}$ in Figure 1 cannot be KD-18-homotopy equivalent to $SC_{18}^{3,6}$. If not, suppose that there is a KD-18-homotopy equivalence $f : SC_{18}^{3,6,*} \to SC_{18}^{3,6}$. Then, let us examine if the map f is a KD-18-homotopy equivalence at the point $b_0 \in SC_{18}^{3,6,*}$. Indeed, the smallest open set containing the point b_0 is the set

$$\{b_0, b_1, b_2, b_4, b_5\} := O_{b_0}$$

and the smallest open set containing the point $f(b_0)$ is the singleton $\{f(b_0)\}$ or $N_{18}(f(b_0), 1)$. Thus we should obtain

$$f(O_{b_0}) \subset \{f(b_0)\} \quad \text{or} \quad f(O_{b_0}) \subset N_{18}(f(b_0), 1),$$

which is a contradiction to the KD-18-homotopy equivalence of f.

In DTC, the recent paper [5] proved the following: SC_k^{n,l_0} cannot be (k, k)-homotopy equivalent to SC_k^{n,l_1} if $l_0 \neq l_1$. Motivated this property, we obtain the following.

Theorem 5.3. *In KDTC, $SC_{k_0}^{n_0,l_0}$ cannot be KD-(k_0, k_1)-homotopy equivalent to $SC_{k_1}^{n_1,l_1}$ if $l_0 \neq l_1$.*

Proof. Owing to the property (2) of Definition 4, the proof is completed. □

By Theorem 5.3, we obtain the following.

Theorem 5.4. *$SC_{k_0}^{n_0,l_0,*}$ cannot be KD-(k_0,k_1)-homotopy equivalent to $SC_{k_1}^{n_1,l_1,*}$ if $l_0 \neq l_1$.*

Proof. We may assume that $f : SC_{k_0}^{n_0,l_0,*} \to SC_{k_1}^{n_1,l_1,*}$ is not a constant map. If not, then we obviously observe that the map f cannot be a KD-(k_0, k_1)-homotopy equivalence.

Next, put $SC_{k_0}^{n_0,l_0,*} := (c_i)_{i\in[0,l_0-1]_{\mathbb{Z}}}$ and $SC_{k_1}^{n_1,l_1,*} := (d_j)_{j\in[0,l_1-1]_{\mathbb{Z}}}$ as sequences. Let us assume $l_0 \lneq l_1$ for convenience. Then, owing to (3.1) and

Theorem 5.3, we observe that there is no KD-(k_0, k_1)-homotopy equivalence $f : SC_{k_0}^{n_0,l_0,*} \to SC_{k_1}^{n_1,l_1,*}$. □

By Remark 4.5, we obtain the following.

Theorem 5.5. *In KDTC, $SC_{k_0}^{n_0,l_0,*}$ need not be KD-(k_0, k_1)-homotopy equivalent to $SC_{k_1}^{n_1,l_1,*}$ even if $l_0 = l_1$.*

Proof. As a counterexample we consider the two spaces

$$SC_4^{2,12,*} := (a_i)_{i\in[0,11]_{\mathbb{Z}}} \quad \text{and} \quad SC_{18}^{3,12,*} := (b_j)_{j\in[0,11]_{\mathbb{Z}}}$$

in Figures 1 and 5(c), respectively. Then, even though the two spaces $SC_4^{2,12,*}$ and $SC_{18}^{3,12,*}$ have the same cardinality, they must not be KD-$(4, 18)$-homotopy equivalent to each other. To be specific, owing to the point a_i which does not have $N_4^*(a_i, 1) \subset SC_4^{2,12,*}$, we cannot have a KD-$(4, 18)$-homotopy equivalence $f : SC_4^{2,12,*} \to SC_{18}^{3,12,*}$ because any map $f : SC_4^{2,12,*} \to SC_{18}^{3,12,*}$ cannot be a KD-$(4, 18)$-homotopy equivalence at the point $a_0 \in SC_4^{2,12,*}$.

Precisely, first in order for the map $f : SC_4^{2,12,*} \to SC_{18}^{3,12,*}$ to be KD-$(4, 18)$-homotopy equivalence, we clearly observe that the map f should be at least a bijection.

Second, consider the point $a_0 \in SC_4^{2,12,*}$ and assume its image $f(a_0) \in SC_{18}^{3,12,*}$. While the three points a_0, a_4, and a_8 in $SC_4^{2,12,*}$ do not have their computer topological 4-neighborhoods with radius 1, only the two points b_0 and b_6 in $SC_{18}^{3,12,*}$ do not have their computer topological 18-neighborhoods with radius 1, either.

While the smallest open set containing the point a_0 is the set $\{a_{10}, a_{11}, a_0, a_1, a_2, a_6\}$ in Figure 1 which is not 4-connected, the open 18-neighborhoods of each point in $SC_{18}^{3,12,*}$ are 18-connected and further, consist of three points or five points, *e.g.,*

$$\sharp N_{18}(b_0, 2) = 5,$$
$$\sharp N_{18}(b_6, 2) = 5,$$
$$\sharp N_{18}(b_i, 1) = \sharp\{b_{i-1(\mathrm{mod}\,12)}, b_{i(\mathrm{mod}\,12)}, b_{i+1(\mathrm{mod}\,12)}\} = 3,$$

where '$\sharp$' means the cardinality of a set and $i \in [0, 11]_{\mathbb{Z}} - \{0, 6\}$.

Let us denote $O_{f(a_0)}$ the smallest open set containing the point $f(a_0)$. Then we obtain the following

$$(5.4) \qquad O_{f(a_0)} = \left\{ \begin{array}{lll} N_{18}(f(a_0), 2) & \text{if} \;\; f(a_0) \in \{b_0, b_6\}; \\ N_{18}(f(a_0), 1) & \text{if} \;\; f(a_0) \;\; \text{is a mixed point;} \\ \{f(a_0)\} & \text{if} \;\; f(a_0) \;\; \text{is an open point.} \end{array} \right\}$$

In order for the map f to be a KD-$(18, 18)$-continuous map, $f(O_{a_0})$ should be a subset of $O_{f(a_0)}$ in (5.4). This leads to the contradiction to the bijection of f and the non-preservation of the 4-connectivity of $SC_4^{2,12,*}$. Thus the two spaces $SC_4^{2,12,*}$ and $SC_{18}^{3,12,*}$ cannot be KD-$(4,18)$-homotopy equivalent to each other. □

By Theorem 5.5, we obtain the following.

Corollary 5.6. *$SC_k^{n,l,*}$ need not be KD-(k,k)-homotopy equivalent to $SC_k^{n,l_1,*}$ even if $l_0 = l_1$.*

Let us now remind again that $C_k^{n,l}$ is a closed k-curve with l elements as a subspace of the Khalimsky n-dimensional space $(\mathbb{Z}^n, T^n)$ whose every element $x \in C_k^{n,l}$ has an $N_k^*(x,1)$ and satisfies the property (5.1) and further, $C_k^{n,l,*}$ is a closed k-curve with l elements as a subspace of $(\mathbb{Z}^n, T^n)$ whose some point $x \in C_k^{n,l,*}$ does not have an $N_k^*(x,1)$ (see the spaces in Figure 6) and every point in $C_k^{n,l,*}$ satisfies the property (5.1).

We observe that topological structures of $SC_{k_0}^{n_0,l_0,*} := (c_i)_{i\in[0,l_0-1]_{\mathbb{Z}}}$ and $C_{k_1}^{n_1,l_1,*} := (d_j)_{j\in[0,l_1-1]_{\mathbb{Z}}}$ influence the arrangement of the points c_i and d_j which do not have $N_{k_0}^*(c_i,1) \subset SC_{k_0}^{n_0,l_0,*}$ and $N_{k_1}^*(d_j,1) \subset SC_{k_1}^{n_1,l_0,*} \subset C_{k_1}^{n_1,l_1,*}$. In addition, they are also related to the cardinalities of the sets of the points c_i and d_j which do not have $N_{k_0}^*(c_i,1) \subset SC_{k_0}^{n_0,l_0,*}$ and $N_{k_1}^*(d_j,1) \subset SC_{k_1}^{n_1,l_0,*} \subset C_{k_1}^{n_1,l_1,*}$.

By Theorem 5.5, we obtain the following.

Corollary 5.7. *$SC_{k_0}^{n_0,l_0,*} := (c_i)_{i\in[0,l_0-1]_{\mathbb{Z}}}$ need not be KD-(k_0,k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1,*} := (d_j)_{j\in[0,l_1-1]_{\mathbb{Z}}}$ even if $l_0 = l_1 - t_1$, where t_1 is the number of the KD-k_1-homotopic thinning points in $C_{k_1}^{n_1,l_1,*}$.*

Example 5.8. Consider the two spaces $SC_4^{2,8,*}$ and $C_{18}^{3,10,*}$ (see Figure 6(a)) and further, the subspace $SC_{18}^{3,8,*} := C_{18}^{3,10,*} - \{a_4, a_6\}$ in Figure 6(a). Then the space $SC_4^{2,8,*}$ cannot be KD-$(4,18)$-homotopy equivalent to $SC_{18}^{3,8,*}$. Since there are points x in $SC_4^{2,8,*}$ and $SC_{18}^{3,8,*}$ such that

$$\sharp\{x \mid \nexists N_4^*(x,1) \subset SC_4^{2,8,*}\} = 2 \quad \text{and} \quad \sharp\{x \mid \nexists N_{18}^*(x,1) \subset SC_{18}^{3,8,*}\} = 1.$$

This difference between the two cardinalities 1 and 2 above makes impossible obtain a KD-$(4,18)$-homotopy equivalence between $SC_4^{2,8,*}$ and $SC_{18}^{3,8,*}$.

As shown in Example 5.8, in general, $C_{k_0}^{n_0,l_0,*}$ need not be KD-(k_0,k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1,*}$ even if $l_0 - t_0 = l_1 - t_1$, where t_i is the number of the KD-k_i-homotopic thinning points in $C_{k_i}^{n_i,l_i,*}, i \in \{0,1\}$. Let us now compare between $C_{k_0}^{n_0,l_0,*}$ and $C_{k_1}^{n_1,l_1,*}$ up to a KD-(k_0,k_1)-homotopy equivalence.

Motivated by Theorems 5.1, 5.3 and 5.5, and Corollary 5.7, we obtain the following.

Theorem 5.9. *$C_{k_0}^{n_0,l_0,*}$ is KD-(k_0,k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1,*}$ if and only if there is $SC_{k_i}^{n_i,l'_i,*} \subset C_{k_i}^{n_i,l_i,*}, i \in \{0,1\}$, such that $SC_{k_0}^{n_0,l'_0,*}$ is KD-(k_0,k_1)-homeomorphic to $SC_{k_1}^{n_1,l'_1,*}$.*

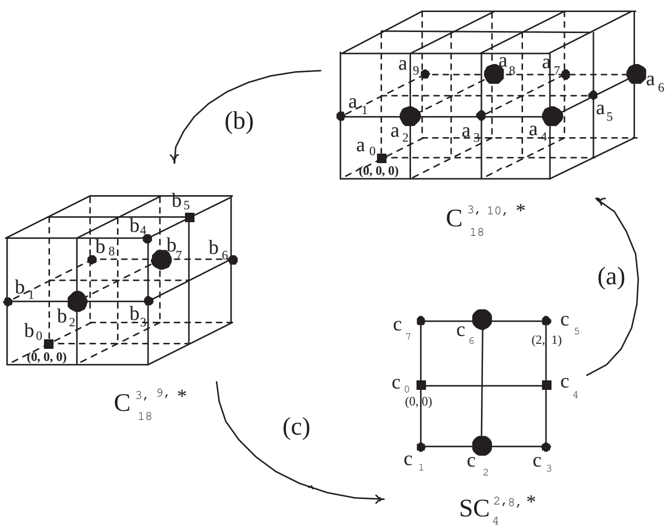

FIGURE 6. Comparison between $C_{k_0}^{n_0,l_0,*}$ and $C_{k_1}^{n_1,l_1,*}$ up to a KD-(k_0, k_1)-homotopy equivalence

Proof. First, let us show a method of obtaining $SC_{k_i}^{n_i,l_i',*} \subset C_{k_i}^{n_i,l_i,*}, i \in \{0, 1\}$. Namely, removing all KD-k_i-homotopic thinning points from $C_{k_i}^{n_i,l_i,*}, i \in \{0, 1\}$, we can obtain a space $SC_{k_i}^{n_i,l_i',*} := C_{k_i}^{n_i,l_i,*} - Y_i$ which is KD-k_i-homotopy equivalent to $C_{k_i}^{n_i,l_i,*}$ and $l_i' = l_i - t_i$, where Y_i is the set of all KD-k_i-homotopic thinning points in $C_{k_i}^{n_i,l_i,*}$ and t_i is the cardinality of Y_i.

Let us now assume that the space $SC_{k_0}^{n_0,l_0',*} \subset C_{k_0}^{n_0,l_0,*}$ is KD-(k_0, k_1)-homeomorphic to the space $SC_{k_1}^{n_1,l_1',*} \subset C_{k_1}^{n_1,l_1,*}$, then by Theorem 4.4, the following composition of KD-k_0-, KD-(k_0, k_1)-, and KD-k_1-homotopy equivalences,

$$C_{k_0}^{n_0,l_0,*} \to SC_{k_0}^{n_0,l_0',*} \to SC_{k_1}^{n_1,l_1',*} \to C_{k_1}^{n_1,l_1,*}$$

is also a KD-(k_0, k_1)-homotopy equivalence because a KD-k_1-homeomorphism is a special kind of a KD-k_1-homotopy equivalences, which proves the assertion.

Conversely, if $C_{k_0}^{n_0,l_0,*} \to C_{k_1}^{n_1,l_1,*}$ is a KD-(k_0, k_1)-homotopy equivalence, then by a removal of each KD-k_i-homotopic thinning point in $C_{k_i}^{n_i,l_i,*}$ shown above, we obtain a subspace $SC_{k_i}^{n_i,l_i',*} \subset C_{k_i}^{n_i,l_i,*}$ which is KD-k_i-homotopy equivalent to $C_{k_i}^{n_i,l_i,*}, i \in \{0, 1\}$. Then, $SC_{k_0}^{n_0,l_0',*}$ is KD-(k_0, k_1)-homeomorphic to $SC_{k_1}^{n_1,l_1',*}$ with $l_i' \leq l_i$.

If not, suppose that

$$SC_{k_0}^{n_0,l_0',*} \text{ is not } KD\text{-}(k_0,k_1)\text{-homeomorphic to } SC_{k_1}^{n_1,l_1',*}. \tag{5.5}$$

Then, we will get a contradiction to the hypothesis of the KD-k_i-homotopy equivalence between $C_{k_0}^{n_0,l_0,*}$ and $C_{k_1}^{n_1,l_1,*}$.

Precisely, from (5.5) we can consider the following two cases (see Definition 5).

(Case 1) In case of $l_0' \neq l_1'$ in (5.5), then, by Theorem 5.4, $SC_{k_0}^{n_0,l_0',*}$ cannot be KD-(k_0,k_1)-homotopy equivalent to $SC_{k_1}^{n_1,l_1',*}$. This implies that $C_{k_0}^{n_0,l_0',*}$ cannot be KD-(k_0,k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1',*}$, which contradicts to the hypothesis.

(Case 2) In case of $l_0' = l_1'$ in (5.5), we can assume a non-existence of either a KD-(k_0,k_1)-continuous map

$$f : SC_{k_0}^{n_0,l_0',*} := (c_i)_{i\in[0,l_0'-1]_{\mathbb{Z}}} \to SC_{k_1}^{n_1,l_1',*} := (d_j)_{j\in[0,l_1'-1]_{\mathbb{Z}}}$$

or a KD-(k_1,k_0)-continuous map

$$g : SC_{k_1}^{n_1,l_1',*} \to SC_{k_0}^{n_0,l_0',*}$$

such that $g \circ f = 1_{SC_{k_0}^{n_0,l_0',*}}$ and $f \circ g = 1_{SC_{k_1}^{n_1,l_1',*}}$.

First, assume that there is no KD-(k_0,k_1)-continuous map

$$f : SC_{k_0}^{n_0,l_0',*} := (c_i)_{i\in[0,l_0'-1]_{\mathbb{Z}}} \to SC_{k_1}^{n_1,l_1',*} := (d_j)_{j\in[0,l_1'-1]_{\mathbb{Z}}}$$

having an inverse map.

Let us now consider the following two cardinalities:

$$\sharp\{c_i' \in SC_{k_0}^{n_0,l_0',*} := (c_i)_{i\in[0,l_0'-1]_{\mathbb{Z}}}\} := \delta_0,$$

where the point c_i' has an $N_{k_0}^*(c_i',\varepsilon_0) \subset SC_{k_0}^{n_0,l_0',*}$ instead of $N_{k_0}^*(c_i',1)$ and the number ε_0 is the least element of $\mathbb{N} - \{1\}$ such that $N_{k_0}^*(c_i',\varepsilon_0)$ contains an open set including the point c_i'; and

$$\sharp\{d_j' \in SC_{k_1}^{n_1,l_1',*} := (d_j)_{j\in[0,l_1'-1]_{\mathbb{Z}}}\} := \delta_1,$$

where the point d_j' has $N_{k_0}^*(d_j',\varepsilon_1) \subset SC_{k_1}^{n_1,l_1',*}$ instead of $N_{k_0}^*(d_i',1)$ and the number ε_1 is the least element of $\mathbb{N} - \{1\}$ such that $N_{k_1}^*(d_j',\varepsilon_1)$ contains an open set including the point d_j'.

Then we can obtain the following three cases:

$$\delta_0 \neq \delta_1; \quad \text{or} \tag{5.6}$$

$$\varepsilon_0 \neq \varepsilon_1; \quad \text{or} \tag{5.7}$$

$$\delta_0 = \delta_1 \quad \text{and} \quad \varepsilon_0 = \varepsilon_1. \tag{5.8}$$

Then, under $l_0 = l_1$ of the hypothesis, (5.6) implies that $C_{k_0}^{n_0,l_0',*}$ cannot be KD-(k_0, k_1)-homotopic to $C_{k_1}^{n_1,l_1',*}$.

Similarly, (5.7) also implies that $C_{k_0}^{n_0,l_0',*}$ cannot be KD-(k_0, k_1)-homotopic to $C_{k_1}^{n_1,l_1',*}$.

Finally, (5.8) implies the difference of the arrangements of the points c_i and d_j which do not have $N_{k_0}^*(c_i, 1) \subset SC_{k_0}^{n_0,l_0,*} \subset C_{k_0}^{n_0,l_0,*}$ and $N_{k_1}^*(d_j, 1) \subset SC_{k_1}^{n_1,l_0,*} \subset C_{k_1}^{n_1,l_1,*}$ (see the map $f : SC_4^{2,12,*} := (a_i)_{i\in[0,11]_{\mathbb{Z}}} \to SC_{18}^{3,12,*} := (b_j)_{j\in[0,11]_{\mathbb{Z}}}$ given by $f(a_i) = b_i, i \in [0, 11]_{\mathbb{Z}}$ in Theorem 5.5).

Therefore, we can obviously observe that $C_{k_0}^{n_0,l_0',*}$ cannot be KD-(k_0, k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1',*}$, which contradicts to the hypothesis.

Second, assume that there is no KD-(k_1, k_0)-continuous map

$$g : SC_{k_1}^{n_1,l_1',*} := (d_j)_{j\in[0,l_1'-1]_{\mathbb{Z}}} \to SC_{k_0}^{n_0,l_0',*} := (c_i)_{i\in[0,l_0'-1]_{\mathbb{Z}}}$$

having an inverse map. Then, by the same method as the just above we can prove the assertion. □

Example 5.10. Consider the three spaces $SC_4^{2,8,*}$, $C_{18}^{3,10,*}$, and $C_{18}^{3,9,*}$ in Figure 6. Even though $SC_4^{2,8,*}$ is not KD-$(4, 18)$-homotopy equivalent to $C_{18}^{3,10,*}$ (see Figure 6(a)), we observe that $C_{18}^{3,9,*}$ is KD-$(18, 4)$-homotopy equivalent to $SC_4^{2,8,*}$ (see Figure 6(c)). Consequently, $C_{18}^{3,10,*}$ cannot be KD-18-homotopy equivalent to $C_{18}^{3,9,*}$ (see Figure 6(b)). Precisely, after removing a KD-18-homotopic thinning point from $C_{18}^{3,9,*}$, we obtain the space $C_{18}^{3,9,*} - \{b_4\}$ which is a simple closed 18-curve with eight elements. Furthermore, we observe that $C_{18}^{3,9,*} - \{b_4\}$ is KD-$(18, 4)$-homeomorphic to $SC_4^{2,8,*}$. Thus it turns out that $C_{18}^{3,9,*}$ is KD-$(18, 4)$-homotopy equivalent to $SC_4^{2,8,*}$.

By the similar method as Theorem 5.9, we obtain the following.

Theorem 5.11. *$C_{k_0}^{n_0,l_0}$ is KD-(k_0, k_1)-homotopy equivalent to $C_{k_1}^{n_1,l_1}$ if and only if $SC_{k_0}^{n_0,l_0'} \subset C_{k_0}^{n_0,l_0}$ is KD-(k_0, k_1)-homeomorphic to $SC_{k_1}^{n_1,l_1'} \subset C_{k_1}^{n_1,l_1}$.*

6. Summary and further work

We have studied KD-(k_0, k_1)-homotopy equivalence in KDTC in relation with the classification of computer topological spaces in KDTC. Indeed, if the current KD-(k_0, k_1)-homotopy equivalence is adapted in DTC, then this is exactly the digital (k_0, k_1)-homotopy equivalence. These have some advantages and a disadvantages, their usages depend on the situation. Furthermore, the notions of KD-k-deformation retraction and KD-(k_0, k_1)-homotopy equivalence will be applied to the image synthesis, image segmentation, image weaving, pattern matching, and image analysis. Moreover, the two notions of KD-(k_0, k_1)-continuity and KD-(k_0, k_1)-homeomorphism will play a significant role in studying the existence of KD-(k, k_1)-covering theory as a further work in relation with the calculation of the k-fundamental group of some space $X_{n,k}$.

Appendix: The general k-adjacency relations of $\mathbb{Z}^n$

Since the notion of k-connectivity for a space in $\mathbb{Z}^n$ is absolutely necessary for the study of spaces in $\mathbb{Z}^n, n \geq 1$, we have used the *general k-adjacency relations* of $\mathbb{Z}^n$, i.e.,

(A.1)

$$k \in \{3^n - 1(n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(2 \leq r \leq n-1, n \geq 3), 2n(n \geq 1)\},$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, we use the notation $[a, b]_{\mathbb{Z}} = \{a \leq n \leq b \,|\, n \in \mathbb{Z}\}$ without topology in Appendix.

We now show precisely the establishment of the general k-adjacency relations of $\mathbb{Z}^n$ as follows.

Consider the two functions $d_n, d_* : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{N} \cup \{0\}$ such that

- $d_n(p, q) = \sum_{i=1}^n |p_i - q_i|$ and
- $d_*(p, q) = \max\{|p_i - q_i|\}_{i \in [1,n]_{\mathbb{Z}}}$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in \mathbb{Z}^n$.

Then the general k-adjacency relations of $\mathbb{Z}^n$ in (A.1) are obtained from the n kinds of digital k-neighbors of $\mathbb{Z}^n$ stemmed from the two functions d_n, d_* above:

(1) For $n \geq 2$, consider the set $\{q \in \mathbb{Z}^n \,|\, d_n(p, q) \leq n, d_*(p, q) = 1\}$ denoted by $N_{3^n-1}(p)$. Then we can observe that $\sharp N_{3^n-1}(p) = 3^n - 1$, where $\sharp$ means the cardinality of a set.

(2) For $n \geq 3$, take the set $\{q \in \mathbb{Z}^n \,|\, d_n(p, q) \leq n - 1, d_*(p, q) = 1\}$ denoted by $N_{3^n-2^n-1}(p)$. Then we observe that $\sharp N_{3^n-2^n-1}(p) = 3^n - 2^n - 1$.

To be specific, for $p = (p_1, p_2, p_3, \ldots, p_n) \in \mathbb{Z}^n$,

$$N_{3^n-2^n-1}(p) := N_{3^n-1}(p) - X_n(p),$$

where $X_n(p) = \{q \in \mathbb{Z}^n \,|\, d_n(p, q) = n, d_*(p, q) = 1\}$. Precisely, we observe that $X_n(p) = \cup_{i=0}^n X_n(p)^i$ in terms of the following notations:

$X_n(p)^0 := \{(p_1 + 1, p_2 + 1, p_3 + 1, \ldots, p_n + 1)\}$ which is a singleton whose all coordinates are $p_i + 1$.

Consequently, $\sharp X_n(p)^0 = C_0^n = 1$; $X_n(p)^1 := \{(p_1+1, p_2+1, p_3+1, \ldots, p_{i-1}+1, p_i - 1, p_{i+1} + 1, \ldots, p_{n-1} + 1, p_n + 1)\}$ consisting of the elements such that only one coordinate is $p_i - 1$ and the others are $p_j + 1, i \neq j$, $i, j \in [1, n]_{\mathbb{Z}}$.

Thus $\sharp X_n(p)^1 = C_1^n$; $X_n(p)^2 := \{(p_1 + 1, p_2 + 1, p_3 + 1, \ldots, p_{i-1} + 1, p_i - 1, p_{i+1} + 1, \ldots, p_{j-1} + 1, p_j - 1, p_{j+1} + 1, \,, \ldots, p_n + 1)\}$ consisting of the elements such that only two tuples of coordinates are $p_i - 1, p_j - 1$, and the others are $p_k + 1, k \notin \{i, j\} \subset [1, n]_{\mathbb{Z}}$. Indeed, we can observe that $\sharp X_n(p)^2 = C_2^n$; Similarly, $X_n(p)^3 := \{(p_1+1, p_2+1, \ldots, p_{i-1}+1, p_i-1, p_{i+1}+1, \ldots, p_{j-1}+1, p_j-1, p_{j+1}+1, \ldots, p_{k-1} + 1, p_k - 1, p_{k+1} + 1, \ldots, p_n + 1)\}$ consisting of the elements such that only three tuples of coordinates are p_i-1, p_j-1 and p_k-1 and the others are $p_m+1, m\notin\{i,j,k\}\subset[1,n]_{\mathbb{Z}}$. Thus $\sharp X_n(p)^3=C_3^n$;

Sequentially, assume that $X_n(p)^{n-1}:=\{(p_1+1,p_2-1,p_3-1,\dots,p_n-1),(p_1-1,p_2+1,p_3-1,p_4-1,\dots,p_n-1),\dots,(p_1-1,p_2-1,p_3-1,\dots,p_{n-1}-1,p_n+1)\}$ in which only one coordinate is p_i+1 and the others are p_j-1 for $i,j\in[1,n]_{\mathbb{Z}}$ with $j\neq i$. Thus $\sharp X_n(p)^{n-1}=C_{n-1}^n$;

Finally, suppose that $X_n(p)^n:=\{(p_1-1,p_2-1,p_3-1,\dots,p_n-1)\}$ which is a singleton whose all coordinates are $p_i-1, i\in[1,n]_{\mathbb{Z}}$. Thus $\sharp X_n(p)^n=C_n^n$.

Then if $i,j\in[1,n]_{\mathbb{Z}}$ with $i\neq j$, then $X_n(p)^i$ and $X_n(p)^j$ above are disjoint. Thus we obtain that $\sharp X_n(p)=\sum_{i=0}^n C_i^n$. Consequently, we observe that

$$\begin{aligned}&\sharp\{q\in\mathbb{Z}^n|d_n(p,q)\leq n-1, d_*(p,q)=1\}\\ &=\sharp(N_{3^n-1}(p)-X_n(p))\\ &=3^n-1-(C_0^n+C_1^n+C_2^n+\cdots+C_n^n)=3^n-2^n-1, n\geq 3.\end{aligned}$$

(3) For $n\geq 4$, consider the set

$$\{q\in\mathbb{Z}^n|d_n(p,q)\leq n-2, d_*(p,q)=1\}:=N_{3^n-2^n-1}(p)-X_{n-1}(p)$$

denoted by $N_{3^n-2^n-n2^{n-1}-1}(p)$, where $X_{n-1}(p)=\{q\in\mathbb{Z}^n\,|\,d_n(p,q)=n-1, d_*(p,q)=1\}$. More precisely, $X_{n-1}(p)=\{(p_1\pm1,p_2\pm1,p_3\pm1,\dots,p_{i-1}\pm1,p_i,p_{i+1}\pm1,\dots,p_n\pm1)\,|\,i\in[1,n]_{\mathbb{Z}}\}=\cup_{i=0}^{n-1}X_{n-1}(p)^i$ in terms of the following notations: $X_{n-1}(p)^0:=\{(p_1+1,p_2+1,\dots,p_{i-1}+1,p_i,p_{i+1}+1,\dots,p_{j-1}+1,p_j+1,p_{j+1}+1,\dots,p_n+1), i,j\in[1,n]_{\mathbb{Z}}\}$ consisting of the elements in which all coordinates are p_k+1 except p_i and only one coordinate is p_j-1 with $i\neq j\neq k$. Thus $\sharp X_{n-1}(p)^0=C_0^{n-1}$. $X_{n-1}(p)^1:=\{(p_1+1,p_2+1,\dots,p_{i-1}+1,p_i,p_{i+1}+1,\dots,p_{j-1}+1,p_j-1,p_{j+1}+1,\dots,p_n+1), i,j\in[1,n]_{\mathbb{Z}}\}$ consisting of the elements such that only one coordinate is p_j-1 except p_i, and the others are $p_k+1, k\in[1,n]_{\mathbb{Z}}$ and $i\neq j\neq k$. Thus we observe that $\sharp X_{n-1}(p)^1=C_1^{n-1}$.

In general, $t\in[2,n-2]_{\mathbb{Z}}$, $X_{n-1}(p)^t$ is assumed to be the set consisting of the elements in which t-tuples of coordinates are p_j-1 except p_i, $i,j\in[1,n]_{\mathbb{Z}}$ with $i\neq j$, and the others are $p_k+1, k\notin\{i,j\}$. Thus we observe that $\sharp X_{n-1}(p)^t=C_t^{n-1}$.

Finally, assume that

$$X_{n-1}(p)^{n-1}:=\{(p_1-1,p_2-1,p_3-1,\dots,p_{i-1}-1,p_i,p_{i+1}-1,\dots,p_n-1)\}.$$

Thus $\sharp X_{n-1}(p)^{n-1}=C_{n-1}^{n-1}$.

Then we observe that if $i,j\in[0,n-1]_{\mathbb{Z}}$ with $i\neq j$, then $X_{n-1}(p)^i$ and $X_{n-1}(p)^j$ are disjoint. Therefore it turns out that

$$\sharp X_{n-1}(p)=C_1^n(C_0^{n-1}+C_1^{n-1}+C_2^{n-1}+\cdots+C_{n-1}^{n-1}).$$

Thus we observe that

$$\begin{aligned} &\sharp\{q \in \mathbb{Z}^n | d_n(p,q) \leq n-2, d_*(p,q) = 1\} \\ &= \sharp(N_{3^n-2^n-1}(p) - X_{n-1}(p)) \\ &= 3^n - 2^n - 1 - C_1^n(C_0^{n-1} + C_1^{n-1} + C_2^{n-1} + \cdots + C_{n-1}^{n-1}) \\ &= 3^n - 2^n - n2^{n-1} - 1. \end{aligned}$$

Generally, for $l \in [4, n-1]_{\mathbb{Z}}$, the following is obtained.

(l) Consider the set $\{q \in \mathbb{Z}^n | d_n(p,q) \leq n-l+1, d_*(p,q) = 1\}$ denoted by $N_{3^n-(\sum_{t=0}^{r-2} C_t^n 2^{n-t})-1}(p)$. Similarly, we observe that

$$\sharp\{q \in \mathbb{Z}^n | d_n(p,q) \leq n-l+1, d_*(p,q) = 1\} = 3^n - (\sum_{t=0}^{r-2} C_t^n 2^{n-t}) - 1, r \in [4, n-1]_{\mathbb{Z}}.$$

(n) Finally, consider the set $\{q \in \mathbb{Z}^n | d_n(p,q) = 1\}$ denoted by $N_{2n}(p)$. Thus $\sharp N_{2n}(p) = 2n$.

At last, the general k-adjacency relations of $\mathbb{Z}^n$ in (A.1) are taken from (1) $\sim$ (n) above with the criterion that p and q are *k-adjacent* if $q \in N_k(p)$.

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