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POSITIVE SOLUTIONS FOR A SYSTEM OF SINGULAR SECOND ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. Sufficient conditions for the existence of positive solutions for a coupled system of nonlinear nonlocal boundary value problems of the type

$$\begin{aligned} -x''(t) &= f(t, y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t)), \quad t \in (0,1), \\ x(0) &= y(0) = 0,\ x(1) = \alpha x(\eta),\ y(1) = \alpha y(\eta), \end{aligned}$$

are obtained. The nonlinearities $f, g : (0,1) \times (0,\infty) \to (0,\infty)$ are continuous and may be singular at $t = 0$, $t = 1$, $x = 0$, or $y = 0$. The parameters η, α satisfy $\eta \in (0,1)$, $0 < \alpha < 1/\eta$. An example is provided to illustrate the results.

1. Introduction

Nonlocal boundary value problems (BVPs) arise in different areas of applied mathematics and physics. For example, the vibration of a guy wire composed of N parts with a uniform cross section and different densities in different parts can be modeled as a nonlocal boundary value problem [18]; problems in the theory of elastic stability can also be modeled as nonlocal boundary value problems [19].

The study of nonlocal BVPs for linear second order ordinary differential equations was initiated by Il'in and Moiseev in [10, 11] and extended to nonlocal linear elliptic boundary value problems by Bitsadze and Samarskiĭ, [2, 3, 4]. Existence theory for nonlinear three-point boundary value problems was initiated by Gupta [9]. Since then the study of nonlinear regular multi-point BVPs has attracted the attention of many researchers; see for example, [5, 9, 13, 14, 15, 17, 18, 20] for scalar equations, and for systems of ordinary differential equations, see [6, 7, 12].

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Recently, the study of singular BVPs has also attracted some attention. An excellent resource with an extensive bibliography was produced by Agarwal and O'Regan [1]. Recently, S. Xie and J. Zhu [21] applied topological degree theory in a cone to study the following two point BVP for a coupled system of nonlinear fourth-order ordinary differential equations

$$
\begin{aligned}
-x^{(4)} &= f_1(t, y), \quad t \in (0,1),\\
-y'' &= f_2(t, x), \quad t \in (0,1),\\
&x(0) = x(1) = x''(0) = x''(1) = 0,\\
&y(0) = y(1) = 0.
\end{aligned}
\tag{1.1}
$$

In [21], the nonlinearities $f_i \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy $f_i(t,0) \equiv 0$ $(i = 1,2)$ and may be singular at $t = 0$ or $t = 1$ only.

More recently, Y. Zhou and Y. Xu [23] studied the following nonlocal BVP for a system of second order regular ordinary differential equations

$$
\begin{aligned}
-x''(t) &= f(t, y), \quad t \in (0,1),\\
-y''(t) &= g(t, x), \quad t \in (0,1),\\
x(0) &= 0, \quad x(1) = \alpha x(\eta),\\
y(0) &= 0, \quad y(1) = \alpha y(\eta),
\end{aligned}
\tag{1.2}
$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$, $f, g \in C([0,1] \times [0,\infty), [0,\infty))$, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$. The above system was extended to the singular case by B. Liu, L. Liu, and Y. Wu [16], where the nonlinearities f, g were assumed to be singular at $t = 0$ or $t = 1$ together with the assumption that $f(t,0) \equiv 0$, $g(t,0) \equiv 0$, $t \in (0,1)$.

In this paper, we generalize the system (1.2) by allowing f, g to be singular at $t = 0$, $t = 1$, $x = 0$, or $y = 0$ and obtain sufficient conditions for the existence of a positive solution of the BVP for the system of singular equations, (1.2). By singularity we mean that the functions $f(t,u)$ or $g(t,u)$ are allowed to be unbounded at $t = 0, t = 1$, or $u = 0$. In general, the assumption that there exist singularities with respect to the dependent variable is not new; see [1, 6], for example. However, in the case of nonlocal boundary conditions and coupled systems of ordinary differential equations, we believe this assumption is new.

Throughout this paper, we shall assume that

$$f, g : (0,1) \times (0,\infty) \to (0,\infty)$$

are continuous and may be singular at $t = 0$, $t = 1$, or $u = 0$. We also assume that $f(t,0), g(t,0)$ are not identically 0. Let $N > \max\{\frac{1}{\eta}, \frac{1}{1-\eta}, \frac{2-\alpha}{1-\alpha\eta}\}$ denote a fixed positive integer. Assume that the following conditions hold:

(A_1) there exist $K, L \in C((0,1), (0,\infty))$ and $F, G \in C((0,\infty), (0,\infty))$ such that

$$f(t,u) \le K(t)F(u), \quad g(t,u) \le L(t)G(u), \quad t \in (0,1),\ u \in (0,\infty)$$

and

$$a := \int_0^1 t(1-t)K(t)dt < +\infty, \quad b := \int_0^1 t(1-t)L(t)dt < +\infty;$$

(A_2) there exist $\alpha_1, \alpha_2 \in (0, \infty)$ with $\alpha_1\alpha_2 \leq 1$ such that

$$\lim_{u\to\infty} \frac{F(u)}{u^{\alpha_1}} \to 0, \quad \lim_{u\to\infty} \frac{G(u)}{u^{\alpha_2}} \to 0;$$

(A_3) there exist $\beta_1, \beta_2 \in (0, \infty)$ with $\beta_1\beta_2 \geq 1$ such that

$$\liminf_{u\to 0^+} \min_{t\in[\eta,1]} \frac{f(t,u)}{u^{\beta_1}} > 0, \quad \liminf_{u\to 0^+} \min_{t\in[\eta,1]} \frac{g(t,u)}{u^{\beta_2}} > 0;$$

(A_4) $f(t,u)$, $G(u)$ are non-increasing with respect to u and for each fixed $n \in \{N, N+1, N+2, \ldots\}$, there exists a constant $M_1 > 0$ such that $t \in [\frac{1}{n}, 1-\frac{1}{n}]$,

$$f\left(t, \tfrac{1}{n} + b\,\mu_n G(\tfrac{1}{n})\right) \geq M_1 \left(\nu_n \int_\eta^{1-1/n} (s-\tfrac{1}{n})(1-\tfrac{1}{n}-s)ds\right)^{-1};$$

(A_5) $F(u)$, $g(t,u)$ are non-increasing with respect to u and for each fixed $n \in \{N, N+1, N+2, \ldots\}$, there exists a constant $M_2 > 0$ such that

$$F\left(\nu_n \int_\eta^{1-1/n} (s-\tfrac{1}{n})(1-\tfrac{1}{n}-s)g(s, M_2)ds\right) \leq \frac{M_2 - \frac{1}{n}}{a\,\mu_n}.$$

The parameters μ_n and ν_n in (A_4) and (A_5) are given by

$$\mu_n = \frac{\max\{1,\alpha\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, \quad \nu_n = \frac{\min\{1,\alpha\}\min\{\eta-\frac{1}{n}, 1-\frac{1}{n}-\eta\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

Since $N > \max\{\frac{1}{\eta}, \frac{1}{1-\eta}, \frac{2-\alpha}{1-\alpha\eta}\}$, $\mu_n, \nu_n > 0$.

We state the main results of this paper here.

Theorem 1.1. *Assume that* $(A_1) - (A_3)$ *hold. Then the system* (1.1) *has at least one positive solution.*

Theorem 1.2. *Assume that* $(A_1), (A_2)$ *and* (A_4) *hold. Then the system* (1.1) *has at least one positive solution.*

Theorem 1.3. *Assume that* $(A_1), (A_3)$ *and* (A_5) *hold. Then the system* (1.1) *has at least one positive solution.*

Theorem 1.4. *Assume that* $(A_1), (A_4)$ *and* (A_5) *hold. Then the system* (1.1) *has at least one positive solution.*

2. Preliminaries

For each $x \in C[0,1]$ we write $\|x\| = \max\{|x(t)| : t \in [0,1]\}$. Clearly, $C[0,1]$ with the norm $\|\cdot\|$ is a Banach space. For $n \geq N$, define a cone P, and a cone K_n of $C[\frac{1}{n}, 1-\frac{1}{n}]$ as follows:

$$P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\},$$
$$P_n = \big\{x \in P : x \text{ is concave on } [0,1], \min_{t\in[\frac{1}{n},1-\frac{1}{n}]} x(t) \geq \tfrac{1}{n}\big\},$$
$$K_n = \big\{x \in C\left[\tfrac{1}{n}, 1-\tfrac{1}{n}\right] : x \text{ is concave on } [0,1]\big\}.$$

For any real constant $r > 0$, define

$$\Omega_r = \{x \in C[0,1] : \|x\| < r\}$$

as an open neighborhood of $0 \in C[0,1]$ of radius r. $(x(t), y(t))$ is called a positive solution of (1.1) if

$$(x,y) \in \big(C[0,1] \cap C^2(0,1)\big) \times \big(C[0,1] \cap C^2(0,1)\big),$$

$x(t) > 0$, $y(t) > 0$ on $(0,1)$ and (x,y) satisfies (1.1).

The proofs of our main results (Theorems 1.1-1.4) are based on the Guo-Krasnosel'skii fixed-point theorem.

Lemma 2.1 ([8, Guo Krasnosel'skii Fixed-Point Theorem])**.** *Let K be a cone of a real Banach space E, and let Ω_1, Ω_2 be bounded open neighborhoods of $0 \in E$, and assume $\Omega_1 \subset \Omega_2$. Suppose that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous such that one of the following conditions holds*:

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_1 \cap K$; $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_2 \cap K$;
(ii) $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_2 \cap K$; $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_1 \cap K$.

Then, T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For fixed $n \geq N$ and $z \in C[0,1]$, the linear boundary value problem

$$\begin{aligned} -u''(t) &= z(t), \quad t \in [\tfrac{1}{n}, 1-\tfrac{1}{n}], \\ u(\tfrac{1}{n}) &= \tfrac{1}{n}, \quad u(1-\tfrac{1}{n}) = \alpha u(\eta) + \tfrac{1-\alpha}{n}, \end{aligned} \tag{2.1}$$

has a unique solution

$$u(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) z(s)ds, \tag{2.2}$$

where $H_n : \left[\frac{1}{n}, 1-\frac{1}{n}\right] \times \left[\frac{1}{n}, 1-\frac{1}{n}\right] \to [0,\infty)$ is an associated Green's function and is defined by

(2.3)

$$H_n(t,s) = \begin{cases} \frac{\left(t-\frac{1}{n}\right)\left(\left(1-\frac{1}{n}-s\right)-\alpha(\eta-s)\right)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s), & \frac{1}{n} \le s \le t \le 1-\frac{1}{n},\ s \le \eta, \\ \frac{\left(t-\frac{1}{n}\right)\left(\left(1-\frac{1}{n}-s\right)-\alpha(\eta-s)\right)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, & \frac{1}{n} \le t \le s \le 1-\frac{1}{n},\ s \le \eta, \\ \frac{\left(t-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, & \frac{1}{n} \le t \le s \le 1-\frac{1}{n},\ s \ge \eta, \\ \frac{\left(t-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s), & \frac{1}{n} \le s \le t \le 1-\frac{1}{n},\ s \ge \eta. \end{cases}$$

We note that $H_n(t,s) \to H(t,s)$ as $n \to \infty$, where

$$H(t,s) = \begin{cases} \frac{t(1-s)}{1-\alpha\eta} - \frac{\alpha t(\eta-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1,\ s \le \eta, \\ \frac{t(1-s)}{1-\alpha\eta} - \frac{\alpha t(\eta-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \le \eta, \\ \frac{t(1-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \ge \eta, \\ \frac{t(1-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1-,\ s \ge \eta, \end{cases}$$

is the Green's function corresponding the boundary value problem

$$\begin{aligned} -u''(t) &= z(t), \quad t \in [0,1], \\ u(0) = 0, &\quad u(1) = \alpha u(\eta) \end{aligned}$$

with

$$u(t) = \int_0^1 H(t,s) z(s) ds,$$

as its integral representation. We need the following properties of the Green's function H_n in the sequel. For the proof, see [22].

Lemma 2.2. *The function H_n can be written as*

$$H_n(t,s) = G_n(t,s) + \frac{\alpha\left(t-\frac{1}{n}\right)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} G_n(\eta,s), \tag{2.4}$$

where

$$G_n(t,s) = \frac{n}{n-2} \begin{cases} \left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-t\right), & \frac{1}{n} \le s \le t \le 1-\frac{1}{n}, \\ \left(t-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right), & \frac{1}{n} \le t \le s \le 1-\frac{1}{n}. \end{cases} \tag{2.5}$$

Lemma 2.3. *Let*

$$\mu_n = \frac{\max\{1,\alpha\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, \quad \nu_n = \frac{\min\{1,\alpha\}\min\{\eta-\frac{1}{n}, 1-\frac{1}{n}-\eta\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

Then

(i) $H_n(t,s) \le \mu_n \left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right), \quad (t,s) \in \left[\frac{1}{n}, 1-\frac{1}{n}\right] \times \left[\frac{1}{n}, 1-\frac{1}{n}\right],$
(ii) $H_n(t,s) \ge \nu_n \left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right), \quad (t,s) \in \left[\eta, 1-\frac{1}{n}\right] \times \left[\frac{1}{n}, 1-\frac{1}{n}\right].$

Now consider the system of nonlinear non-singular BVPs

$$\begin{aligned} -x''(t) &= f(t, \max\{\tfrac{1}{n}, y(t)\}), \quad t \in [\tfrac{1}{n}, 1-\tfrac{1}{n}], \\ -y''(t) &= g(t, \max\{\tfrac{1}{n}, x(t)\}), \quad t \in [\tfrac{1}{n}, 1-\tfrac{1}{n}], \\ x(\tfrac{1}{n}) &= \tfrac{1}{n}, \quad x(1-\tfrac{1}{n}) = \alpha x(\eta) + \tfrac{1-\alpha}{n}, \\ y(\tfrac{1}{n}) &= \tfrac{1}{n}, \quad y(1-\tfrac{1}{n}) = \alpha y(\eta) + \tfrac{1-\alpha}{n}, \end{aligned} \tag{2.6}$$

where $n > N$. Write (2.6) as an equivalent system of integral equations

$$\begin{aligned} x(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{\tfrac{1}{n}, y(s)\}) ds, \\ y(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{\tfrac{1}{n}, x(s)\}) ds. \end{aligned} \tag{2.7}$$

Thus, (x, y) is a solution of (2.6) if and only if

$$(x,y) \in C[\tfrac{1}{n}, 1-\tfrac{1}{n}] \times C[\tfrac{1}{n}, 1-\tfrac{1}{n}]$$

and (x, y) is a solution of (2.7).

Define operators $A_n, B_n, T_n : K_n \to K_n$ by

$$\begin{aligned} (A_n y)(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{\tfrac{1}{n}, y(s)\}) ds, \\ (B_n x)(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{\tfrac{1}{n}, x(s)\}) ds, \\ (T_n x)(t) &= (A_n(B_n x))(t). \end{aligned} \tag{2.8}$$

If $u_n \in K_n$ is a fixed point of T_n, then the system of BVPs (2.6) has a solution (x_n, y_n) given by

$$\begin{cases} x_n(t) = u_n(t), \\ y_n(t) = (B_n u_n)(t). \end{cases}$$

By construction, the system of BVPs (2.6) is regular and so the following lemma is standard.

Lemma 2.4. *Assume* $f, g : (0,1) \times (0,\infty) \to [0,\infty)$ *are continuous. Then* $T_n : K_n \to K_n$ *is completely continuous.*

3. Main results

Proof of Theorem 1.1. By (A_2), there exist constants $C_1, C_2, N_1, N_2 > 0$ such that

$$4^{\alpha_1} a b^{\alpha_1} \mu_n^{\alpha_1+1} C_1 C_2^{\alpha_1} < 1, \tag{3.1}$$

and

$$F(x) \le C_1 x^{\alpha_1} + N_1, \quad G(x) \le C_2 x^{\alpha_2} + N_2 \text{ for } x \ge \frac{1}{n}. \tag{3.2}$$

Choose a constant $R > 0$ such that

$$R \ge \frac{\frac{1}{n} + \frac{2^{\alpha_1} a\mu_n C_1}{n^{\alpha_1}} + a\mu_n N_1 + 4^{\alpha_1} a b^{\alpha_1} \mu_n^{\alpha_1+1} C_1 N_2^{\alpha_1}}{1 - 4^{\alpha_1} a b^{\alpha_1} \mu_n^{\alpha_1+1} C_1 C_2^{\alpha_1}}. \tag{3.3}$$

For any $u \in \partial\Omega_R \cap K_n$, using (2.8) and (A_1), we have

$$\begin{aligned}
(T_n u)(t) = (A_n(B_n u))(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, (B_n u)(s)) ds \\
&= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau) ds \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau) ds.
\end{aligned}$$

In view of (3.2) and (A_2), it follows that

$$\begin{aligned}
&(T_n u)(t) \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) K(s) (C_1 (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau)^{\alpha_1} + N_1) ds \\
&= \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau)^{\alpha_1} ds \\
&\quad + N_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) ds \\
&\le \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) L(\tau) G(u(\tau)) d\tau)^{\alpha_1} ds \\
&\quad + N_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) ds \\
&\le \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) \\
&\quad \cdot \left(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) L(\tau) (C_2 (u(\tau))^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} ds
\end{aligned}$$

$$+ N_1 \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds.$$

Employing (i) of Lemma 2.3, we obtain

$$\begin{aligned}
&(T_n u)(t) \\
&\le \frac{1}{n} + C_1\mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds \\
&\quad \cdot \left(\frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau-\frac{1}{n})(1-\frac{1}{n}-\tau)L(\tau)(C_2(u(\tau))^{\alpha_2} + N_2)d\tau\right)^{\alpha_1} \\
&\quad + N_1\mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds \\
&\le \frac{1}{n} + C_1\mu_n \int_{1/n}^{1-1/n} s(1-s)K(s)ds \\
&\quad \cdot \left(\frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} \tau(1-\tau)L(\tau)(C_2(u(\tau))^{\alpha_2} + N_2)d\tau\right)^{\alpha_1} \\
&\quad + N_1\mu_n \int_{1/n}^{1-1/n} s(1-s)K(s)ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(T_n u)(t) \\
&\le \frac{1}{n} + C_1\mu_n \int_{1/n}^{1-1/n} s(1-s)K(s)ds \\
&\quad \cdot \left(\frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} \tau(1-\tau)L(\tau)(C_2\|u\|^{\alpha_2} + N_2)d\tau\right)^{\alpha_1} \\
&\quad + N_1\mu_n \int_{1/n}^{1-1/n} s(1-s)K(s)ds \\
&\le \frac{1}{n} + \mu_n C_1 \int_0^1 s(1-s)K(s)ds \\
&\quad \cdot \left(\frac{1}{n} + \mu_n \int_0^1 \tau(1-\tau)L(\tau)d\tau(C_2\|u\|^{\alpha_2} + N_2)\right)^{\alpha_1} \\
&\quad + \mu_n N_1 \int_0^1 s(1-s)K(s)ds \\
&\le \frac{1}{n} + a\mu_n N_1 + 2^{\alpha_1} a\mu_n C_1 (\frac{1}{n^{\alpha_1}} + b^{\alpha_1}\mu_n^{\alpha_1}(C_2\|u\|^{\alpha_2} + N_2)^{\alpha_1}) \\
&\le \frac{1}{n} + \frac{2^{\alpha_1} a\mu_n C_1}{n^{\alpha_1}} + a\mu_n N_1 + 2^{2\alpha_1} ab^{\alpha_1}\mu_n^{\alpha_1+1} C_1 (C_2^{\alpha_1}\|u\|^{\alpha_1\alpha_2} + N_2^{\alpha_1}).
\end{aligned}$$

Using (3.3), we obtain

$$\|T_n u\| \le \|u\| \text{ for all } u \in \partial\Omega_R \cap K_n. \tag{3.4}$$

Now, by (A_3), there exist constants $C_3, C_4 > 0$ and $\rho \in (0, R)$ such that

$$f(t,x) \ge C_3 x^{\beta_1}, g(t,x) \ge C_4 x^{\beta_2} \quad \text{for } x \in [0,\rho] \text{ and } t \in [\eta, 1]. \tag{3.5}$$

Choose

$$r_n = \min\left\{\rho, \frac{C_3 C_4^{\beta_1} \nu_n^{\beta_1+1}}{n^{\beta_1\beta_2}} \left(\int_\eta^{1-1/n} (s - \tfrac{1}{n})(1 - \tfrac{1}{n} - s) ds\right)^{\beta_1+1}\right\}. \tag{3.6}$$

For any $u \in \partial\Omega_{r_n} \cap K_n$, using (2.8), (3.5) and (ii) of Lemma 2.3, we have

$$\begin{aligned}
(T_n u)(t) &= (A_n(B_n u))(t) \\
&= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau) ds \\
&\ge \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau) ds \\
&\ge \int_{\eta}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau) ds \\
&\ge C_3 \int_{\eta}^{1-1/n} H_n(t,s) \left(\int_{\eta}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau)) d\tau\right)^{\beta_1} ds \\
&\ge C_3 \nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) ds \\
&\quad \cdot \left(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) g(\tau, u(\tau)) d\tau\right)^{\beta_1} \\
&\ge C_3 \nu_n^{\beta_1+1} \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) ds \\
&\quad \cdot \left(C_4 \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)(u(\tau))^{\beta_2} d\tau\right)^{\beta_1} \\
&\ge \frac{C_3 C_4^{\beta_1} \nu_n^{\beta_1+1}}{n^{\beta_1\beta_2}} \left(\int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) ds\right)^{\beta_1+1}.
\end{aligned}$$

Thus, in view of (3.6), it follows that

$$\|T_n u\| \ge \|u\| \text{ for } u \in \partial\Omega_{r_n} \cap K_n. \tag{3.7}$$

By Lemma 2.1, T_n has a fixed point $u_n \in K_n \cap (\overline{\Omega}_R \backslash \Omega_{r_n})$.

Note that

$$r_n \le u_n(t) \le R \text{ for all } t \in [\tfrac{1}{n}, 1 - \tfrac{1}{n}] \tag{3.8}$$

and $r_n \to 0$ as $n \to \infty$. Thus, we have exhibited a uniform bound for each $u_n \in [\frac{1}{n}, 1-\frac{1}{n}]$ and for $m \geq n$, $\{u_m\}$ is uniformly bounded on $[\frac{1}{n}, 1-\frac{1}{n}]$.

To show that $\{u_m\}$ for $m \geq n$, is equicontinuous on $[\frac{1}{n}, 1-\frac{1}{n}]$, consider for $t \in [\frac{1}{n}, 1-\frac{1}{n}]$, the integral equation

$$u_m(t) = u_m(\tfrac{1}{m}) + \int_{1/m}^{1-1/m} H_m(t,s) f(s,(B_m u_m)(s))ds.$$

Employ Lemma 2.2 to obtain

$$\begin{aligned}
&u_m(t)\\
&= u_m(\tfrac{1}{m}) + \int_{1/m}^{1-1/m} \big[G_m(t,s) + \frac{\alpha(t-\frac{1}{m})}{1-\frac{2}{m}+\frac{\alpha}{m}-\alpha\eta} G_m(\eta,s)\big] f(s,(B_m u_m)(s))ds\\
&= u_m(\tfrac{1}{m}) + \frac{m}{m-2}\int_{1/m}^{t} (s-\frac{1}{m})(1-\frac{1}{m}-t) f(s,(B_m u_m)(s))ds\\
&\quad + \frac{m}{m-2}\int_{t}^{1-1/m} (t-\frac{1}{m})(1-\frac{1}{m}-s) f(s,(B_m u_m)(s))ds\\
&\quad + \frac{\alpha(t-\frac{1}{m})}{1-\frac{2}{m}+\frac{\alpha}{m}-\alpha\eta}\int_{1/m}^{1-1/m} G_m(\eta,s) f(s,(B_m u_m)(s))ds.
\end{aligned}$$

Differentiate with respect to t to obtain

$$\begin{aligned}
u_m'(t) = &-\frac{m}{m-2}\int_{1/m}^{t} (s-\frac{1}{m}) f(s,(B_m u_m)(s))ds\\
&+ \frac{m}{m-2}\int_{t}^{1-1/m} (1-\frac{1}{m}-s) f(s,(B_m u_m)(s))ds\\
&+ \frac{\alpha}{1-\frac{2}{m}+\frac{\alpha}{m}-\alpha\eta}\int_{1/m}^{1-1/m} G_m(\eta,s) f(s,(B_m u_m)(s))ds,
\end{aligned}$$

which implies that for $t \in [\frac{1}{n}, 1-\frac{1}{n}]$

$$\begin{aligned}
|u_m'(t)| \leq &\int_{1/m}^{1-1/m} f(s,(B_m u_m)(s))ds\\
&+ \frac{\alpha}{1-\frac{2}{m}+\frac{\alpha}{m}-\alpha\eta}\int_{1/m}^{1-1/m} G_m(\eta,s) f(s,(B_m u_m)(s))ds.
\end{aligned} \tag{3.9}$$

Hence, for $m \geq n$, $\{u_m\}$ is equicontinuous on $[\frac{1}{n}, 1-\frac{1}{n}]$.

For $m \geq n$, define

$$v_m = \begin{cases} u_m(\frac{1}{n}), & \text{if } 0 \leq t \leq \frac{1}{n},\\ u_m(t), & \text{if } \frac{1}{n} \leq t \leq 1-\frac{1}{n},\\ \alpha u_m(\eta), & \text{if } 1-\frac{1}{n} \leq t \leq 1.\end{cases}$$

Since v_m is a constant extension of u_m to $[0,1]$, the sequence $\{v_m\}$ is uniformly bounded and equicontinuous on $[0,1]$. Thus, there exists a subsequence $\{v_{n_k}\}$ of $\{v_m\}$ converging uniformly on $[0,1]$ to $v \in P \cap (\overline{\Omega}_R \backslash \Omega_r)$.

We introduce the notation

$$x_{n_k}(t) = v_{n_k}(t), \quad y_{n_k}(t) = \frac{1}{n_k} + \int_{1/n_k}^{1-1/n_k} H_{n_k}(t,s) g(s, v_{n_k}(s)) ds,$$
$$\overline{x}(t) = \lim_{n_k \to \infty} x_{n_k}(t), \quad \overline{y}(t) = \lim_{n_k \to \infty} y_{n_k}(t),$$

and for $t \in [0,1]$ consider the integral equation

$$x_{n_k}(t) = x_{n_k}(\tfrac{1}{n_k}) + \int_{1/n_k}^{1-1/n_k} H_{n_k}(t,s) f(t, y_{n_k}(s)) ds.$$

Letting $n_k \to \infty$, we have

$$\overline{x}(t) = \overline{x}(0) + \int_0^1 H(t,s) f(t, \overline{y}(s)) ds,$$

and

$$\overline{y}(t) = \int_0^1 H(t,s) g(s, \overline{x}(s)) ds, \; t \in [0,1].$$

Moreover,

$$\overline{x}(0) = 0, \quad x(1) = \alpha \overline{x}(\eta), \quad \overline{y}(0) = 0, \quad \overline{y}(1) = \alpha \overline{y}(\eta).$$

Hence, $(\overline{x}(t), \overline{y}(t))$ is a solution of the system (1.2).

Since

$$f, g : (0,1) \times (0, \infty) \to (0, \infty),$$

$f(t,0), g(t,0)$ are not identically 0, and H is of fixed sign on $(0,1) \times (0,1)$, it follows that $\overline{x}, \overline{y} > 0$ on $(0,1)$. □

Example 3.1. Let

$$f(t,y) = \frac{1}{t(1-t)} \left(\frac{1}{y} + 3y^{1/3} \right), \quad g(t,x) = \frac{1}{t(1-t)} \left(\frac{1}{x} + 4x \right)$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{y} + 3y^{1/3}, \quad G(x) = \frac{1}{x} + 4x,$$

and $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 2$, $\beta_1 = \beta_2 = 1$. Then $(A_1) - (A_3)$ are satisfied. Hence, by Theorem 1.1, system (1.2) has a positive solution.

Proof of Theorem 1.2. For $u \in \partial\Omega_{M_1} \cap K_n$, using (2.8), we obtain for $t \in [\frac{1}{n}, 1-\frac{1}{n}]$

$$\begin{aligned}(T_n u)(t) = (A_n(B_n u))(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s,(B_n u)(s))ds \\ &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau))d\tau)ds \\ &\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau))d\tau)ds.\end{aligned}$$

Using (A_1), (A_4) and Lemma 2.3, we have

$$\begin{aligned}&(T_n u)(t) \\ &\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) g(\tau, u(\tau))d\tau)ds \\ &\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) L(\tau) G(u(\tau))d\tau)ds \\ &\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n G(\frac{1}{n}) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) L(\tau)d\tau)ds \\ &\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + b\, \mu_n\, G(\frac{1}{n}))ds \\ &\geq M_1 \int_{1/n}^{1-1/n} H_n(t,s)ds (\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)d\tau)^{-1} \geq M_1,\end{aligned}$$

which implies that

$$\|T_n u\| \geq \|u\| \text{ for all } u \in \partial\Omega_{M_1} \cap K_n. \tag{3.10}$$

In view of (A_2), we can choose $R > M_1$ such that (3.4) holds. Hence, by Lemma 2.1, T_n has a fixed point $u_n \in K_n \cap (\overline{\Omega}_R \backslash \Omega_{M_1})$. By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. □

Example 3.2. Let

$$f(t,y) = \frac{e^{\frac{1}{y}}}{t(1-t)}, \quad g(t,x) = \frac{e^{\frac{1}{x}}}{t(1-t)}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = e^{\frac{1}{y}}, \quad G(x) = e^{\frac{1}{x}}.$$

Choose constant M_1 such that $M_1 \leq \frac{4(n-3)}{n} e^{\frac{n}{1+6ne^n}} \int_{1/3}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds$. Then (A_1), (A_2) and (A_4) are satisfied. Hence, by Theorem 1.2, system (1.2) has a positive solution.

Proof of Theorem 1.3. For $u \in \partial\Omega_{M_2} \cap K_n$, using (2.8), we have

$$
\begin{aligned}
(T_n u)(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s,(B_n u)(s))ds \\
&= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau))d\tau)ds.
\end{aligned}
$$

In view of (A_1), (A_5) and Lemma 2.3, we obtain

$$
\begin{aligned}
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,u(\tau))d\tau)ds \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,u(\tau))d\tau)ds \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,M_2)d\tau)ds \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{\eta}^{1-1/n} H_n(s,\tau)g(\tau,M_2)d\tau)ds \\
&\le \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M_2)d\tau)ds \\
&= \frac{1}{n} + F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M_2)d\tau) \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds \\
&\le \frac{1}{n} + \mu_n F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M_2)d\tau) \\
&\quad \cdot \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds \\
&\le \frac{1}{n} + a\mu_n F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M_2)d\tau) \le M_2,
\end{aligned}
$$

which implies that

$$\|T_n u\| \le \|u\| \quad \text{for all } u \in \partial\Omega_{M_2} \cap K_n. \tag{3.11}$$

By (A_3), we can choose $\rho \in (0, M_2)$ such that (3.7) holds. Hence, T_n has a fixed point $u_n \in K_n \cap (\overline{\Omega}_{M_2} \backslash \Omega_\rho)$. By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. $\square$

Example 3.3. Let

$$
f(t,y) = \begin{cases} \frac{ye^{\frac{1}{y}}}{t(1-t)}, & y \le 1, \\ \frac{e}{t(1-t)}, & y > 1, \end{cases} \qquad g(t,x) = \begin{cases} \frac{xe^{\frac{1}{x}}}{t(1-t)}, & x \le 1, \\ \frac{e}{t(1-t)}, & x > 1, \end{cases}
$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \begin{cases} ye^{\frac{1}{y}}, & y \le 1, \\ e, & y > 1, \end{cases} \quad G(x) = \begin{cases} xe^{\frac{1}{x}}, & x \le 1, \\ e, & x > 1, \end{cases}$$

and $\beta_1 = \beta_2 = 1$. Choose constant M_2 such that

$$M_2 \ge \max\left\{1, \frac{1}{n} + 6F(e(1-3/n)\int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)}ds)\right\}.$$

Then (A_1), (A_3) and (A_5) are satisfied. Hence, by Theorem 1.3, system (1.2) has a positive solution.

Proof of Theorem 1.4. By (A_1) and (A_4), we obtain (3.10). By (A_5) we can choose a constant $M_2 > M_1$ such that (3.11) holds. Then T_n has a fixed point $u_n \in K_n \cap (\overline{\Omega}_{M_2} \backslash \Omega_{M_1})$. By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. □

Example 3.4. Let

$$f(t, y) = \frac{1}{t(1-t)}\frac{1}{\sqrt{y}}, \quad g(t, x) = \frac{1}{t(1-t)}\frac{1}{x^2}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{\sqrt{y}}, \quad G(x) = \frac{1}{x^2}.$$

Choose constants M_1 and M_2 such that $M_1 \le \frac{4(n-3)}{\sqrt{n(6n^3+1)}} \int_{1/3}^{1-\frac{1}{n}} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds$ and $M_2 \ge \frac{1}{6n}(\frac{1}{6} - \sqrt{\frac{n}{n-3}}(\int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)}ds)^{-1/2})^{-1}$. Then (A_1), (A_4) and (A_5) are satisfied. Hence, by Theorem 1.4, system (1.2) has a positive solution.

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