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MIXED BRIGHTNESS-INTEGRALS OF CONVEX BODIES

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Abstract. The mixed width-integrals of convex bodies are defined by E. Lutwak. In this paper, the mixed brightness-integrals of convex bodies are defined. An inequality is established for the mixed brightness-integrals analogous to the Fenchel-Aleksandrov inequality for the mixed volumes. An isoperimetric inequality (involving the mixed brightness-integrals) is presented which generalizes an inequality recently obtained by Chakerian and Heil. Strengthened version of this general inequality is obtained by introducing indexed mixed brightness-integrals.

1. Introduction and main results

The setting for this paper will be the n-dimensional Euclidean space, $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subset with non-empty interiors) and $\mathcal{K}^n_o$ denote the subspace of $\mathcal{K}^n$ consisting of all convex bodies that contain the origin in their interiors. Let $\mathcal{S}^n_o$ denote the set of star bodies about the origin (star-shaped, continuous radial function) in $\mathbb{R}^n$. The unit n-ball and its surface will be denoted by U and S^{n-1}, respectively. The volume of the n-ball, U, will be denoted by ω_n.

Lutwak introduced the notion of the mixed width-integrals of convex bodies in [8, p. 250]: For $k \in \mathcal{K}^n$ and $u \in S^{n-1}$, $b(K,u)$ is half the width of K in the direction u. Mixed width-integrals $A(K_1, K_2, \ldots, K_n)$ of $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ was defined by

$$A(K_1, K_2, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} b(K_1,u)b(K_2,u)\cdots b(K_n,u)dS(u). \tag{1.1}$$

More in general, for a real number $p \neq 0$, the mixed width-integrals of order p, $A_p(K_1, K_2, \ldots, K_n)$, of $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ was also defined by Lutwak [8,

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p. 251],

$$A_p(K_1, K_2, \dots, K_n) = \omega_n \left[\frac{1}{n\omega_n} \int_{S^{n-1}} b(K_1, u)^p \cdots b(K_n, u)^p dS(u) \right]^{\frac{1}{p}}. \tag{1.2}$$

And the properties of the mixed width-integrals of convex bodies were listed, such as positive, continuous, translation invariant, monotone under set inclusion, and homogeneous of degree one in each variable.

After that, the mixed chord-integrals of star bodies are defined by Fenghong Lu in [5]. For $L \in \mathcal{S}_o^n$ and $u \in S^{n-1}$, let

$$d(L, u) = \frac{1}{2}\rho(L, u) + \frac{1}{2}\rho(L, -u) \tag{1.3}$$

denote half the chord of L in the direction u. The mixed chord-integral, $B(L_1, \dots, L_n)$, of $L_1, \dots, L_n \in \mathcal{S}_o^n$ is defined by

$$B(L_1, \dots, L_n) = \frac{1}{n} \int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u) dS(u). \tag{1.4}$$

Lutwak established some inequalities for mixed width-integrals in [6, 8]:

Theorem A. *If $K_1, \dots, K_n \in \mathcal{K}^n$ and $1 < m \le n$, then*

$$A^m(K_1, \dots, K_n) \le \prod_{i=0}^{m-1} A(K_1, \dots, K_{n-m}, K_{n-i}, \dots, K_{n-i}), \tag{1.5}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \dots, K_n$ are all of similar width.

Theorem B. *If $K_1, \dots, K_n \in \mathcal{K}^n$, then*

$$V(K_1) \cdots V(K_n) \le A^n(K_1 \cdots K_n), \tag{1.6}$$

with equality if and only if $K_1, K_2, \dots, K_n$ are n-ball.

Strengthened versions of inequality (1.6) are obtained by introducing indexed mixed width-integrals.

Theorem C. *If $K_1, \dots, K_n \in \mathcal{K}^n$, $p \ne 0$ and $-1 \le p \le \infty$, then*

$$V(K_1) \cdots V(K_n) \le A_p^n(K_1 \cdots K_n), \tag{1.7}$$

with equality if and only if $K_1, K_2, \dots, K_n$ are n-ball.

In this paper, the mixed brightness-integrals of convex bodies are defined. Half the brightness is defined by

$$\delta(K, u) = \frac{1}{2} h(\Pi K, u). \tag{1.8}$$

The mixed brightness-integral $D(K_1, \dots, K_n)$ of $K_1, \dots, K_n \in \mathcal{K}^n$ is defined by

$$D(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u). \tag{1.9}$$

Further, some inequalities for the mixed brightness-integrals analogous to the Fenchel-Aleksandrov inequality for the mixed volumes are established. And we obtain strengthened versions of the general inequality established by Chakerian. We mainly obtain the following results:

Theorem 1. *If* $K_1, \ldots, K_n \in \mathcal{K}^n$ *and* $1 < m \leq n$, *then*

$$D^m(K_1, \ldots, K_n) \leq \prod_{i=0}^{m-1} D(K_1, \ldots, K_{n-m}, K_{n-i}, \ldots, K_{n-i}), \tag{1.10}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ *are all of similar brightness.*

A strengthened version of inequality (1.10) is obtained:

Theorem 2. *If* $K_1, \ldots, K_n \in \mathcal{K}^n$ *and* $1 < m \leq n$, *then for* $p > 0$

$$D_p^m(K_1, \ldots, K_n) \leq \prod_{i=0}^{m-1} D_p(K_1, \ldots, K_{n-m}, K_{n-i}, \ldots, K_{n-i}), \tag{1.11}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ *are all of similar brightness. For* $p < 0$, *inequality* (1.11) *is reversed.*

Theorem 1 and Theorem 2 are just analogs of the Fenchel-Aleksandrov inequality for the mixed volumes.

For $K \in \mathcal{K}^n$, and $u \in S^{n-1}$, let K^u denote the image of the orthogonal projection of K onto ξ_u, the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u. $v(K_1^u, \ldots, K_n^u)$ denote the mixed volume of $K_1^u, \ldots, K_n^u$ and $v(K^u)$ denote the volume of K^u.

Theorem 3. *If* $K_1, \ldots, K_n \in \mathcal{K}^n$, *and* $u \in S^{n-1}$, *then*

$$D^n(K_1, \ldots, K_n) \leq v(K_1^u) \cdots v(K_n^u), \tag{1.12}$$

with equality if and only if $K_1, K_2, \ldots, K_n$ *are all of similar brightness.*

A strengthened version of inequality (1.12) is obtained:

Theorem 4. *If* $K_1, \ldots, K_n \in \mathcal{K}^n$, $u \in S^{n-1}$ *and* $-\infty \leq p \leq 1$, *then*

$$D_p^n(K_1, \ldots, K_n) \leq v(K_1^u) \cdots v(K_n^u), \tag{1.13}$$

with equality if and only if $K_1, K_2, \ldots, K_n$ *are all of similar brightness and have constant joint brightness.*

2. Preliminaries

2.1. Support function and radial function

Let $h(K, u)$ denote the support function (restricted to the unit sphere) of $K \in \mathcal{K}^n$; i.e., for $u \in S^{n-1}$,

$$h(K, u) = \max\{u \cdot x : x \in K\}, \tag{2.1}$$

where $u \cdot x$ denote the usual inner product of x and u in $\mathbb{R}^n$.

For $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_n \geq 0$, the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_n K_n \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_n K_n = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \in \mathcal{K}^n : x_i \in K_i, 1 \leq i \leq n\}. \tag{2.2}$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_n K_n, u) = \lambda_1 h(K_1, u) + \cdots + \lambda_n h(K_n, u). \tag{2.3}$$

The support function $h(K, u)$ is a sublinear function, which satisfies

$$h(K, \lambda u) = \lambda h(K, u),\ \ h(K, u + v) \leqq h(K, u) + h(K, v) \tag{2.4}$$

for $\lambda \geqq 0$.

It is very clear from the definition that $K \subset L$ if and only if

$$h(K, u) \leq h(L, u). \tag{2.5}$$

A compact convex set K is centered if and only if

$$h(K, u) = h(K, -u). \tag{2.6}$$

The group of nonsingular linear transformations is denoted by $GL(n)$. Let $\phi \in GL(n)$, the transpose and inverse are denoted by ϕ^t and ϕ^{-1}. Then

$$h(\phi K, u) = h(K, \phi^t u) = \|\phi^t u\| h\left(K, \frac{\phi^t u}{\|\phi^t u\|}\right) \tag{2.7}$$

for all $u \in S^{n-1}$.

The radial function $\rho(K, u)$ of the convex body K is

$$\rho(K, u) = \sup\{\lambda > 0 : \lambda u \in K\},\ \ u \in S^{n-1}. \tag{2.8}$$

2.2. Brightness and mixed brightness

For convex bodies $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and a direction $u \in S^{n-1}$, the mixed brightness of $K_1, \ldots, K_{n-1}$ in the direction u, $\sigma(K_1, \ldots, K_{n-1}; u)$, is defined by

$$\sigma(K_1, \ldots, K_{n-1}; u) = nV(K_1, \ldots, K_{n-1}, \langle u \rangle), \tag{2.9}$$

where $V(K_1, \ldots, K_{n-1}, \langle u \rangle)$ denote the mixed volume of $K_1, \ldots, K_{n-1}, \langle u \rangle$ and $\langle u \rangle$ denote the closed line segment.

Since $h(\langle u \rangle, \bar{u}) = \frac{1}{2}|u \cdot \bar{u}|$, we obtain

$$\sigma(K_1, \ldots, K_{n-1}; u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot \bar{u}| d(K_1, \ldots, K_{n-1}; \bar{u}). \tag{2.10}$$

For $K \in \mathcal{K}^n$, and $u \in S^{n-1}$, the mixed brightness of $K_1, \ldots, K_{n-1}$ in the direction u can be written as

$$\sigma(K_1, \ldots, K_{n-1}; u) = v(K_1^u, \ldots, K_{n-1}^u), \tag{2.11}$$

If $K_1 = \cdots = K_{n-i-1} = K$ and $K_{n-i} = \cdots = K_{n-1} = \bar{K}$, then the mixed brightness $\sigma(K_1, \ldots, K_{n-1}; u)$ is written as $\sigma_i(K, \bar{K}; u)$. If $i = 0$, then $\sigma(K, u)$ is called the brightness of K in the direction u. From (2.11) we have

$$\sigma_i(K, \bar{K}; u) = u(K^u, \bar{K}^u),\ \sigma(K, u) = v(K^u). \tag{2.12}$$

2.3. Projection and mixed projection bodies

The projection body, ΠK, of the body $K \in \mathcal{K}^n$ is defined as the convex figure whose support function is given, for $u \in S^{n-1}$, by

$$h(\Pi K, u) = v(K^u). \tag{2.13}$$

From (2.10), (2.12), and (2.13), we can see that the homogeneous extension of degree 1 of $h(\Pi K, u)$ is a convex function and hence ΠK is a convex figure. From (2.13), it is easy to see that a projection body is always centered (symmetric about the origin), and if K has interior points, then ΠK will have interior points as well.

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then the mixed projection body of $K_1, \ldots, K_{n-1}$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and defined by

$$h(\Pi(K_1, \ldots, K_{n-1}), u) = \sigma(K_1, \ldots, K_{n-1}; u). \tag{2.14}$$

It is easy to see that the mixed projection body, $\Pi(K_1, \ldots, K_{n-1})$, must be a convex body that is symmetric with respect to the origin from (2.10) and (2.14).

The following is a list of the basic properties of the mixed projection operator.

The projection operator is multilinear with respect to Minkowski linear combinations; i.e., if $K_1, K_1', K_2, \ldots, K_{n-1} \in \mathcal{K}^n$ and $\lambda, \lambda' \geq 0$, then

$$\begin{aligned} &\Pi(\lambda K_1 + \lambda' K_1', K_2, \ldots, K_{n-1}) \\ &= \lambda \Pi(K_1, K_2, \ldots, K_{n-1}) + \lambda' \Pi(K_1', K_2, \ldots, K_{n-1}). \end{aligned} \tag{2.15}$$

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, and $\phi \in GL(n)$, then

$$\Pi(\phi K_1, \ldots, \phi K_{n-1}) = |\det \phi| \phi^{-t}(\Pi(K_1, \ldots, K_{n-1})). \tag{2.16}$$

The mixed projection operator is monotone nondecreasing with respect to set inclusion (by seeing [10, p. 907]); i.e., if $K_i, L_i \in \mathcal{K}^n$, and $K_i \subset L_i$, $1 \leq i \leq n-1$, then

$$\Pi(K_1, \ldots, K_{n-1}) \subset \Pi(L_1, \ldots, L_{n-1}). \tag{2.17}$$

From the corresponding properties of the $(n-1)$-dimensional mixed volumes and (2.9) or (2.11), it follows that the mixed projection bodies $\Pi(K_1, \ldots, K_{n-1})$ is symmetric in its argument, and for $x_1, \ldots, x_n \in \mathbb{R}^n$, we have

$$\Pi(x_1 + K_1, \ldots, x_n + K_n) = \Pi(K_1, \ldots, K_n). \tag{2.18}$$

2.4. Mixed volumes

For $K_1,\dots,K_n \in \mathcal{K}^n$, and $u \in S^{n-1}$, then the following equation relates mixed volumes $V(K_1,\dots,K_n)$ and mixed area measures $S(K_1,\dots,K_{n-1},u)$:

$$V(K_1,\dots,K_n) = \frac{1}{n}\int_{S^{n-1}} h(K_n,u)dS(K_1,\dots,K_{n-1},u). \tag{2.19}$$

3. Mixed brightness-integrals of convex bodies

3.1. Half the brightness

Definition 1. If $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, we let

$$\delta(K,u) = \frac{1}{2}h(\Pi K,u),$$

i.e., $\delta(K,u)$ denotes half the brightness of K in the direction u. Convex bodies $K_1,\dots,K_n$ are said to have similar brightness if there exist constants $\lambda_1,\lambda_2,\dots,\lambda_n > 0$ such that $\lambda_1\delta(K_1,u) = \cdots = \lambda_n\delta(K_n,u)$ for all $u \in S^{n-1}$; they are said to have constant joint brightness if the product $\delta(K_1,u)\cdots\delta(K_n,u)$ is constant for all $u \in S^{n-1}$. For reference see Gardner [3] and schneider [11].

3.2. Mixed brightness-integral

Definition 2. Following Lutwak, we define the mixed brightness-integrals of convex bodies:

For $K_1,K_2,\dots,K_n \in \mathcal{K}^n$, the mixed brightness-integral

$$D(K_1,\dots,K_n) = \frac{1}{n}\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_n,u)dS(u).$$

By this definition, D is a map

$$D : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n} \to \mathbb{R}.$$

3.3. The properties of mixed brightness-integrals

We list some of its elementary properties.

(1) (*Positively homogeneous*) If $K_1,\dots,K_n \in \mathcal{K}^n$ and $\lambda_1,\dots,\lambda_n > 0$, then

$$D(\lambda_1K_1,\dots,\lambda_nK_n) = \lambda_1\cdots\lambda_n D(K_1,\dots,K_n).$$

(2) (*Continuity*) The mixed brightness-integrals $D(K_1,\dots,K_n)$ is a continuous function of $K_1,\dots,K_{n-1} \in \mathcal{K}^n$.

(3) (*Monotonicity for set inclusion*) If $K_i,L_i \in \mathcal{K}^n$, $K_i \subset L_i$ and $1 \le i \le n$, then

$$D(K_1,\dots,K_n) \le D(L_1,\dots,L_n),$$

with equality if and only if $K_i = L_i$ for $1 \le i \le n$.

(4) (*Nonnegativity*) For $K_1,\dots,K_n \in \mathcal{K}^n$, $D(K_1,\dots,K_n) \ge 0$.

(5) (*Invariance under individual translation*) If $x \in \mathbb{R}^n$, then

$$D(K_1 + x, K_2, \dots, K_n) = D(K_1, \dots, K_n).$$

(6) (*Invariance under linear transformation*) If $K_1, \dots, K_n \in \mathcal{K}^n$, and $\phi \in GL(n)$, then

$$D(\phi K_1, \dots, \phi K_n) = D(K_1, \dots, K_n).$$

Proof. (1) From (2.3) and (2.15), we can get

$$\delta(\lambda_i K_i, u) = \frac{1}{2} h(\Pi \lambda_i K_i, u) = \frac{1}{2} h(\lambda_i \Pi K_i, u) = \frac{1}{2} \lambda_i h(\Pi K_i, u) = \lambda_i \delta(K_i, u)$$

for $1 \le i \le n$. Then, from the definition, we can obtain,

$$\begin{aligned} D(\lambda_1 K_1, \dots, \lambda_n K_n) &= \frac{1}{n} \int_{S^{n-1}} \delta(\lambda_1 K_1, u) \cdots \delta(\lambda_n K_n, u) dS(u) \\ &= \lambda_1 \cdots \lambda_n \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u) dS(u) \\ &= \lambda_1 \cdots \lambda_n D(K_1, \dots, K_n). \end{aligned}$$

(2) The polar coordinate formula for mixed volume of bodies $K_1, \dots, K_n$ in $\mathbb{R}^n$ is

$$\tilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) d(u).$$

From the continuity of the mixed volume and Minikowski addition, we can see the support function is continuous. Hence, the mixed brightness-integral is a continuous function.

(3) For $K_i, L_i \in \mathcal{K}^n$, $K_i \subset L_i$ and $1 \le i \le n$, from (2.17) and (2.5), we have

$$\Pi K_i \subset \Pi L_i,$$

then

$$h(\Pi K_i, u) \le h(\Pi L_i, u),$$

hence,

$$D(K_1, \dots, K_n) \le D(L_1, \dots, L_n),$$

with equality if and only if $K_i = L_i$ for $1 \le i \le n$.

(4) The mixed projection body, $\Pi(K_1, \dots, K_n)$, is a convex body that is symmetric with respect to the origin, then $h(\Pi K_i, u) > 0$ for $1 \le i \le n$. From the definition,

$$D(K_1, \dots, K_n) > 0.$$

In particular, if any K_i is a single point, then ΠK_i is the origin. In this case, $h(\Pi K_i) = 0)$, then

$$D(K_1, \dots, K_n) = 0.$$

Hence,

$$D(K_1, \dots, K_n) \ge 0.$$

(5) From (2.18), we have

$$\delta(K_1+x,u)=\frac{1}{2}h(\Pi(K_1+x),u)=\frac{1}{2}h(\Pi(K_1),u)=\delta(K_1,u),$$

hence,

$$D(K_1+x,K_2,\ldots,K_n)=D(K_1,\ldots,K_n).$$

(6) From (2.16) and (2.7), we can get

$$\begin{aligned}\delta(\phi K,u)&=\frac{1}{2}h(\Pi(\phi K),u)\\&=\frac{1}{2}h(|\det\phi|\phi^{-t}(\Pi K),u)\\&=\frac{1}{2}|\det\phi|h(\phi^{-t}(\Pi K),u)\\&=\frac{1}{2}|\det\phi|h(\Pi K,\phi^{-1}u)\\&=\frac{1}{2}|\det\phi|\|\phi^{-1}u\|h\left(\Pi K,\frac{\phi^{-1}u}{\|\phi^{-1}u\|}\right)\\&=\frac{1}{2}h(\Pi K,u)\\&=\delta(\phi K,u'),\end{aligned}$$

where $u'\in S^{n-1}$.

Hence,

$$D(\phi K_1,\ldots,\phi K_n)=D(K_1,\ldots,K_n).$$

□

3.4. Mixed brightness-integral of order *p*

Just as the width-integral $B_i(K)$ [6] of $K\in\mathcal{K}^n$, are defined to be the special mixed width-integral

$$A(\underbrace{K,\ldots,K}_{n-i},\underbrace{U,\ldots,U}_{i}),$$

the brightness-integral $C_i(K)$ of $K\in\mathcal{K}^n$, can be defined as the special mixed brightness-integral

$$D(\underbrace{K,\ldots,K}_{n-i},\underbrace{U,\ldots,U}_{i}).$$

Now we generalize the notion of the mixed brightness-integral of convex bodies: For $K_1,\ldots,K_n\in\mathcal{K}^n$, and a real number $p\neq 0$, the mixed brightness-integral of order p, $D_p(K_1,\ldots,K_n)$ of $K_1,\ldots,K_n$ is defined by

$$D_p(K_1,K_2,\ldots,K_n)=\omega_n\left[\frac{1}{n\omega_n}\int_{S^{n-1}}\delta(K_1,u)^p\cdots\delta(K_n,u)^p dS(u)\right]^{\frac{1}{p}}.$$

Specially $p=1$, this definition is just Definition 1.

4. Inequalities for the mixed-brightness integrals

In order to prove the conclusions in the introduction, we require the following simply extension of Hölder' s inequality.

Lemma 1. *If $f_0, f_1, \ldots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \ldots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u)f_1(u)\cdots f_m(u)dS(u) \le \prod_{i=1}^{m}\left[\int_{S^{n-1}} f_0(u)f_i^{\lambda_i}(u)dS(u)\right]^{\frac{1}{\lambda_i}},$$

with equality if and only if there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1. For $K_1, \ldots, K_n \in \mathcal{K}^n$, let

$$\begin{aligned}\lambda_i &= m(1 \le i \le m),\\ f_0 &= \delta(K_1,u)\cdots\delta(K_{n-m},u)\ (f_0 = 1 \text{ if } m = n),\\ f_i &= \delta(K_{n-i+1},u)(1 \le i \le m).\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}&\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_n,u)dS(u)\\ \le& \prod_{i=1}^{m}\left[\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_{n-m},u)\delta(K_{n-i+1},u)^m dS(u)\right]^{\frac{1}{m}},\end{aligned}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ are all of similar brightness, i.e.,

$$D^m(K_1,\ldots,K_n) \le \prod_{i=0}^{m-1} D(K_1,\ldots,K_{n-m},K_{n-i},\ldots,K_{n-i}),$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ are all of similar brightness. □

Proof of Theorem 2. For $K_1, \ldots, K_n \in \mathcal{K}^n$, let

$$\begin{aligned}\lambda_i &= m(1 \le i \le m),\\ f_0 &= \delta^p(K_1,u)\cdots\delta^p(K_{n-m},u)\ (f_0 = 1 \text{ if } m = n),\\ f_i &= \delta^p(K_{n-i+1},u)(1 \le i \le m).\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}&\int_{S^{n-1}} \delta^p(K_1,u)\cdots\delta^p(K_n,u)dS(u)\\ \le& \prod_{i=1}^{m}\left[\int_{S^{n-1}} \delta^p(K_1,u)\cdots\delta^p(K_{n-m},u)\delta^{pm}(K_{n-i+1},u)dS(u)\right]^{\frac{1}{m}},\end{aligned}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ are all of similar brightness.

For $p > 0$, we get

$$\begin{aligned}&\omega_n \left[\frac{1}{n\omega_n}\int_{S^{n-1}} \delta^p(K_1,u)\cdots\delta^p(K_n,u)dS(u)\right]^{\frac{1}{p}}\\ &\leq \omega_n \prod_{i=1}^{m}\left[\frac{1}{n\omega_n}\int_{S^{n-1}} \delta^p(K_1,u)\cdots\delta^p(K_{n-m},u)\delta^{pm}(K_{n-i+1},u)dS(u)\right]^{\frac{1}{pm}},\end{aligned}$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \ldots, K_n$ are all of similar brightness. For $p < 0$, inequality above is reversed.

Thus we obtain the conclusion. □

Proof of Theorem 3. For $K_1, \ldots, K_n \in \mathcal{K}^n$, from (2.12), (2.19) and Lemma 1, we can get

$$\begin{aligned}&D(K_1,\ldots,K_n)\\ &= \frac{1}{n}\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_n,u)dS(u)\\ &\leq n^{-\frac{1}{n}}\left[\int_{S^{n-1}} \delta^n(K_1,u)dS(u)\right]^{\frac{1}{n}}\cdots\left[\int_{S^{n-1}} \delta^n(K_n,u)dS(u)\right]^{\frac{1}{n}}\\ &= n^{-\frac{1}{n}}\left\{\int_{S^{n-1}}\left[\frac{1}{2}h(\Pi K_1,u)\right]^n dS(u)\right\}^{\frac{1}{n}}\cdots\left\{\int_{S^{n-1}}\left[\frac{1}{2}h(\Pi K_n,u)\right]^n dS(u)\right\}^{\frac{1}{n}}\\ &= v^{\frac{1}{n}}(K_1^u)\cdots v^{\frac{1}{n}}(K_n^u),\end{aligned}$$

with equality if and only if $K_1, \ldots, K_n$ are all of similar brightness.

Thus we get

$$D^n(K_1,\ldots,K_n) \leq v(K_1^u)\cdots v(K_n^u). \qquad \square$$

For p equal to $-\infty$, 0 or ∞, we respectively define the mixed brightness-integral of order p by

$$D_p(K_1,\ldots,K_n) = \lim_{s\to p} D_s(K_1,\ldots,K_n).$$

As a direct consequence of Jensen's inequality [4] we have:

Proposition 1. *If $K_1, \ldots, K_n \in \mathcal{K}^n$ and $-\infty \leq p \leq q \leq \infty$, then*

$$D_p(K_1,\ldots,K_n) \leq D_q(K_1,\ldots,K_n),$$

with equality if and only if $K_1, \ldots, K_n$ have constant joint brightness.

Proof of Theorem 4. For $K_1, \ldots, K_n \in \mathcal{K}^n$ and $-\infty \leq p \leq 1$. By combining Theorem 3 with Proposition 1 we obtain

$$\begin{aligned}D_p^n(K_1,\ldots,K_n) &\leq D_1^n(K_1,\ldots,K_n)\\ &\leq v(K_1^u)\cdots v(K_n^u).\end{aligned}$$

In view of the equality conditions of Theorem 1 with Proposition 1, equality holds if and only if $K_1, K_2, \ldots, K_n$ are all of similar brightness and have constant joint brightness. Thus we obtain the conclusion. □

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