

**INCLUSION AND INTERSECTION THEOREMS WITH
APPLICATIONS IN EQUILIBRIUM THEORY
IN G -CONVEX SPACES**

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ABSTRACT. In this paper we obtain a very general theorem of ρ -compatibility for three multivalued mappings, one of them from the class \mathfrak{B} . More exactly, we show that given a G -convex space Y , two topological spaces X and Z , a (binary) relation ρ on 2^Z and three mappings $P : X \multimap Z$, $Q : Y \multimap Z$ and $T \in \mathfrak{B}(Y, X)$ satisfying a set of conditions we can find $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x})\rho Q(\tilde{y})$. Two particular cases of this general result will be then used to establish existence theorems for the solutions of some general equilibrium problems.

1. Introduction

Since Park and Kim introduced the concept of generalized convex space (simply, G -convex space) [30], a large number of works contributing mainly to the KKM theory and equilibrium theory on these spaces have appeared. On the other hand, in the fixed point theory of multivalued mappings, Park introduced a new class of multivalued mappings, called the “better” admissible class \mathcal{B} , first in topological vector spaces [23] and later in G -convex spaces [25].

In this paper, after a preliminary section, we obtain a very general theorem of ρ -compatibility for three multivalued mappings, one of them from the class \mathfrak{B} . More exactly, we show that given a G -convex space Y , two topological spaces X and Z , a (binary) relation ρ on 2^Z and three mappings $P : X \multimap Z$, $Q : Y \multimap Z$ and $T \in \mathfrak{B}(Y, X)$ satisfying a set of conditions we can find $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x})\rho Q(\tilde{y})$. Two particular cases of this general result will be then used to establish existence theorems for the solutions of some general equilibrium problems described below.

In many recent papers (see [1], [2], [4], [8]-[10], [12], [13], [15], [17]-[21], [29]) and the references therein) one studies one of more of the following equilibrium problems:

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For a suitable choice of the sets X, Z and V and of the maps $F : X \times Z \rightarrow V$ and $C : X \rightarrow V$ we are interesting in finding $x_0 \in X$ such that one of the following situations occurs:

- (I) $F(x_0, z) \subseteq C(x_0)$ for all $z \in Z$;
- (II) $F(x_0, z) \cap C(x_0) \neq \emptyset$ for all $z \in Z$;
- (III) $F(x_0, z) \not\subseteq C(x_0)$ for all $z \in Z$;
- (IV) $F(x_0, z) \cap C(x_0) = \emptyset$ for all $z \in Z$.

Each existence result concerning problem (I) (respectively, (II)) yields an existence theorem for problem IV (respectively III), if we take into account the following equivalences: $F(x, z) \subseteq C(x) \Leftrightarrow F(x, z) \cap (V \setminus C(x)) = \emptyset$ and $F(x, z) \cap C(x) \neq \emptyset \Leftrightarrow F(x, z) \not\subseteq (V \setminus C(x))$. For this reason we can fix our attention on problems (I) and (II), only.

In [3] and [4] the first author extends and unifies these problems by considering a (binary) relation ρ on 2^Z and looking for a point $x_0 \in X$ such that $F(x_0, z) \rho C(x_0)$ for all $z \in Z$. In Section 4 the following more general problem is considered:

Let $(Y, D; \Gamma)$ and $(Z, D'; \Gamma')$ be G -convex spaces, X and V be topological spaces, $C : X \rightarrow V$, $F : X \times Z \rightarrow V$ and $Q : Y \rightarrow Z$ be three mappings and ρ be a relation on 2^V . Find $x_0 \in X$ such that

- (V) $F(x_0, z) \rho C(x_0)$ for all $z \in Q(D)$.

In a very recent paper [5] the authors study the following problem:

Given a G -convex space $(Y, D; \Gamma)$, two topological spaces Z and V , three mappings $P : Y \rightarrow Z$, $F : Y \times Z \rightarrow V$, $C : Z \rightarrow V$ and a relation ρ on 2^V , find $y_0 \in Y$ such that

$$\forall u \in D \quad \exists z \in P(y_0) \text{ such that } F(u, z) \rho C(z),$$

and, respectively

$$\forall y \in Y \quad \exists z \in P(y_0) \text{ such that } F(y, z) \rho C(z).$$

Each of these two problems is called, in the paper mentioned above, weak ρ -equilibrium problem with respect to P . In the last section of the paper the two problems will be generalized as follows:

Given a G -convex space $(Y, D; \Gamma)$, three topological spaces X, Z and V , three mappings $P : X \rightarrow Z$, $F : Y \times Z \rightarrow V$, $C : Z \rightarrow V$ and a relation ρ on 2^V , find $x_0 \in X$ such that

- (VI₁) $\forall u \in D \quad \exists z \in P(x_0) \text{ such that } F(u, z) \rho C(z),$

and, respectively

- (VI₂) $\forall y \in Y \quad \exists z \in P(x_0) \text{ such that } F(y, z) \rho C(z).$

2. Preliminaries

Let X and Y be nonempty sets. A multivalued mapping (or simply, a mapping) $T : X \rightarrow Y$ is a function from X into the power set of a set Y . If $y \in Y$, the sets $T^-(y) = \{x \in X : y \in T(x)\}$, $T^*(y) = \{x \in X : y \notin T(x)\}$ are called the *fiber* and respectively, the *cofiber* of T on y . For $A \subseteq X$, $T(A) =$

$\bigcup_{x \in A} T(x)$ is the image of A under T . If X and Y are topological spaces a mapping $T : X \multimap Y$ is said to be: (i) *upper semicontinuous* (in short, u.s.c.) (respectively, *lower semicontinuous* (in short, l.s.c.)) if for every closed subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) *closed* if its graph (that is, the set $GrT = \{(x, y) \in X \times Y : y \in T(x)\}$) is a closed subset of $X \times Y$; (iii) *compact* if $T(X)$ is contained in a compact subset of Y .

The following lemma collects known facts about u.s.c. or l.s.c. mappings (see for instance [16] for assertions (i) (ii) and (iii), respectively [31] for assertion (iv)).

Lemma 1. *Let X and Y be topological spaces and $T : X \multimap Y$ be a mapping.*

- (i) *If Y is compact and T is closed, then T is u.s.c..*
- (ii) *If Y is regular and T is u.s.c. with closed values, then T is closed.*
- (iii) *If X is compact and T is u.s.c. with compact values, then $T(X)$ is compact.*
- (iv) *T is l.s.c. if and only if for any $x \in X$, $y \in T(x)$ and any net $\{x_\alpha\}$ converging to x , there exists a net $\{y_\alpha\}$ converging to y , with $y_\alpha \in T(x_\alpha)$ for each α .*

A *generalized convex space* or a *G-convex space* $(Y, D; \Gamma)$ (see [28]) consists of a topological space Y and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ there exists a subset $\Gamma(A)$ of Y and a continuous function $\Phi_A : \Delta_n \rightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subseteq \Gamma(J)$. Here $\langle D \rangle$ denotes the set of all nonempty finite subsets of D , Δ_n denotes the standard n -dimensional simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J the face of Δ_n corresponding to J ; that is, if $A = \{u_0, u_1, \dots, u_n\}$ and $J = \{u_{i_0}, u_{i_1}, \dots, u_{i_k}\} \subseteq A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. When D is a subset of Y , as matters stands throughout this paper, a subset C of Y is said to be *G-convex* if for each $A \in \langle C \cap D \rangle$, $\Gamma(A) \subseteq C$.

The main example of *G-convex space* corresponds to the case when $Y = D$ is a convex subset of a topological vector space, and for each $A \in \langle A \rangle$, $\Gamma(A)$ is the convex hull of A . For other examples of *G-convex* see [26] and [27].

Let X be a topological space and $(Y, D; \Gamma)$ be a *G-convex space*. The *better admissible class* \mathfrak{B} of mappings from Y into X (see [25]) is defined as follows:

$T \in \mathfrak{B}(Y, X) \Leftrightarrow T : Y \multimap X$ is a mapping such that for any $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ and any continuous mapping $p : T(\Gamma(A)) \rightarrow \Delta_n$ the composition $p \circ T|_{\Gamma(A)} \circ \phi_A : \Delta_n \multimap \Delta_n$ has a fixed point. The class $\mathfrak{B}(Y, X)$ includes many important classes of mappings, such as $\mathfrak{A}_c^\kappa(Y, X)$ in [24], $KKM(Y, X)$ in [7] and $A(Y, X)$ in [6], as proper subclasses.

As usual, for a subset A of a topological space, we denote the interior and the closure of A by $\text{int } A$ and \overline{A} , respectively. From now on all topological spaces will be assumed Hausdorff.

3. Inclusion and intersection theorems

In this section, unless otherwise specified, we shall assume that $(Y, D; \Gamma)$ is a G -convex space, X and Z are topological spaces, $T \in \mathfrak{B}(Y, X)$ is compact, $P : X \rightarrow Z$, $Q : Y \rightarrow Z$ are two mappings and ρ is a (binary) relation on 2^Z . Motivated by the considerations from the previous section we will consider the following two particular relations on 2^Z : if $A, B \subseteq Z$, then

- (i) $A\rho_1 B \Leftrightarrow A \subseteq B$;
- (ii) $A\rho_2 B \Leftrightarrow A \cap B \neq \emptyset$;

Denote by ρ^c the complementary relation of ρ , that is, for any $A, B \subseteq Z$ exactly one of the following assertions $A\rho B$, $A\rho^c B$ holds.

Theorem 2. *Suppose that the following conditions are satisfied:*

- (i) $X = \bigcup_{u \in D} \text{int} \{x \in X : P(x)\rho Q(u)\}$;
- (ii) *for each $x \in X$, $\{y \in Y : P(x)\rho Q(y)\}$ is G -convex.*

Then there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x})\rho Q(\tilde{y})$.

Proof. For $u \in D$ set $S(u) = \{x \in X : P(x)\rho Q(u)\}$. Let $X_0 = \overline{T(Y)}$. Since X_0 is compact there exists a finite set $A = \{u_0, u_1, \dots, u_n\} \subseteq D$ such that $X_0 = \bigcup_{i=0}^n (\text{int} S(u_i) \cap X_0)$. Let $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be a partition of unity on X_0 subordinated to the cover $\{\text{int} S(u_i) \cap X_0 : 0 \leq i \leq n\}$. Recall that this means that

$$\begin{cases} \alpha_i : X_0 \rightarrow [0, 1] \text{ is continuous for each } i \in \{0, 1, \dots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in \text{int} S(u_i); \\ \sum_{i=0}^n \alpha_i(x) = 1 \text{ for each } x \in X_0. \end{cases}$$

Define $f : T(\Gamma(A)) \rightarrow \Delta_n$ by

$$f(x) = \sum_{i=0}^n \alpha_i(x)e_i \text{ for all } x \in T(\Gamma(A)).$$

Clearly f is continuous. Since $(Y, D; \Gamma)$ is a G -convex space there exists a continuous function $\phi_A : \Delta_n \rightarrow \Gamma(A)$ such that $\phi_A(\Delta_J) \subset \Gamma(J)$ for each $J \in \langle A \rangle$. Since $T \in \mathfrak{B}(Y, X)$, there exists $\tilde{t} \in \Delta_n$ such that $\tilde{t} \in (f \circ T|_{\Gamma(A)} \circ \phi_A)(\tilde{t})$. Let $\tilde{y} = \phi_A(\tilde{t})$. Then $\tilde{y} \in (\phi_A \circ f \circ T)(\tilde{y})$. Thus for some $\tilde{x} \in T(\tilde{y})$ we have $\tilde{y} = \phi_A(f(\tilde{x}))$. Let $\tilde{J} = \{i \in \{0, 1, \dots, n\} : \alpha_i(\tilde{x}) > 0\}$. Then $f(\tilde{x}) \in \Delta_{\tilde{J}}$ and

$$(1) \quad \tilde{y} = \phi_A(f(\tilde{x})) \in \phi_A(\Delta_{\tilde{J}}) \subset \Gamma(\{u_i : i \in \tilde{J}\}).$$

For each $i \in \tilde{J}$ we have $\tilde{x} \in \text{int} S(u_i) \subset S(u_i)$, hence $\{u_i : i \in \tilde{J}\} \subset S^-(\tilde{x})$. By (ii) and (1) it follows that $\tilde{y} \in S^-(\tilde{x})$, that is $P(\tilde{x})\rho Q(\tilde{y})$ and the proof is complete. \square

Remark 1. In the particular case $X = Z$, $P = 1_X$ and ρ is either ρ_1 or ρ_2 the previous theorem generalizes Theorems 2.5 and 2.6 in [17], 3.1 and 3.5 in [18], and 3.1 in [20]. In the above mentioned results Y is a convex set in a topological vector space and $T \in KKM(Y, X)$. Also a particular case of Theorem 2 can

be compared with some closely related results obtained in FC -spaces by Ding (Theorems 4.4 in [8] and 3.1 in [9]) and Fang and Huang (Theorem 3.1 in [14]).

As usual the compactness hypothesis on T can be replaced by some coercivity conditions. The next result is an example in this direction.

Theorem 3. *Suppose that condition (i) and (ii) in Theorem 2 hold. Moreover, assume that $\overline{T(Y)}$ is paracompact and there exist a compact subset K of X and $B \in \langle D \rangle$ such that*

$$X \setminus K \subset \bigcup_{u \in B} \text{int} \{x \in X : P(x) \rho Q(u)\}.$$

Then there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x}) \rho Q(\tilde{y})$.

Proof. Since K is compact there exists $B' \in \langle D \rangle$ such that $K \subset \bigcup_{u \in B'} \text{int} \{x \in X : P(x) \rho Q(u)\}$. Let $A = B \cup B'$. We have $Y = \bigcup_{u \in A} \text{int} \{x \in X : P(x) \rho Q(u)\}$. Keeping the notations from the previous proof, $\mathcal{U} = \{\text{int } S(u) \cap X_0 : u \in A\}$ is an open cover of X_0 . Since X_0 is paracompact, there exists a partition of unity on X_0 subordinates to \mathcal{U} and the proof is now similar to that of Theorem 2. □

We say that a mapping $P : X \rightarrow Z$ is ρ -semicontinuous if for each open subset G of Z the set $\{x \in X : P(x) \rho G\}$ is open.

Remark 2. It's clear that ρ_1 (respectively, ρ_2)-semicontinuity means upper (respectively, lower)-semicontinuity.

Theorem 4. *Assume that the following conditions are fulfilled:*

- (i) *for each $x \in X$ there exists $u \in D$ such that $P(x) \rho Q(u)$;*
- (ii) *P is ρ -semicontinuous;*
- (iii) *$Q|_D$ has open values;*
- (iv) *for each $x \in X$, $\{y \in Y : P(x) \rho Q(y)\}$ is G -convex.*

Then there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x}) \rho Q(\tilde{y})$.

Proof. By (i), $X = \bigcup_{u \in D} \{x \in X : P(x) \rho Q(u)\}$. By (ii) and (iii), for each $u \in D$, $\{x \in X : P(x) \rho Q(u)\}$ is open. Thus, condition (i) in Theorem 2 is fulfilled. The conclusion follows from Theorem 2. □

When $\rho = \rho_1$, Theorem 4 yields the following result:

Corollary 5. *Assume that the following conditions are fulfilled:*

- (i) *for each $x \in X$ there exists $u \in D$ such that $P(x) \subseteq Q(u)$;*
- (ii) *P is u.s.c.;*
- (iii) *Q has G -convex fibers and $Q|_D$ has open values.*

Then there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x}) \subseteq Q(\tilde{y})$.

Proof. The desired conclusion follows from Theorem 4 as soon as we prove that for $\rho = \rho_1$ the requirement (iv) in Theorem 4 is fulfilled. Let $x \in X$ and $A \in \langle \{u \in D : P(x) \subseteq Q(u)\} \rangle$. Suppose that for some $y \in \Gamma(A)$ there exists $z \in P(x) \setminus Q(y)$. For each $u \in A$, $z \in P(x) \subseteq Q(u)$, hence $A \subset Q^-(z)$. Since $Q^-(z)$ is G -convex, it follows that $y \in Q^-(z)$, that is $z \in Q(y)$. The contradiction obtained show that the set $\{y \in Y : P(x) \subseteq Q(y)\}$ is G -convex. \square

Definition 1. A mapping $Q : Y \multimap Z$ is said to be:

- (a) G -strongly quasiconvex if for each $A \in \langle D \rangle$ and any $y \in \Gamma(A)$, there is $u \in A$ such that $Q(u) \subseteq Q(y)$.
- (b) G -quasiconvex [22] if Z is a G -convex space and for each subset G -convex M of Z the set $\{y \in Y : Q(y) \cap M \neq \emptyset\}$ is G -convex.

It's clear that a mapping G -strongly quasiconvex is G -quasiconvex.

The next result is obtained from Theorem 4, in a similar manner, when Z is a G -convex space and $\rho = \rho_2$.

Corollary 6. *Suppose that $(Z, \Gamma'; D')$ is a G -convex space and the following conditions are fulfilled:*

- (i) for each $x \in X$ there exists $u \in D$ such that $P(x) \cap Q(u) \neq \emptyset$;
- (ii) P is l.s.c.;
- (iii) $Q|_D$ has open values;
- (iv) either Q is G -strongly quasiconvex or Q is G -quasiconvex and P is G -convex valued.

Then there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x}) \cap Q(\tilde{y}) \neq \emptyset$.

Remark 3. Corollary 6 generalizes Theorem 4 in [4].

4. Existence results for Problem (V)

Throughout this section, unless otherwise specified, we shall assume that $(Y, D; \Gamma)$ and $(Z, D'; \Gamma')$ are G -convex spaces, X and V are topological spaces, $C : X \multimap V$, $F : X \times Z \multimap V$ and $Q : Y \multimap Z$ are three mappings and ρ is a relation on 2^V .

Theorem 7. *Suppose that the following conditions are satisfied:*

- (i) for each $z \in Z$ the set $\{x \in X : F(x, z) \rho C(x)\}$ is closed;
- (ii) $Q|_D$ has open values;
- (iii) one of the following situations occurs:
 - (iii₁) Q is strongly G -quasiconvex; or
 - (iii₂) Q is G -quasiconvex and for each $x \in X$ the set $\{z \in Z : F(x, z) \rho^c C(x)\}$ is G -convex;
- (iv) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$, $x \in T(y)$ and $z \in Q(y)$ we have $F(x, z) \rho C(x)$.

Then there exists $x_0 \in X$ such that $F(x_0, z) \rho C(x_0)$ for all $z \in Q(D)$.

Proof. Let $P : X \rightarrow Z$ be the mapping defined by

$$P(x) = \{z \in Z : F(x, z) \rho^c C(x)\}.$$

Assume that the conclusion does not hold. Then for each $x \in X$ there exists $u \in D$ and $z \in Q(u)$ such that $F(x, z) \rho^c C(x)$, hence $z \in P(x) \cap Q(u)$. Since P has open fibers (by (i)), it follows easily that P is l.s.c.. By (iii), it follows that condition (iv) in Corollary 6 is fulfilled. By Corollary 6, there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$ and $P(\tilde{x}) \cap Q(\tilde{y}) \neq \emptyset$. Let $z \in P(\tilde{x}) \cap Q(\tilde{y})$. Then $z \in Q(\tilde{y})$ and $F(\tilde{x}, z) \rho^c C(\tilde{x})$, which contradicts (iv). \square

The concepts introduced below extend to G -convex spaces the corresponding notions from vector spaces (see [18]).

Definition 2. Suppose that V is a topological vector space. The mapping $F(x, \cdot)$ is said to be:

- (i) $C(x)$ - G -quasiconvex if for each $x \in X$, $A \in \langle D' \rangle$ and $z \in \Gamma'(A)$ there exists $u' \in A$ such that $F(x, u') \subseteq F(x, z) + C(x)$;
- (ii) $C(x)$ - G -quasiconvex-like if for each $x \in X$, $A \in \langle D' \rangle$ and $z \in \Gamma'(A)$ there exists $u' \in A$ such that $F(x, z) \subseteq F(x, u') - C(x)$.

Theorem 8. Assume that:

- (i) C is a closed mapping;
- (ii) for each $z \in Z$, $F(\cdot, z)$ is l.s.c.;
- (iii) $Q|_D$ has open values;
- (iv) one of the following situations holds:
 - (iv₁) Q is G -strongly quasiconvex;
 - (iv₂) Q is G -quasiconvex and for each $x \in X$ $F(x, \cdot)$ is G -strongly quasiconvex;
 - (iv₃) V is a G -convex space, Q is G -quasiconvex, and for each $x \in X$, the set $V \setminus C(x)$ is G -convex and the mapping $F(x, \cdot)$ is G -quasiconvex;
 - (iv₄) V is a topological vector space, Q is G -quasiconvex, for each $x \in X$ $C(x)$ is a convex cone and $F(x, \cdot)$ is $C(x)$ - G -quasiconvex;
- (v) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$, $x \in T(y)$ and $z \in Q(y)$ we have $F(x, z) \subseteq C(x)$.

Then there exists $x_0 \in X$ such that $F(x_0, z) \subseteq C(x_0)$ for all $z \in Q(D)$.

Proof. It suffices to prove that for $\rho = \rho_1$ conditions (i) and (iii) in Theorem 7 are fulfilled. Let $z \in Z$ and $x \in \overline{P^*(z)}$, where $P^*(z) = \{x \in X : F(x, z) \subseteq C(x)\}$. Then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $P^*(z)$ such that $x_\alpha \rightarrow x$. For each $\alpha \in \Lambda$, $F(x_\alpha, z) \subseteq C(x_\alpha)$. Let $v \in F(x, z)$. By (ii), via Lemma 1 (iv), there is a net $\{v_\alpha\}_{\alpha \in \Lambda}$ in V converging to v such that $v_\alpha \in F(x_\alpha, z)$ for all $\alpha \in \Lambda$. Since $C : X \rightarrow V$ is closed, $v \in C(x)$ hence $F(x, z) \subseteq C(x)$. This shows that $\{x \in X : F(x, z) \subseteq C(x)\}$ is closed.

One can easily check that in each of the cases (iv₂) and (iv₃) for $x \in X$ arbitrarily fixed the set $\{z \in Z : F(x, z) \not\subseteq C(x)\}$ is G -convex and consequently

in each of these cases condition (iii) in Theorem 7 holds. We show that the situation is the same in case (iv₄). Let $A \in \langle D' \rangle$ such that for each $u' \in A$, $F(x, u') \not\subseteq C(x)$. Suppose that for a $z \in \Gamma'(A)$, $F(x, z) \subseteq C(x)$. Since $F(x, \cdot)$ is $C(x)$ - G -quasiconvex, for some $u' \in A$ we have $F(x, u') \subseteq F(x, z) + C(x) \subseteq C(x) + C(x) = C(x)$, which implies a contradiction. \square

Theorem 9. *Suppose that X is compact and the following conditions are satisfied:*

- (i) C is a closed mapping;
- (ii) for each $z \in Z$, $F(\cdot, z)$ is u.s.c. with nonempty compact values;
- (iii) $Q|_D$ has open values;
- (iv) one of the following situations holds:
 - (iv₁) Q is G -strongly quasiconvex;
 - (iv₂) Q is G -quasiconvex and for each $x \in X$ $F(x, \cdot)$ has G -convex cofibers;
 - (iv₃) V is a topological vector space, Q is G -quasiconvex, for each $x \in X$, $C(x)$ is a convex cone and $F(x, \cdot)$ is $C(x)$ - G -quasiconvex-like;
- (v) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$, $x \in T(y)$ and $z \in Q(y)$ we have $F(x, z) \cap C(x) \neq \emptyset$.

Then there exists $x_0 \in X$ such that $F(x_0, z) \cap C(x_0) \neq \emptyset$ for all $z \in Q(D)$.

Proof. We show first that for $\rho = \rho_2$ condition (i) in Theorem 7 holds. Let $z \in Z$ and $x \in \overline{P^*(z)}$, where $P^*(z) = \{x \in X : F(x, z) \cap C(x) \neq \emptyset\}$. Then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $P^*(z)$ such that $x_\alpha \rightarrow x$. For each $\alpha \in \Lambda$ there is $v_\alpha \in F(x_\alpha, z) \cap C(x_\alpha)$. Since $F(\cdot, z)$ is u.s.c. with nonempty compact values and X is compact, $F(X, z) = \bigcup_{x \in X} F(x, z)$ is compact hence without loss of generality we may suppose that $\{v_\alpha\}_{\alpha \in \Lambda}$ converges to $v \in F(X, z)$. The space $F(X, z)$ being compact it is regular and by Lemma 1 (ii), $F(\cdot, z)$ is closed. Hence $v \in F(x, z)$. Since C is closed, it follows that $v \in C(x)$. Thus $v \in F(x, z) \cap C(x)$ hence $\{x \in X : F(x, z) \cap C(x) \neq \emptyset\}$ is closed.

We show that in each of the cases (iv₂) and (iv₃) for $x \in X$ arbitrarily fixed, the set $\{z \in Z : F(x, z) \cap C(x) = \emptyset\}$ is G -convex and consequently in each of these cases condition (iii) in Theorem 7 holds.

Case (iv₂). Let $A \in \langle D' \rangle$ such that for each $u' \in A$, $F(x, u') \cap C(x) = \emptyset$. Suppose that for a $z \in \Gamma'(A)$ there exists $v \in F(x, z) \cap C(x)$. Then $v \in C(x)$ and for each $u' \in A$, since $F(x, u') \cap C(x) = \emptyset$, $v \notin F(x, u')$. It follows that $A \subseteq [F(x, \cdot)]^*(v)$. The cofiber $[F(x, \cdot)]^*(v)$ being convex, $z \in [F(x, \cdot)]^*(v)$. This implies $v \notin F(x, z)$; a contradiction.

Case (iv₃). Let $A \in \langle D' \rangle$ such that for each $u' \in A$, $F(x, u') \cap C(x) = \emptyset$. Suppose that for a $z \in \Gamma'(A)$, $F(x, z) \cap C(x) \neq \emptyset$. Since $F(x, \cdot)$ is $C(x)$ - G -quasiconvex-like, for some $u' \in A$ we have $F(x, z) \subseteq F(x, u') - C(x)$, hence $(F(x, u') - C(x)) \cap C(x) \neq \emptyset$. This implies $\emptyset \neq F(x, u') \cap (C(x) + C(x)) = F(x, u') \cap C(x)$; a contradiction.

The desired conclusion follows now from Theorem 7. \square

Remark 4. When Y and Z are convex sets in topological vector spaces our theorems from this section can be compared with similar results obtained by Lin et al. such as Theorems 4.1-4.6 in [18] and 4.1-4.4 in [20]. However, there are at least two major differences between our results and the above mentioned results: 1) Lin's results involve only three mappings F , C and T ; 2) In the above mentioned results $T \in KKM(X, Y)$. This condition is stronger than $T \in \mathfrak{B}(X, Y)$. One can also compare Theorems 8 and 9 with Theorems 11 and 13 in [4] and Theorem 4.1 and Corollaries 4.1 and 4.3 in [14].

Example. The following example shows that although the assumptions of Theorems 8 and 9 are not always easy to verify in practice, these theorems could be important tools in establishing the existence of solutions for the corresponding problems.

Let $X = [-2, 2]$, $Y = D = [-1, 1]$, $Z = D' = [-4, 4]$, $V = \mathbb{R}$ and for each $A \in \langle D \rangle$ (respectively, $A \in \langle D' \rangle$) $\Gamma(A) = coA$ (respectively $\Gamma'(A) = coA$). Let the mappings $C : X \multimap V$, $F : X \times Z \multimap V$, $T : Y \multimap X$ and $Q : Y \multimap Z$ be defined by:

$$C(x) = [x^2 - 3, \infty), \quad F(x, z) = [-z^2 + 2xz + 6, \infty), \\ T(y) = [y - 1, y + 1], \quad Q(y) = (y - 2, y + 2).$$

Observe that $F(x, z) \subseteq C(x)$ if and only if $|x - z| \leq 3$. Since T is a Kakutani mapping (that is, u.s.c with compact convex values) we have $T \in \mathfrak{B}(Y, X)$. For each $y \in Y$, $x \in T(y)$ and $z \in Q(y)$ we have $|x - z| \leq |x - y| + |y - z| \leq 3$, hence $F(x, z) \subseteq C(x)$. It is easy to check that all the requirements of Theorem 8 are fulfilled. An x_0 satisfying the conclusion of Theorem 8 is $x_0 = 0$.

5. Weak equilibrium problems

Theorem 10. Let X , Z and V be topological spaces and $(Y, D; \Gamma)$ be a G -convex space. Let $C : Z \multimap V$, $F : Y \times Z \multimap V$, $P : X \multimap Z$, be three mappings and ρ be a relation on 2^V . Assume that:

- (i) P is u.s.c.;
- (ii) for each $z \in Z$ the set $\{y \in Y : F(y, z)\rho^c C(z)\}$ is G -convex;
- (iii) for each $u \in D$ the set $\{z \in Z : F(u, z)\rho C(z)\}$ is closed;
- (iv) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$ and any $x \in T(y)$ there exists $z \in P(x)$ such that $F(y, z)\rho C(z)$.

Then there exists $x_0 \in X$ such that for each $u \in D$ there is $z \in P(x_0)$ satisfying $F(u, z)\rho C(z)$.

Proof. Define the mapping $Q : Y \multimap Z$ by

$$Q(y) = \{z \in Z : F(y, z)\rho^c C(z)\}.$$

Suppose that the conclusion does not hold. Then for each $x \in X$ there exists $u \in D$ such that $F(u, z)\rho^c C(z)$ for all $z \in P(x)$, that is, $P(x) \subseteq Q(u)$. By (ii), Q has G -convex values and by (iii), $Q|_D$ has open values. The requirements of Corollary 5 are verified, hence there exists $(\tilde{x}, \tilde{y}) \in X \times Y$ such that $\tilde{x} \in T(\tilde{y})$

and $P(\tilde{x}) \subseteq Q(\tilde{y})$. Then for each $z \in P(\tilde{x})$ we have $F(\tilde{y}, z) \rho^c C(z)$, but this contradicts (iv). \square

Theorem 11. *Let X, Y, Z, V, C, F and P be as in Theorem 10. Suppose that D is dense in Y , P has compact values, the mapping $S : Y \rightarrow Z$ defined by $S(y) = \{z \in Z : F(y, z) \rho C(z)\}$ is closed and conditions (i), (ii) and (iv) in Theorem 10 are fulfilled. Then there exists $x_0 \in X$ such that for each $y \in Y$ there is $z \in P(x_0)$ satisfying $F(y, z) \rho C(z)$, i.e., x_0 is a solution of the weak ρ -equilibrium problem with respect to P , (VI_2) .*

Proof. Since S is a closed mapping, it has closed values hence condition (iii) in Theorem 10 holds. By the previous theorem there is an $x_0 \in X$ which is solution of the weakly ρ -equilibrium problem with respect to P , (VI_1) . Let $y \in Y$. Since D is dense in Y there is a net $\{u_\alpha\}_{\alpha \in \Lambda}$ converging to y . For each $\alpha \in \Lambda$ there is $z_\alpha \in P(x_0)$ such that $F(u_\alpha, z_\alpha) \rho C(z_\alpha)$. Since $P(x_0)$ is compact we may suppose that $\{z_\alpha\}_{\alpha \in \Lambda}$ converges to a point $z \in P(x_0)$. Therefore $\{(u_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ is a net in GrS converging to (y, z) . Since S is closed, it follows that $z \in S(y)$, that is $F(y, z) \rho C(z)$. Hence x_0 is a solution of the weak ρ -equilibrium problem with respect to P , (VI_2) . \square

In the next results from this section we will assume that $(Z, D'; \Gamma')$ is a G -convex spaces and X, Y, V, C, F and P are as in Theorem 10.

Theorem 12. *Suppose that:*

- (i) P is u.s.c.;
- (ii) C is a closed mapping;
- (iii) for each $u \in D$, $F(u, \cdot)$ is l.s.c.;
- (iv) one of the following situations holds:
 - (iv₁) for each $z \in Z$, $F(\cdot, z)$ is G -strongly quasiconvex;
 - (iv₂) V is a G -convex space and for each $z \in Z$, $V \setminus C(z)$ is G -convex and $F(\cdot, z)$ G -quasiconvex;
 - (iv₃) V is a topological vector space and for each $z \in Z$, $C(z)$ is a convex cone and $F(\cdot, z)$ is $C(z)$ - G -quasiconvex;
- (v) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$ and any $x \in T(y)$ there exists $z \in P(x)$ such that $F(y, z) \subseteq C(z)$.

Then there exists $x_0 \in X$ such that for each $u \in D$ there is $z \in P(x_0)$ satisfying $F(u, z) \subseteq C(z)$.

Proof. The mappings P, F and C satisfy all the requirements of Theorem 10 when $\rho = \rho_1$. In order to prove that for each $u \in D$ the set $\{z \in Z : F(u, z) \subseteq C(z)\}$ is closed see the first part of the proof of Theorem 8. In order to prove that for each $z \in Z$ the set $\{y \in Y : F(y, z) \not\subseteq C(z)\}$ is G -convex see the last part of the same theorem. The conclusion follows from Theorem 10. \square

Theorem 13. *Suppose that D is dense in Y , conditions (i), (ii), (iv) and (v) in Theorem 12 are fulfilled and condition (iii) is replaced by the following condition:*

(iii') F is l.s.c. on $Y \times Z$.

Then there exists $x_0 \in X$ such that for each $y \in Y$ there is $z \in P(x_0)$ satisfying $F(y, z) \subseteq C(z)$.

Proof. It's clear that condition (iii') implies condition (iii) in Theorem 12. According to Theorem 11 it suffices to prove that the map $S : X \dashrightarrow Y$ defined by $S(y) = \{z \in Z : F(y, z) \subseteq C(z)\}$ is closed. Let $(y, z) \in \overline{GrS}$. Then there exists a net $\{(y_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ in GrS such that $(y_\alpha, z_\alpha) \rightarrow (y, z)$. Let $v \in F(y, z)$ be arbitrarily fixed. Since F is l.s.c. on $Y \times Z$, by Lemma 1 (iv), there is a net $\{v_\alpha\}_{\alpha \in \Lambda}$ such that $v_\alpha \in F(y_\alpha, z_\alpha)$ and $v_\alpha \rightarrow v$. For each $\alpha \in \Lambda$, since $(y_\alpha, z_\alpha) \in GrS$, we have $v_\alpha \in F(y_\alpha, z_\alpha) \subseteq C(z_\alpha)$. Since C is a closed mapping we infer that $v \in C(z)$, hence $(y, z) \in GrS$. Therefore S is closed. \square

Theorem 14. *Suppose that*

- (i) P is u.s.c.;
- (ii) C is a closed mapping;
- (iii) for each $u \in D$, $F(u, \cdot)$ is u.s.c.;
- (iv) one of the following situations holds:
 - (iv₁) for each $z \in Z$, $F(\cdot, z)$ has G -convex cofibers;
 - (iv₂) V is a topological vector space and for each $z \in Z$, $C(z)$ is a convex cone and $F(\cdot, z)$ is $C(z)$ - G -quasiconvex-like;
- (v) there exists a compact mapping $T \in \mathfrak{B}(Y, X)$ such that for each $y \in Y$ and any $x \in T(y)$ there exists $z \in P(x)$ such that $F(y, z) \cap C(z) \neq \emptyset$.

Then there exists $x_0 \in X$ such that for each $u \in D$ there is $z \in P(x_0)$ satisfying $F(u, z) \cap C(z) \neq \emptyset$.

Proof. Arguing as in the proof of Theorem 12 one can show that the mappings P, F, C satisfy all the requirements of Theorem 10 when $\rho = \rho_2$. Conclusion follows from Theorem 10. \square

Theorem 15. *Suppose that D is dense in Y , V is compact, conditions (i), (ii), (iv) and (iv) in Theorem 14 are fulfilled and condition (iii) is replaced by the following condition:*

(iii') F is a closed mapping.

Then there exists $x_0 \in X$ such that for each $y \in Y$ there is $z \in P(x_0)$ satisfying $F(y, z) \cap C(z) \neq \emptyset$.

Proof. According to Theorem 14 it suffices to prove that the map $S : Y \dashrightarrow Z$ defined by $S(y) = \{z \in Z : F(y, z) \cap C(z) \neq \emptyset\}$ is closed. Let $(y, z) \in \overline{GrS}$. Then there exists a net $\{(y_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ in GrS such that $(y_\alpha, z_\alpha) \rightarrow (y, z)$. For each $\alpha \in \Lambda$ let $v_\alpha \in F(y_\alpha, z_\alpha) \cap C(z_\alpha)$. Since V is compact we may suppose that $\{v_\alpha\}$ converges to a $v \in V$. Since F and C are closed it follows that $v \in F(y, z) \cap C(z)$, hence $(y, z) \in GrS$. \square

The following result is a version of Theorem 4 in [5]. It can be obtained from Theorem 10 taking $Y = Z$ and $T(y) = \{y\}$ for all $y \in Y$. In the same manner

from Theorems 11, 12, 13, 14 and 15 we can obtain versions of Theorems 5, 6, 7, 9 and 10 in the mentioned paper.

Theorem 16. *Let $(Y, D; \Gamma)$, $(Z, D'; \Gamma')$ be G -convex spaces, Y being compact, V be a topological space, ρ be a relation on 2^V and $P : Y \multimap Z$, $F : Y \times Z \multimap V$, $C : Y \multimap Z$ be three mappings satisfying the following conditions:*

- (i) P is u.s.c.;
- (ii) for each $z \in Z$ the set $\{y \in Y : F(y, z) \rho^c C(z)\}$ is G -convex;
- (iii) for each $u \in D$ the set $\{z \in Z : F(u, z) \rho C(z)\}$ is closed;
- (iv) for each $y \in Y$ there exists $z \in P(y)$ such that $F(y, z) \rho C(z)$.

Then there exists $y_0 \in Y$ such that for each $u \in D$ there is $z \in P(y_0)$ satisfying $F(u, z) \rho C(z)$.

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