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AVERAGES AND COMPACT, ABSOLUTELY SUMMING AND NUCLEAR OPERATORS ON $C(\Omega)$

Dumitru Popa

Abstract. In the paper we introduce averages of each type and use these averages to construct examples of weakly compact operators on the space $C(\Omega)$ for which the necessary and sufficient conditions that they be compact, absolutely summing or nuclear are distinct. A great number of concrete examples, in various situations, are given.

1. Introduction

Let Ω be a compact Hausdorff space, Σ_Ω the σ-field of Borel subsets of Ω, $C(\Omega)$ the space of all scalar-valued continuous functions on Ω under the uniform norm, X a Banach space and $U : C(\Omega) \to X$ a bounded linear operator. It is well-known, see [3, Chapter VI], that U has a representing vector measure G, and that U is weakly compact if and only if G takes its values in X; U is compact if and only if G has norm compact range; U is absolutely summing if and only if G has bounded variation; U is nuclear if and only if G has a Bochner integrable Radon-Nikodym derivative with respect to its variation $|G|$.

In [8] are given explicit examples of bounded linear operators on $C[0,1]$ with values in c_0 which distinguish certain ideals of operators. In this paper we complete the results and examples in [8] by giving many other examples.

We fix now some notations and terminology. Let X be a Banach space, Σ a σ-field of sets and $G : \Sigma \to X$ a vector measure. We denote by $|G|$ the variation measure of G, $\|G\|$ the semivariation, $\|G\|(E) = \sup_{\|x^*\|\le 1} |x^* G|(E)$, $E \in \Sigma$, see [3, Chapter I, pp. 3–4]. If (S, Σ, μ) is a finite measure space, X a Banach space and $f : S \to X$ a μ-Bochner integrable function we write $\int_{(\cdot)} f d\mu$ for the Bochner integral; if $f : S \to X$ is a μ-Pettis integrable function, the Pettis norm of f is defined by $\|f\|_{\text{Pettis}} = \sup_{\|x^*\|\le 1} \int_S |x^* f| \, d\mu$, see [3, Chapter II].

If $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces, we denote $c_0(X_n \mid n \in \mathbb{N})$, the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$, $x_n \in X_n$ for every $n \in \mathbb{N}$, $\|x_n\| \to 0$, endowed to the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\| = \sup_{n\in\mathbb{N}} \|x_n\|$ and similarly, $l_\infty(X_n \mid n \in \mathbb{N})$

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denote the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$, $x_n \in X_n$ for every $n \in \mathbb{N}$, with $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$, endowed to the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\| = \sup_{n\in\mathbb{N}} \|x_n\|$.

When $X_n = X$, we write $c_0(X)$ resp. $l_\infty(X)$. By $l_\infty^n(X)$ we denote

$$(\underbrace{X \times \cdots \times X}_{n \text{ times}}, \| \ \|_\infty).$$

The scalar field $\mathbb{R}$ (or $\mathbb{C}$) is denoted $\mathbb{K}$ and if $n \in \mathbb{N}$, $1 \leq p \leq \infty$, then

$$l_p^n = \left(\mathbb{K}^n, \| \ \|_p\right),$$

where $\|(\alpha_1, \ldots, \alpha_n)\|_p = \left(\sum_{i=1}^n |\alpha_i|^p\right)^{\frac{1}{p}}$ if $p < \infty$ and $\|(\alpha_1, \ldots, \alpha_n)\|_\infty = \max_{1\leq i\leq n} |\alpha_i|$. Further p^* is the conjugate of p and by $(e_{ni})_{1\leq i\leq n}$ we denote the standard basis in l_p^n.

If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are two real sequences we write $a_n \asymp b_n$ if and only if there exist $m, M > 0$ such that $mb_n \leq a_n \leq Mb_n$ for every $n \in \mathbb{N}$. If $k \in \mathbb{N}$ and $(a_{nk})_{n\in\mathbb{N}}$, $(b_{nk})_{n\in\mathbb{N}}$ are two real sequences we write $a_{nk} \asymp b_{nk}$ if and only if there exist $m_k, M_k > 0$ such that $m_k b_{nk} \leq a_{nk} \leq M_k b_{nk}$ for every $n \in \mathbb{N}$.

If X is a Banach space, $1 \leq p < \infty$, $m \in \mathbb{N}$ and $x_1, x_2, \ldots, x_m$ a finite system of vectors in X, we write

$$\begin{aligned} w_p(x_i \mid 1 \leq i \leq m; X) &= \sup_{\|x^*\|\leq 1} \left(|x^*(x_1)|^p + \cdots + |x^*(x_m)|^p\right)^{\frac{1}{p}} \\ &= \left\|T : X^* \to l_p^m\right\|, \end{aligned}$$

where $T(x^*) = (x^*(x_1), \ldots, x^*(x_m))$.

In the rest of the paper, $\mathcal{B}$ denotes the σ-algebra of all Borel sets in $[0,1]$, $\lambda : \mathcal{B} \to [0,1]$ the Lebesgue measure, $(r_n)_{n\in\mathbb{N}}$ the sequence of Rademacher functions and $C[0,1]$ the space of all scalar-valued continuous functions on $[0,1]$ under the uniform norm. If X is a Banach space, $L_1(\lambda, X)$ is the space of λ-Bochner integrable functions. If μ, ν are two positive measures we denote $\mu \otimes \nu$ their product.

All notation and terminology, not otherwise explained, are as in [2, 3].

2. Scalar and vector averages

Let X be a Banach space, $m \in \mathbb{N}$ and $x_1, x_2, \ldots, x_m$ a finite system of vectors in X. As in [8] we define *Average* $(x_i \mid 1 \leq i \leq m)$ as the finite system with 2^m elements obtained by arranging in the lexicographical order of $\{-1,1\}^m$, the set of all the elements of the form $\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m$ for $(\varepsilon_1, \ldots, \varepsilon_m) \in \{-1,1\}^m = D_m$ (On $\{-1,1\}$ we consider the natural order). We will consider *Average* $(x_i \mid 1 \leq i \leq m)$ as an element of the space X^{2^m} and as sets we have the equality

$$\textit{Average}\,(x_i \mid 1 \leq i \leq m) = \{\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m \mid (\varepsilon_1, \ldots, \varepsilon_m) \in D_m\}.$$

The idea of considering these averages was suggested to the author by the well-known discrete form of Rademacher means, namely the equality

$$\int_0^1 \|x_1 r_1(t) + \cdots + x_m r_m(t)\| \, dt = \frac{1}{2^m} \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in D} \|\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m\|$$

see [1], [2]. Further, in [1, Exercise 8.18(a), p. 107], or [9, p. 64] appear also these averages.

Lemma 1. *Let $m \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a finite system of scalars. Then*

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_\infty \asymp \|(\alpha_1, \ldots, \alpha_m)\|_1 ,$$

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_1 \asymp 2^m \|(\alpha_1, \ldots, \alpha_m)\|_2 ,$$

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_2 = \left(\sqrt{2}\right)^m \|(\alpha_1, \ldots, \alpha_m)\|_2 .$$

Proof. Indeed, in the real case, we have obvious

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_\infty = \max_{(\varepsilon_1,\ldots,\varepsilon_m)\in D_m} |\varepsilon_1 \alpha_1 + \cdots + \varepsilon_m \alpha_m| = \sum_{i=1}^m |\alpha_i|$$

and from here, taking the real and imaginary part, we deduce, in the complex case

$$\frac{1}{2}\sum_{i=1}^m |\alpha_i| \le \|Average\,(\alpha_i \mid 1 \le i \le m)\|_\infty \le \sum_{i=1}^m |\alpha_i|$$

see also [1, Exercise 8.18(a), p. 107], or [9, p. 64].

For the second, by Khichin's inequality, see [1], [2], [5], we have

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_1 = \sum_{\varepsilon\in D_m} |\varepsilon_1 \alpha_1 + \cdots + \varepsilon_m \alpha_m| \asymp 2^m \|(\alpha_1, \ldots, \alpha_m)\|_2 .$$

The last equality follows from the well-known equality

$$\|Average\,(\alpha_i \mid 1 \le i \le m)\|_2 = \left(\sum_{\varepsilon\in D_m} |\varepsilon_1 \alpha_1 + \cdots + \varepsilon_m \alpha_m|^2\right)^{\frac{1}{2}} = \left(\sqrt{2}\right)^m \|(\alpha_1, \ldots, \alpha_m)\|_2 .$$

□

Our next definition is a natural iteration for averages.

Definition 2. For $k \in \mathbb{N} \bigcup \{0\}$ define $f_k : \mathbb{N} \to \mathbb{N}$ by

$$\begin{cases} f_0(n) = n, \\ f_{k+1}(n) = 2^{f_k(n)}, \ k \ge 0. \end{cases}$$

Let X be a Banach space, $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n$ a finite system of vectors in X. Define

$$Average_1\,(x_i \mid 1 \le i \le n;\, X) = Average\,(x_i \mid 1 \le i \le n;\, X) .$$

Let also $k \in \mathbb{N}$. For the $f_k(n)$ finite system

$$Average_k(x_i \mid 1 \leq i \leq n;\, X) = \left\{\beta_1, \dots, \beta_{f_k(n)}\right\},$$

say, we apply the same procedure and denote

$$Average_{k+1}(x_i \mid 1 \leq i \leq n;\, X) = Average\left(\beta_i \mid 1 \leq i \leq f_k(n);\, X\right).$$

We consider $Average_k(x_i \mid 1 \leq i \leq n; X)$ as an element of the space $X^{f_k(n)}$.

Lemma 3. *Let $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_n$ be a finite system of scalars and $k \in \mathbb{N}$. Then*

$$\|Average(\alpha_i \mid 1 \leq i \leq n)\|_\infty \asymp \|(\alpha_1, \dots, \alpha_n)\|_1,$$
$$\|Average_2(\alpha_i \mid 1 \leq i \leq n)\|_\infty \asymp 2^n \|(\alpha_1, \dots, \alpha_n)\|_2,$$

$$\|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_\infty$$
$$\asymp f_{k-1}(n) \sqrt{f_1(n) f_2(n) \cdots f_{k-2}(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2,\ k \geq 3.$$

Proof. With the same notations as in Definition 2, by Lemma 1 we have

$$\|Average_{k+1}(\alpha_i \mid 1 \leq i \leq n)\|_\infty = \|Average(\beta_i \mid 1 \leq i \leq f_k(n))\|_\infty$$
$$\asymp \left\|\left(\beta_1, \dots, \beta_{f_k(n)}\right)\right\|_1 = \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_1,$$

$$\|Average_{k+1}(\alpha_i \mid 1 \leq i \leq n)\|_1 = \|Average(\beta_i \mid 1 \leq i \leq f_k(n))\|_1$$
$$\asymp 2^{f_k(n)} \left\|\left(\beta_1, \dots, \beta_{f_k(n)}\right)\right\|_2 = f_{k+1}(n) \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_2$$

and

$$\begin{aligned}\|Average_{k+1}(\alpha_i \mid 1 \leq i \leq n)\|_2 &= \|Average(\beta_i \mid 1 \leq i \leq f_k(n))\|_2\\ &= \sqrt{2^{f_k(n)}} \left\|\left(\beta_1, \dots, \beta_{f_k(n)}\right)\right\|_2\\ &= \sqrt{f_{k+1}(n)}\, \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_2.\end{aligned}$$

Denote

$$\begin{aligned}a_k &= \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_\infty,\\ b_k &= \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_1,\\ c_k &= \|Average_k(\alpha_i \mid 1 \leq i \leq n)\|_2.\end{aligned}$$

Then from the above proved relations for each $k \geq 1$ we have

$$a_{k+1} \asymp b_k;\ b_{k+1} \asymp f_{k+1}(n)\, c_k;\ c_{k+1} = c_k \sqrt{f_{k+1}(n)}.$$

Because by Lemma 1

$$c_1 = \|Average(\alpha_i \mid 1 \leq i \leq n)\|_2 = \sqrt{f_1(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2$$

we deduce

$$c_k = \sqrt{f_1(n) f_2(n) \cdots f_k(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2,\ k \geq 1.$$

From $b_{k+1} \asymp f_{k+1}(n)\, c_k$ we get

$$b_{k+1} \asymp f_{k+1}(n) \sqrt{f_1(n) f_2(n) \cdots f_k(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2, \ k \geq 1,$$

i.e.,

$$b_k \asymp f_k(n) \sqrt{f_1(n) f_2(n) \cdots f_{k-1}(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2, \ k \geq 2,$$

and by Lemma 1

$$b_1 = \|Average(\alpha_i \mid 1 \leq i \leq n)\|_1 \asymp 2^n \|(\alpha_1, \dots, \alpha_n)\|_2 .$$

From $a_{k+1} \asymp b_k$, $k \geq 1$ we get

$$a_{k+1} \asymp f_k(n) \sqrt{f_1(n) f_2(n) \cdots f_{k-1}(n)}\, \|(\alpha_1, \dots, \alpha_n)\|_2, \ k \geq 2,$$

i.e., for $k \geq 3$ we get the evaluations from the statement.

Also by Lemma 1,

$$a_2 \asymp b_1 = 2^n \|(\alpha_1, \dots, \alpha_n)\|_2,$$
$$a_1 = \|Average(\alpha_i \mid 1 \leq i \leq n)\|_\infty \asymp \|(\alpha_1, \dots, \alpha_n)\|_1 . \qquad \square$$

We state now a result which is a well-known consequence of the Hahn-Banach theorem.

Result. *Let X be a Banach space. Then for each $x \in X$ we have*

$$\|x\| = \sup_{\|x^*\| \leq 1} |x^*(x)| .$$

Lemma 4. *Let X be a Banach space, $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n$ a finite system of vectors in X and $k \in \mathbb{N}$. Then*

$$\|Average(x_i \mid 1 \leq i \leq n;\, X)\|_\infty \asymp w_1(x_i \mid 1 \leq i \leq n;\, X),$$
$$\|Average_2(x_i \mid 1 \leq i \leq n;\, X)\|_\infty \asymp 2^n w_2(x_i \mid 1 \leq i \leq n;\, X),$$
$$\|Average_k(x_i \mid 1 \leq i \leq n;\, X)\|_\infty$$
$$\asymp f_{k-1}(n) \sqrt{f_1(n) f_2(n) \cdots f_{k-2}(n)}\, w_2(x_i \mid 1 \leq i \leq n;\, X), \ k \geq 3.$$

Proof. We will use the notations from Definition 2. From Result we have

$$\begin{aligned}
&\|Average(x_i \mid 1 \leq i \leq n;\, X)\|_\infty \\
&= \max_{\varepsilon \in D_n} \|\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\| \\
&= \max_{\varepsilon \in D_n} \sup_{\|x^*\| \leq 1} |\varepsilon_1 x^*(x_1) + \varepsilon_2 x^*(x_2) + \cdots + \varepsilon_n x^*(x_n)| \\
&= \sup_{\|x^*\| \leq 1} \max_{\varepsilon \in D_n} |\varepsilon_1 x^*(x_1) + \varepsilon_2 x^*(x_2) + \cdots + \varepsilon_n x^*(x_n)| \\
&= \sup_{\|x^*\| \leq 1} \|Average(x^*(x_i) \mid 1 \leq i \leq n)\|_\infty .
\end{aligned}$$

By Lemma 1, for each $\|x^*\| \leq 1$ we have

$$\|Average(x^*(x_i) \mid 1 \leq i \leq n)\|_\infty \asymp \|(x^*(x_i) \mid 1 \leq i \leq n)\|_1$$

thus

$$\|Average\,(x_i \mid 1 \le i \le n;\, X)\|_\infty \asymp w_1\,(x_i \mid 1 \le i \le n;\, X)\,.$$

We prove now that for each $k \ge 1$,

$$\|Average_{k+1}\,(x_i\,|\,1 \le i \le n;\, X)\|_\infty \asymp \sup_{\|x^*\| \le 1} \|Average_k\,(x^*\,(x_i)\,|\,1 \le i \le n)\|_1\,.$$

Indeed, by Lemma 1 and from what we have proved above we deduce

$$\begin{aligned} &\|Average_{k+1}\,(x_i \mid 1 \le i \le n;\, X)\|_\infty \\ &= \|Average\,(\beta_i \mid 1 \le i \le f_k\,(n)\,;\, X)\|_\infty \\ &\asymp w_1\,(\beta_i \mid 1 \le i \le f_k\,(n)\,;\, X) \\ &= \sup_{\|x^*\| \le 1} \|(x^*\,(\beta_i) \mid 1 \le i \le f_k\,(n))\|_1 \\ &= \sup_{\|x^*\| \le 1} \|Average_k\,(x^*\,(x_i) \mid 1 \le i \le n)\|_1\,. \end{aligned}$$

The Lemma 3, implies, for each $k \ge 2$ and each $\|x^*\| \le 1$

$$\begin{aligned} &\|Average_k\,(x^*\,(x_i) \mid 1 \le i \le n)\|_1 \\ &\asymp f_k\,(n)\,\sqrt{f_1\,(n)\, f_2\,(n) \cdots f_{k-1}\,(n)}\,\|(x^*\,(x_1)\,, \ldots, x^*\,(x_n))\|_2\,. \end{aligned}$$

Hence for $k \ge 2$

$$\begin{aligned} &\|Average_{k+1}\,(x_i \mid 1 \le i \le n;\, X)\|_\infty \\ &\asymp f_k\,(n)\,\sqrt{f_1\,(n)\, f_2\,(n) \cdots f_{k-1}\,(n)}\,w_2\,(x_i \mid 1 \le i \le n;\, X) \end{aligned}$$

i.e., for $k \ge 3$ we get the evaluations from the statement.

Also, from Lemma 1

$$\begin{aligned} \|Average_2\,(x_i \mid 1 \le i \le n;\, X)\|_\infty &\asymp \sup_{\|x^*\| \le 1} \|Average\,(x^*\,(x_i) \mid 1 \le i \le n)\|_1 \\ &\asymp 2^n \sup_{\|x^*\| \le 1} \|(x^*\,(x_i) \mid 1 \le i \le n)\|_2 \\ &= 2^n w_2\,(x_i \mid 1 \le i \le n;\, X)\,. \end{aligned}$$

□

Notation. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of Banach spaces, $(x_{ni})_{1 \le i \le n} \subset X_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $Average\,(x_{ni} \mid 1 \le i \le n;\, X_n)$ is an element of the space $X_n^{2^n}$ and we consider the sequence

$$(*) \qquad (Average\,(x_{ni} \mid 1 \le i \le n;\, X_n))_{n \in \mathbb{N}}\,.$$

From Lemma 4, the sequence $(*)$ is an element of the space

$$(**) \qquad c_0\,(X_1, X_1, \ldots, X_n, \ldots, X_n, \ldots)$$

(each X_n appears 2^n for each $n \in \mathbb{N}$) if and only if $w_1\,(x_{ni} \mid 1 \le i \le n;\, X_n) \to 0$.

In order to avoid unpleasant writings, instead $(**)$ we write simply $c_0\,(X_n \mid n \in \mathbb{N})$.

In the rest of the paper for a natural number $k \geq 2$ we denote

$$b_{n2} = 2^n,$$
$$b_{nk} = f_{k-1}(n)\sqrt{f_1(n) f_2(n) \cdots f_{k-2}(n)}, \text{ if } k \geq 3.$$

Using the same convention as above, from Lemma 4, the sequence $(Average_k(x_{ni} \mid 1 \leq i \leq n; X_n))_{n\in\mathbb{N}}$ is an element of the space $c_0(X_n \mid n \in \mathbb{N})$ if and only if $b_{nk} w_2(x_i \mid 1 \leq i \leq n;\, X_n) \to 0$.

3. The main results

We begin with a well-known fact:

Fact. *Let (S, Σ, μ) be a finite measure space, X, Y Banach spaces, $g : S \to L(X,Y)$ a μ-Bochner integrable function and $G : \Sigma \to L(X,Y)$,*

$$G(E) = \int_E g d\mu \quad \text{for } E \in \Sigma.$$

Then

$$\|g\|_{\text{Pettis}} = \|G\|(T) = \sup_{\|x\|\leq 1,\ \|y^*\|\leq 1} \int_S |\langle g(s)x, y^*\rangle| \, d\mu(s).$$

This follows from the definition of semivariation and the Pettis norm and the fact that $\{x \otimes y^* \mid \|x\| \leq 1,\ \|y^*\| \leq 1\}$ is norming for $L(X,Y)$.

Proposition 5. *Let Ω be a compact Hausdorff space, μ a nonnegative finite regular Borel measure on Ω, $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ two sequences of Banach spaces, $g_n : \Omega \to L(X_n, Y_n)$ μ-Bochner integrable functions such that for each $E \in \Sigma_\Omega$*

$$\int_E g_n d\mu \to 0 \quad \text{in the operator norm.}$$

Let $U : C(\Omega) \to c_0(L(X_n, Y_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(\int_\Omega f(\omega) g_n(\omega) \, d\mu(\omega)\right)_{n\in\mathbb{N}}.$$

Then

(i) *U is weakly compact.*

(ii) *U is compact if and only if $\sup\limits_{\|x\|\leq 1,\ \|y^*\|\leq 1} \int_\Omega |\langle g_n(\omega)x, y^*\rangle| \, d\mu(\omega) \to 0$.*

(iii) *U is absolutely summing if and only if $\int_\Omega \sup\limits_{n\in\mathbb{N}} \|g_n(\omega)\| \, d\mu(\omega) < \infty$.*

(iv) *U is nuclear if and only if U is absolutely summing and $\|g_n(\omega)\| \to 0$ μ-a.e..*

Proof. Let $G_n : \Sigma_\Omega \to L(X_n, Y_n)$, $G_n(E) = \int_E g_n d\mu$ for $E \in \Sigma_\Omega$. The hypothesis and Nikodym's boundedness theorem, see [3], gives us that the family

$(\langle G_n x, y^* \rangle)_{\|x\|\le 1,\ \|y^*\|\le 1,\ n\in\mathbb{N}}$ is uniformly bounded, i.e., there exists $L > 0$ such that

(1)
$$\int_\Omega |\langle g_n(\omega)x, y^*\rangle| \, d\mu(\omega) = |\langle G_n x, y^*\rangle|(\Omega) \le L, \forall \|x\| \le 1, \|y^*\| \le 1, n \in \mathbb{N}.$$

From hypothesis we get $\int_\Omega f(\omega) g_n(\omega) d\mu(\omega) \to 0$ for each simple function f and from this fact and (1) we deduce $\int_\Omega f(\omega) g_n(\omega) d\mu(\omega) \to 0$ for each $f \in B(\Sigma_\Omega)$, so U is well defined. The representing measure of U is

$$G(E) = \left(\int_E g_n(\omega) \, d\mu(\omega) \right)_{n\in\mathbb{N}}, \ E \in \Sigma_\Omega$$

which, by hypothesis, takes its values in $c_0(L(X_n, Y_n) \mid n \in \mathbb{N})$ and thus, see [3, Chapter VI], U is weakly compact.

By [3, Chapter VI], U is compact if and only if the range of G is relatively norm compact and this by Proposition 1(ii) in [8] is equivalent to $\|g_n\|_{\text{Pettis}} \to 0$, which by Fact gives (ii).

By [3, Chapter VI], U is absolutely summing if and only if G is of bounded variation, which by Proposition 1(iii) in [8] is equivalent to (iii).

By [3, Chapter VI], U is nuclear if and only if U is absolutely summing and G has a μ-Bochner integrable derivative, and this by Proposition 1(iv) in [8], is equivalent to (iv). □

In view of Example 3 in [8] it is a natural question to apply average technique for a triangular matrix of functions. Since by Lemma 4 there is a delineation between averages of first order and averages of order greater or equal than two, we analyze these two situations.

Proposition 6. *Let Ω be a compact Hausdorff space, μ a nonnegative finite regular Borel measure on Ω, $(X_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces and $(h_{ni})_{1\le i\le n} \subset L_1(\mu, X_n)$.*

(a) *Suppose that*

$$w_1\left(\int_E h_{ni} d\mu \mid 1 \le i \le n;\ X_n \right) \to 0 \quad \text{for each } E \in \Sigma_\Omega.$$

Let $U : C(\Omega) \to c_0(X_n \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average\left(\int_\Omega f(\omega) h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\ X_n \right) \right)_{n\in\mathbb{N}}.$$

Then

(i) *U is weakly compact.*

(ii) *U is compact if and only if*

$$\sup_{|\varepsilon_i|\le 1} \|h_{n1}\varepsilon_1 + \cdots + h_{nn}\varepsilon_n\|_{\text{Pettis}} \to 0.$$

(iii) *U is absolutely summing if and only if*

$$\int_\Omega \sup_{n\in\mathbb{N}} w_1\left(h_{ni}(\omega) \mid 1 \le i \le n;\ X_n\right) d\mu(\omega) < \infty.$$

(iv) *U is nuclear if and only if U is absolutely summing and*

$$w_1\left(h_{ni}(\omega) \mid 1 \le i \le n;\ X_n\right) \to 0 \quad \textit{for } \mu\textit{-a.e. } \omega \in \Omega.$$

(b) *Let $k \ge 2$ be a natural number. Suppose that*

$$b_{nk} w_2\left(\int_E h_{ni} d\mu \mid 1 \le i \le n;\ X_n\right) \to 0 \quad \textit{for each } E \in \Sigma_\Omega.$$

Let $U : C(\Omega) \to c_0(X_n \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average_k\left(\int_\Omega f(\omega) h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\ X_n\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) *U is weakly compact.*

(ii) *U is compact if and only if*

$$b_{nk} \sup_{\|\xi\|_2 \le 1} \|h_{n1}\xi_1 + \cdots + h_{nn}\xi_n\|_{\text{Pettis}} \to 0.$$

(iii) *U is absolutely summing if and only if*

$$\int_\Omega \sup_{n\in\mathbb{N}} b_{nk} w_2\left(h_{ni}(\omega) \mid 1 \le i \le n;\ X_n\right) d\mu(\omega) < \infty.$$

(iv) *U is nuclear if and only if U is absolutely summing and*

$$b_{nk} w_2\left(h_{ni}(\omega) \mid 1 \le i \le n;\ X_n\right) \to 0 \quad \textit{for } \mu\textit{-a.e. } \omega \in \Omega.$$

Proof. (a) Let $g_n : \Omega \to L(X_n^*, l_1^n)$ be the function defined by

$$(g_n(\omega))(x^*) = \left(x^*(h_{n1}(\omega)), \ldots, x^*(h_{nn}(\omega))\right) \quad \text{for } \omega \in \Omega, x^* \in X_n^*$$

i.e., $g_n = \sum_{i=1}^{n} h_{ni} \otimes e_{ni}$, and $V_n : C(\Omega) \to L(X_n^*, l_1^n)$

$$(V_n(f))(x^*) = \left(\int_\Omega f(\omega) x^*(h_{n1}(\omega)) d\mu(\omega), \ldots, \int_\Omega f(\omega) x^*(h_{nn}(\omega)) d\mu(\omega)\right).$$

Observe that

$$V_n(f) = \int_\Omega f(\omega) g_n(\omega) d\mu(\omega).$$

Further, because g_n is obvious Bochner integrable, by Hille's theorem, see [3, Chapter II, Theorem 2.6, p. 47], for each $E \in \Sigma_\Omega$, $x^* \in X_n^*$, we have

$$\left(\int_E g_n(\omega) d\mu(\omega)\right)(x^*) = \left(\int_E x^*(h_{n1}(\omega)) d\mu(\omega), \ldots, \int_E x^*(h_{nn}(\omega)) d\mu(\omega)\right)$$

and thus

$$\left\|\int_E g_n(\omega) d\mu(\omega)\right\|_{L\left(X_n^*, l_1^n\right)} = w_1\left(\int_E h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\ X_n\right)$$

which, by hypothesis, is convergent to zero. Proposition 5 assures that the operator $V : C(\Omega) \to c_0(L(X_n^*, l_1^n) \mid n \in \mathbb{N})$ defined by

$$V(f) = (V_n(f))_{n\in\mathbb{N}}$$

takes its values in $c_0(L(X_n^*, l_1^n) \mid n \in \mathbb{N})$.

Let $f \in C(\Omega)$. From Lemma 4 we have

$$\begin{aligned} &\left\| Average\left(\int_\Omega f(\omega) h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\, X_n\right)\right\|_\infty \\ &\asymp w_1\left(\int_\Omega f(\omega) h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\, X_n\right) = \|V_n(f)\| \end{aligned} \tag{1}$$

and hence U takes its values in $c_0(X_n \mid n \in \mathbb{N})$ if and only if V takes its values in $c_0(L(X_n^*, l_1^n) \mid n \in \mathbb{N})$, which, as we already proved, is true.

From (1) we deduce

$$c\|V(f)\| \le \|U(f)\| \le C\|V(f)\|$$

for some constants c, $C > 0$ independent of f.

This shows that U is compact (resp. U is absolutely summing) if and only if V is compact (resp. V is absolutely summing) which by Proposition 5 gives (ii) and (iii).

Since (i) and (iv) do not follow from Proposition 5, we argue as follows. The representing measure of U is

$$G(E) = \left(Average\left(\int_E h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\, X_n\right)\right)_{n\in\mathbb{N}} \quad \text{for } E \in \Sigma_\Omega.$$

From Lemma 4 and hypothesis, for each $E \in \Sigma_\Omega$

$$\begin{aligned} &\left\| Average\left(\int_E h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\, X_n\right)\right\|_\infty \\ &\asymp w_1\left(\int_E h_{ni}(\omega) d\mu(\omega) \mid 1 \le i \le n;\, X_n\right) \to 0, \end{aligned}$$

thus U is weakly compact.

By [3, Chapter VI], U is nuclear if and only if U is absolutely summing and G has a μ-Bochner integrable derivative, and this, by Proposition 1(iv) in [8], is equivalent to

$$\|Average(h_{ni}(\omega) \mid 1 \le i \le n; X_n)\|_\infty \to 0 \;\text{ for } \mu\text{-a.e. } \omega \in \Omega.$$

Then (iv) follows, because by Lemma 4, for each $\omega \in \Omega$

$$\|Average(h_{ni}(\omega) \mid 1 \le i \le n; X_n)\|_\infty \asymp w_1(h_{ni}(\omega) \mid 1 \le i \le n;\, X_n).$$

(b) As we will see in the sequel the proof of (b) is similar to that of (a). Indeed, in this case, let $g_n : \Omega \to L(X_n^*, l_2^n)$ be the function defined by

$$(g_n(\omega))(x^*) = b_{nk}(x^*(h_{n1}(\omega)), \ldots, x^*(h_{nn}(\omega))) \quad \text{for } \omega \in \Omega, x^* \in X_n^*,$$

$V_n : C(\Omega) \to L(X_n^*, l_2^n)$ the operator defined by

$$(V_n(f))(x^*) = b_{nk} \left(\int_\Omega f(\omega) x^*(h_{n1}(\omega)) \, d\mu(\omega), \ldots, \int_\Omega f(\omega) x^*(h_{nn}(\omega)) \, d\mu(\omega) \right)$$

and observe that

$$V_n(f) = \int_\Omega f(\omega) g_n(\omega) \, d\mu(\omega).$$

Further, because g_n is Bochner integrable, as in (a) we deduce that for each $E \in \Sigma_\Omega$

$$\left\| \int_E g_n(\omega) \, d\mu(\omega) \right\|_{L\left(X_n^*, l_2^n\right)} = b_{nk} w_2 \left(\int_E h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right)$$

which, by hypothesis, is convergent to zero. Proposition 5 assures that the operator $V : C(\Omega) \to c_0(L(X_n^*, l_2^n) \mid n \in \mathbb{N})$ defined by

$$V(f) = (V_n(f))_{n \in \mathbb{N}}$$

takes its values in $c_0(L(X_n^*, l_2^n) \mid n \in \mathbb{N})$.

Let $f \in C(\Omega)$. From Lemma 4 we have

$$\begin{aligned} &\left\| Average_k \left(\int_\Omega f(\omega) h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right) \right\|_\infty \\ &\asymp b_{nk} w_2 \left(\int_\Omega f(\omega) h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right) = \|V_n(f)\| \end{aligned} \tag{2}$$

and hence U takes its values in $c_0(X_n \mid n \in \mathbb{N})$ if and only if V takes its values in $c_0(L(X_n^*, l_2^n) \mid n \in \mathbb{N})$, which is true.

From (2) we deduce

$$c \|V(f)\| \le \|U(f)\| \le C \|V(f)\|$$

for some constants $c, C > 0$ independent of f.

This shows that U is compact (resp. U is absolutely summing) if and only if V is compact (resp. V is absolutely summing) and Proposition 5 gives (ii) and (iii).

The representing measure of U is

$$G(E) = \left(Average_k \left(\int_E h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right) \right)_{n \in \mathbb{N}} \quad \text{for } E \in \Sigma_\Omega.$$

From Lemma 4, for each $E \in \Sigma_\Omega$

$$\begin{aligned} &\left\| Average_k \left(\int_E h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right) \right\|_\infty \\ &\asymp b_{nk} w_2 \left(\int_E h_{ni}(\omega) \, d\mu(\omega) \mid 1 \le i \le n;\, X_n \right) \to 0 \end{aligned}$$

by hypothesis, thus U is weakly compact.

By [3, Chapter VI], U is nuclear if and only if U is absolutely summing and G has a μ-Bochner integrable derivative, and this by Proposition 1(iv) in [8], is equivalent to

$$\left\| Average_k \left(h_{ni}\left(\omega\right) \mid 1 \leq i \leq n; X_n \right) \right\|_\infty \to 0 \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

Then (iv) follows, because by Lemma 4, for each $\omega \in \Omega$

$$\left\| Average_k \left(h_{ni}\left(\omega\right) \mid 1 \leq i \leq n; X_n \right) \right\|_\infty \asymp b_{nk} w_2 \left(h_{ni}\left(\omega\right) \mid 1 \leq i \leq n;\, X_n \right). \square$$

In the next corollary, item (a) is an obvious extension of Example 3 in [8]. In addition to [8], it is natural to study the same problem for averages of order greater or equal than two, i.e., item (b).

Corollary 7. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces,* $(x_{ni})_{1\leq i\leq n} \subset X_n$.

(a) *Suppose* $\sup_{n\in\mathbb{N}} w_2 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) < \infty$ *and let* $U : C\left[0,1\right] \to c_0\left(X_n \mid n \in \mathbb{N} \right)$ *be the operator defined by*

$$U\left(f\right) = \left(Average \left(x_{ni} \int_0^1 f\left(t\right) r_{n+i}\left(t\right) dt \mid 1 \leq i \leq n;\, X_n \right) \right)_{n\in\mathbb{N}}.$$

Then

(i) U *is weakly compact.*

(ii) U *is compact if and only if* $w_2 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) \to 0$.

(iii) U *is absolutely summing if and only if* $\sup_{n\in\mathbb{N}} w_1 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) < \infty$.

(iv) U *is nuclear if and only if* $w_1 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) \to 0$.

(b) *Let* $k \geq 2$ *be a natural number. Suppose* $\sup_{n\in\mathbb{N}} \left(b_{nk} \max\limits_{1\leq i\leq n} \|x_{ni}\| \right) < \infty$ *and let* $U : C\left[0,1\right] \to c_0\left(X_n \mid n \in \mathbb{N} \right)$ *be the operator defined by*

$$U\left(f\right) = \left(Average_k \left(x_{ni} \int_0^1 f\left(t\right) r_{n+i}\left(t\right) dt \mid 1 \leq i \leq n;\, X_n \right) \right)_{n\in\mathbb{N}}.$$

Then

(i) U *is weakly compact.*

(ii) U *is compact if and only if* $b_{nk} \max\limits_{1\leq i\leq n} \|x_{ni}\| \to 0$.

(iii) U *is absolutely summing if and only if* $\sup\limits_{n\in\mathbb{N}} b_{nk} w_2 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) < \infty$.

(iv) U *is nuclear if and only if* $b_{nk} w_2 \left(x_{ni} \mid 1 \leq i \leq n;\, X_n \right) \to 0$.

Proof. Take $h_{ni} = x_{ni} r_{n+i} \in L_1\left(\lambda, X_n\right)$ in Proposition 6.

(a) For each $E \in \mathcal{B}$, by Cauchy-Buniakowski-Schwartz's inequality and hypothesis we have

$$\begin{aligned} & w_1 \left(\int_E x_{ni} r_{n+i}\left(t\right) dt \mid 1 \leq i \leq n;\, X_n \right) \\ = & \sup_{\|x^*\|\leq 1} \left(\left| x^*\left(x_{n1}\right) \right| \left| \int_E r_{n+1}\left(t\right) dt \right| + \cdots + \left| x^*\left(x_{nn}\right) \right| \left| \int_E r_{2n}\left(t\right) dt \right| \right) \end{aligned}$$

$$\leq w_2\left(x_{ni} \mid 1 \leq i \leq n;\, X_n\right)\left(\left|\int_E r_{n+1}(t)\,dt\right|^2 + \cdots + \left|\int_E r_{2n}(t)\,dt\right|^2\right)^{\frac{1}{2}} \to 0.$$

(i) follows from Proposition 6(a)(i).

From the definition of Pettis norm and Khinchin's inequality we get

$$\sup_{|\varepsilon_i|\leq 1} \left\|x_{n1} r_{n+1} \varepsilon_1 + \cdots + x_{nn} r_{2n} \varepsilon_n\right\|_{\text{Pettis}} \asymp w_2\left(x_{ni} \mid 1 \leq i \leq n;\, X_n\right)$$

and (ii) follows from Proposition 6(a)(ii).

Further for each $t \in [0,1]$, $|r_n(t)| = 1$,

$$w_1\left(x_{ni} r_{n+i}(t) \mid 1 \leq i \leq n;\, X_n\right) = w_1\left(x_{ni} \mid 1 \leq i \leq n;\, X_n\right)$$

and (iii), (iv) follow from Proposition 6(a)(iii), (iv).

(b) We observe that the hypothesis in Proposition 6(b) are satisfies because, in our hypothesis, for each $E \in \mathcal{B}$

$$b_{nk} w_2\left(x_{ni} \int_E r_{n+i}(t)\,dt \mid 1 \leq i \leq n;\, X_n\right)$$

$$= b_{nk} \sup_{\|x^*\|\leq 1}\left(\left|x^*(x_{n1})\right|^2 \left|\int_E r_{n+1}(t)\,dt\right|^2 + \cdots + \left|x^*(x_{nn})\right|^2 \left|\int_E r_{2n}(t)\,dt\right|^2\right)^{\frac{1}{2}}$$

$$\leq \left(b_{nk} \max_{1\leq i\leq n} \|x_{ni}\|\right)\left(\left|\int_E r_{n+1}(t)\,dt\right|^2 + \cdots + \left|\int_E r_{2n}(t)\,dt\right|^2\right)^{\frac{1}{2}} \to 0.$$

(i) follows from Proposition 6(b)(i). From the definition of Pettis norm, Khinchin's inequality and Result we get

$$\begin{aligned}
&\sup_{\|\xi\|_2\leq 1} \left\|x_{n1} r_{n+1} \xi_1 + \cdots + x_{nn} r_{2n} \xi_n\right\|_{\text{Pettis}} \\
\asymp\ &\sup_{\|x^*\|\leq 1} \sup_{\|\xi\|_2\leq 1} \left\|\left(\xi_1 x^*(x_{n1}), \ldots, \xi_n x^*(x_{nn})\right)\right\|_2 \\
=\ &\sup_{\|x^*\|\leq 1} \left\|\left(x^*(x_{n1}), \ldots, x^*(x_{nn})\right)\right\|_\infty \\
=\ &\max_{1\leq i\leq n} \sup_{\|x^*\|\leq 1} \left|x^*(x_{ni})\right| \\
=\ &\max_{1\leq i\leq n} \|x_{ni}\|\,.
\end{aligned}$$

From Proposition 6(b)(ii) we get (ii).

(iii), (iv) follow from Proposition 6(b)(iii), (iv), because for each $t \in [0,1]$

$$w_2\left(x_{ni} r_{n+i}(t) \mid 1 \leq i \leq n;\, X_n\right) = w_2\left(x_{ni} \mid 1 \leq i \leq n;\, X_n\right). \qquad \square$$

Remark. As in the proof of Example 3 in [8], it can be proved that for the operator U defined as in Corollary 7, either U is absolutely summing, or its representing measure is of everywhere infinite variation, see [4].

4. The examples

In our examples, in view of Corollary 7, we need the following well-known result. For the sake of completeness we include a short proof.

Proposition 8. (i) *Let X be a Banach space, $A \subset B_{X^*}$ norming for X, $(x_i)_{1\le i\le n} \subset X$ and $1 \le p < \infty$. Then*

$$w_p\left(x_i \mid 1 \le i \le n;\, X\right) = \sup_{x^* \in A} \left\| \left(x^*\left(x_1\right), \ldots, x^*\left(x_n\right)\right) \right\|_p .$$

(ii) *Let Ω be a compact Hausdorff space, $(f_i)_{1\le i\le n} \subset C(\Omega)$, $f : \Omega \to \mathbb{K}^n$, $f(\omega) = \left(f_1(\omega), \ldots, f_n(\omega)\right)$ and $1 \le p < \infty$. Then*

$$w_p\left(f_i \mid 1 \le i \le n;\, C(\Omega)\right) = \|f\|_{C\left(\Omega, l_p^n\right)} .$$

(iii) *Let X, Y be Banach spaces, $U \in L(X, Y)$, $M > 0$ such that $\|x\| \le \|U(x)\| \le M\|x\|$ for any $x \in X$. Let $(x_i)_{1\le i\le n} \subset X$ and $1 \le p < \infty$. Then*

$$\begin{aligned} w_p\left(x_i \mid 1 \le i \le n;\, X\right) &\le w_p\left(U\left(x_i\right) \mid 1 \le i \le n;\, Y\right) \\ &\le M w_p\left(x_i \mid 1 \le i \le n;\, X\right). \end{aligned}$$

(iv) *Let $1 \le r < \infty$, $1 \le p \le \infty$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$. Then*

$$w_r\left(\lambda_i e_{ni} \mid 1 \le i \le n\,;\, l_p^n\right) = \left\|M_\lambda : l_{r^*}^n \to l_p^n\right\| = \|\lambda\|_\infty \ \text{ if } p \ge r^*,$$

or $\|\lambda\|_s$ if $p < r^*$, where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{s}$.

(v) *Let $1 \le p < \infty$, $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{K}^n$, $\beta_j = (x_{1j}, \ldots, x_{nj})$ and $\beta = (\beta_1, \ldots, \beta_n)$. Then*

$$w_p\left(x_i \mid 1 \le i \le n;\, l_\infty^n\right) = \|\beta\|_{l_\infty^n\left(l_p^n\right)} .$$

(vi) *Let $1 \le p, r < \infty$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$. Then*

$$w_r\left(\lambda_i r_i \mid 1 \le i \le n;\, L_p[0,1]\right) \asymp \|\lambda\|_\infty \ \text{ if } 2 \le r,$$

or $\|\lambda\|_s$ if $1 \le r < 2$, where $\frac{1}{s} = \frac{1}{r} - \frac{1}{2}$.

(vii) *Let (S, Σ, μ) be a measure space, $\{E_1, \ldots, E_n\} \subset \Sigma$ pairwise disjoint with $\mu(E_i) < \infty$ for each $1 \le i \le n$, $1 \le p, r < \infty$. Then*

$$w_r\left(\lambda_i \chi_{E_i} \mid 1 \le i \le n;\, L_p(\mu)\right) = \max_{1\le i\le n} \left(\left|\lambda_i\right| \left[\mu\left(E_i\right)\right]^{\frac{1}{p}}\right) \ \text{ if } r^* \le p,$$

or $\left(\sum\limits_{i=1}^{n} \left|\lambda_i\right|^s \left[\mu\left(E_i\right)\right]^{\frac{s}{p}}\right)^{\frac{1}{s}}$ if $r^* > p$, where $\frac{1}{p} = \frac{1}{r^*} + \frac{1}{s}$.

(viii) *Let (S, Σ, μ) be a measure space, $1 \le r < \infty$, $(g_i)_{1\le i\le n} \subset L_\infty(\mu)$, $g : S \to \mathbb{K}^n$, $g(s) = \left(g_1(s), \ldots, g_n(s)\right)$. Then*

$$w_r\left(g_i \mid 1 \le i \le n;\, L_\infty(\mu)\right) = \|g\|_{L_\infty\left(\mu, l_r^n\right)} .$$

(ix) *Let (S, Σ, μ) be a finite measure space, $1 \le r < \infty$, $(g_i)_{1\le i\le n} \subset L_1(\mu)$ such that each g_i takes positive values. Then*

$$w_r\left(g_i \mid 1 \le i \le n;\, L_1(\mu)\right) = \left\| \left(\int_S g_1 d\mu, \ldots, \int_S g_n d\mu \right) \right\|_r .$$

(x) *Let* (S, Ξ, ν) *be a finite measure space,* $1 \leq r < \infty$, $(g_i)_{1 \leq i \leq n} \subset L_1(\nu)$ *and* $g : S \to \mathbb{K}^n$ *defined by* $g(s) = (g_1(s), \ldots, g_n(s))$*. Then*

$$w_1(r_i g_i \mid 1 \leq i \leq n;\, L_1(\lambda \otimes \nu)) \asymp \int_S \|g(s)\|_2 \, d\nu(s)$$

and if $r > 1$

$$w_r(r_i g_i \mid 1 \leq i \leq n;\, L_1(\lambda \otimes \nu)) \asymp \sup_{\beta \in l_{r^*}^n, \|\beta\|_{r^*} \leq 1} \int_S \|M_\beta(g(s))\|_2 \, d\nu(s).$$

(xi) *Let* (S, Ξ, ν) *be a finite measure space,* $1 \leq r < \infty$, $(E_i)_{1 \leq i \leq n} \subset \Xi$ *a partition of* S, $(a_i)_{1 \leq i \leq n} \subset \mathbb{K}$*. Then*

$$w_r\left(a_i r_i \chi_{E_i} \mid 1 \leq i \leq n;\, L_1(\lambda \otimes \nu)\right) \asymp \|(a_1 \nu(E_1), \ldots, a_n \nu(E_n))\|_r.$$

Proof. (i) See [6, Lemma 1.1.15, p. 40] where the proof use the equality

$$w_p(x_i \mid 1 \leq i \leq n;\, X) = \sup_{\|\lambda\|_{p^*} \leq 1} \|\lambda_1 x_1 + \cdots + \lambda_n x_n\|. \tag{1}$$

(ii) See [6, Example 1.1.16, p. 40]; it is a particular case of (i), $\{\delta_\omega \mid \omega \in \Omega\}$ being norming for $C(\Omega)$.

(iii) and (iv) follow from hypothesis and (1).

(v) By (1) we have

$$\begin{aligned} w_p(x_i \mid 1 \leq i \leq n;\, l_\infty^n) &= \sup_{\|\lambda\|_{p^*} \leq 1} \|\lambda_1 x_1 + \cdots + \lambda_n x_n\|_\infty \\ &= \sup_{\|\lambda\|_{p^*} \leq 1} \max_{1 \leq j \leq n} |\lambda_1 x_{1j} + \cdots + \lambda_n x_{nj}| \\ &= \max_{1 \leq j \leq n} \|(x_{1j}, \ldots, x_{nj})\|_p = \|\beta\|_{l_\infty^n(l_p^n)}. \end{aligned}$$

(vi) From (1) and Khinchin's inequality

$$w_r(\lambda_i r_i \mid 1 \leq i \leq n;\, L_p[0,1]) = \sup_{\beta \in l_{r^*}^n,\ \|\beta\| \leq 1} \left\| \sum_{i=1}^n \lambda_i \beta_i r_i \right\|_{L_p[0,1]}$$

$$\asymp \sup_{\beta \in l_{r^*}^n, \|\beta\| \leq 1} \left(\sum_{i=1}^n |\lambda_i \beta_i|^2 \right)^{\frac{1}{2}} = \|M_\lambda : l_{r^*}^n \to l_2^n\|.$$

The assertion follows from well-known formula of the norm of the multiplication operator.

(vii) Again (1) gives

$$\begin{aligned} &w_r\left(\lambda_i \chi_{E_i} \mid 1 \leq i \leq n;\, L_p(\mu)\right) \\ = &\sup_{\beta \in l_{r^*}^n, \|\beta\| \leq 1} \left\| \sum_{i=1}^n \lambda_i \beta_i \chi_{E_i} \right\|_{L_p(\mu)} \end{aligned}$$

$$= \sup_{\beta \in l^n_{r^*},\ \|\beta\| \le 1} \left(\sum_{i=1}^n |\lambda_i \beta_i|^p \mu(E_i) \right)^{\frac{1}{p}} = \left\| M_\nu : l^n_{r^*} \to l^n_p \right\|,$$

where $\nu = \left(\lambda_1 [\mu(E_1)]^{\frac{1}{p}}, \dots, \lambda_n [\mu(E_n)]^{\frac{1}{p}} \right)$. We use again the norm of the multiplication operator.

(viii) See [6, Example 1.1.17, p. 40].

(ix) By (1) we have

$$w_r(g_i \mid 1 \le i \le n;\ L_1(\mu)) = \sup_{\|\beta\|_{r^*} \le 1} \int_S \left| \sum_{i=1}^n \beta_i g_i \right| d\mu.$$

For each $\|\beta\|_{r^*} \le 1$ we have

$$\left| \sum_{i=1}^n \beta_i \int_S g_i d\mu \right| \le \int_S \left| \sum_{i=1}^n \beta_i g_i \right| d\mu$$

and from here, by Hölder's inequality

$$\left\| \left(\int_S g_1 d\mu, \dots, \int_S g_n d\mu \right) \right\|_r \le w_r(g_i \mid 1 \le i \le n;\ L_1(\mu)).$$

For the right inequality, from $\int_S \left| \sum_{i=1}^n \beta_i g_i \right| d\mu \le \sum_{i=1}^n |\beta_i| \int_S |g_i| d\mu$, Hölder's inequality gives

$$w_r(g_i \mid 1 \le i \le n;\ L_1(\mu)) \le \left\| \left(\int_S |g_1| d\mu, \dots, \int_S |g_n| d\mu \right) \right\|_r.$$

Because each g_i takes positive values the statement follows.

(x) By (1)

$$w_r(r_i g_i \mid 1 \le i \le n;\ L_1(\lambda \otimes \nu)) = \sup_{\|\beta\|_{r^*} \le 1} \int_{[0,1] \times S} \left| \sum_{i=1}^n \beta_i r_i(t) g_i(s) \right| dt d\nu(s).$$

The Fubini theorem gives

$$\int_{[0,1] \times S} \left| \sum_{i=1}^n \beta_i r_i(t) g_i(s) \right| dt d\nu(s) = \int_S \left(\int_0^1 \left| \sum_{i=1}^n \beta_i r_i(t) g_i(s) \right| dt \right) d\nu(s).$$

Since for each $s \in S$, by Khinchin's inequality,

$$\int_0^1 \left| \sum_{i=1}^n \beta_i r_i(t) g_i(s) \right| dt \asymp \sqrt{|\beta_1|^2 |g_1(s)|^2 + \cdots + |\beta_n|^2 |g_n(s)|^2}$$

by integration we obtain

$$w_r(r_i g_i \mid 1 \le i \le n\ ; L_1(\lambda \otimes \nu)) \asymp \sup_{\|\beta\|_{r^*} \le 1} \int_S \|M_\beta(g(s))\|_2 d\nu(s).$$

In case $r = 1$, we have $\sup_{\|\beta\|_\infty \le 1} \int_S \|M_\beta(g(s))\|_2 d\nu(s) = \int_S \|g(s)\|_2 d\nu(s)$ and the statement follows.

(xi) is a particular case of (x). In this situation $g : S \to \mathbb{K}^n$ is defined by $g(s) = \left(a_1\chi_{E_1}(s), \ldots, a_n\chi_{E_n}(s)\right)$. In case $r = 1$, since $(E_i)_{1\le i\le n}$ is a partition of S, $\|g(s)\|_2 = |a_1|\chi_{E_1}(s) + \cdots + |a_n|\chi_{E_n}(s)$ and the statement follows. In case $r > 1$, for each $\beta \in l_{r^*}^n$,

$$\|M_\beta(g(s))\|_2 = |a_1||\beta_1|\chi_{E_1}(s) + \cdots + |a_n||\beta_n|\chi_{E_n}(s)$$

and thus by (x)

$$w_r\left(a_i r_i \chi_{E_i} \mid 1 \le i \le n;\, L_1(\lambda \otimes \nu)\right)$$

$$\asymp \sup_{\beta\in l_{r^*}^n, \|\beta\|_{r^*}\le 1} \left(\sum_{i=1}^{n} |a_i||\beta_i|\nu(E_i)\right) = \|(a_1\nu(E_1), \ldots, a_n\nu(E_n))\|_r .$$

In the next example item (a) is a natural extension of Example 3 in [8]; for $\Omega_n = \{1\}$, a singleton, $C(\Omega_n) = \mathbb{K}$. Further, item (b) is a natural completion of the same example. The proof follows from Corollary 7 and Proposition 8(ii). □

Example 9. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of compact Hausdorff spaces,

$$(\varphi_{ni})_{1\le i\le n} \subset C(\Omega_n),\ \varphi_n : \Omega_n \to \mathbb{K}^n,\ \varphi_n(\omega) = (\varphi_{n1}(\omega), \ldots, \varphi_{nn}(\omega)).$$

(a) Denote $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ and suppose that $\varphi \in l_\infty(C(\Omega_n, l_2^n) \mid n \in \mathbb{N})$. Let $U : C[0,1] \to c_0(C(\Omega_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average\left(\varphi_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \le i \le n;\, C(\Omega_n)\right)\right)_{n\in\mathbb{N}} .$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\varphi \in c_0(C(\Omega_n, l_2^n) \mid n \in \mathbb{N})$.

(iii) U is absolutely summing if and only if $\varphi \in l_\infty(C(\Omega_n, l_1^n) \mid n \in \mathbb{N})$.

(iv) U is nuclear if and only if $\varphi \in c_0(C(\Omega_n, l_1^n) \mid n \in \mathbb{N})$.

(aa) Let $k \ge 2$ be a natural number. Denote $\varphi_{\text{mod}} = (b_{nk}\varphi_n)_{n\in\mathbb{N}}$ and suppose that $\varphi_{\text{mod}} \in l_\infty(C(\Omega_n, l_\infty^n) \mid n \in \mathbb{N})$. Let $U : C[0,1] \to c_0(C(\Omega_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average_k\left(\varphi_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \le i \le n;\, C(\Omega_n)\right)\right)_{n\in\mathbb{N}} .$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\varphi_{\text{mod}} \in c_0(C(\Omega_n, l_\infty^n) \mid n \in \mathbb{N})$.

(iii) U is absolutely summing if and only if $\varphi_{\text{mod}} \in l_\infty(C(\Omega_n, l_2^n) \mid n \in \mathbb{N})$.

(iv) U is nuclear if and only if $\varphi_{\text{mod}} \in c_0(C(\Omega_n, l_2^n) \mid n \in \mathbb{N})$.

To avoid repetitions in Examples 10, 13-16, if $(\alpha_{ni})_{1\le i\le n, n\in\mathbb{N}}$ is a triangular matrix of scalars, which in the statement of these examples will be written as $(\alpha_{ni})_{i,n}$, we denote $\alpha_n = (\alpha_{n1}, \ldots, \alpha_{nn})$, $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and $\alpha_{\text{mod}} = (b_{nk}\alpha_n)_{n\in\mathbb{N}}$ for a natural number $k \ge 2$.

Also, in Examples 13(c), 14(c), 15(cc), if $(\alpha_{nij})_{1\leq i\leq n,1\leq j\leq n,n\in\mathbb{N}}\subset\mathbb{K}$, which in the statement of these examples will be written as $(\alpha_{nij})_{i,j,n}$, we denote $\beta_{nj}=(\alpha_{n1j},\ldots,\alpha_{nnj})$, $\beta_n=(\beta_{n1},\ldots,\beta_{nn})$, $\beta=(\beta_n)_{n\in\mathbb{N}}$ and $\beta_{\mathrm{mod}}=(b_{nk}\beta_n)_{n\in\mathbb{N}}$ for a natural number $k\geq 2$.

The proof of the next example in case $1\leq p<\infty$ (resp. $p=\infty$) follows from Corollary 7 and Proposition 8(vi) (resp. (viii)).

Example 10. (a) Let $1\leq p<\infty$, $(\alpha_{ni})_{i,n}$ be such that $\alpha\in l_\infty(l_\infty^n\mid n\in\mathbb{N})$ and $U:C[0,1]\to c_0(L_p[0,1])$

$$U(f)=\left(Average\left(\alpha_{ni}r_i\int_0^1 f(t)\,r_{n+i}(t)\,dt\mid 1\leq i\leq n;\,L_p[0,1]\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\alpha\in c_0(l_\infty^n\mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $\alpha\in l_\infty(l_2^n\mid n\in\mathbb{N})$.

(iv) U is nuclear if and only if $\alpha\in c_0(l_2^n\mid n\in\mathbb{N})$.

(aa) Let $k\geq 2$ be a natural number, $1\leq p<\infty$, $(\alpha_{ni})_{i,n}$ such that $\alpha_{\mathrm{mod}}\in l_\infty(l_\infty^n\mid n\in\mathbb{N})$ and $U:C[0,1]\to c_0(L_p[0,1])$

$$U(f)=\left(Average_k\left(\alpha_{ni}r_i\int_0^1 f(t)\,r_{n+i}(t)\,dt\mid 1\leq i\leq n;\,L_p[0,1]\right)\right)_{n\in\mathbb{N}}.$$

Then U is absolutely summing; U is compact if and only if U is nuclear if and only if $\alpha_{\mathrm{mod}}\in c_0(l_\infty^n\mid n\in\mathbb{N})$.

(b) Let $(\alpha_{ni})_{i,n}$ be such that $\alpha\in l_\infty(l_2^n\mid n\in\mathbb{N})$ and $U:C[0,1]\to c_0(L_\infty[0,1])$ the operator defined by

$$U(f)=\left(Average\left(\alpha_{ni}r_i\int_0^1 f(t)\,r_{n+i}(t)\,dt\mid 1\leq i\leq n;\,L_\infty[0,1]\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\alpha\in c_0(l_2^n\mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $\alpha\in l_\infty(l_1^n\mid n\in\mathbb{N})$.

(iv) U is nuclear if and only if $\alpha\in c_0(l_1^n\mid n\in\mathbb{N})$.

(bb) Let $k\geq 2$ be a natural number, $(\alpha_{ni})_{i,n}$ such that $\alpha_{\mathrm{mod}}\in l_\infty(l_\infty^n\mid n\in\mathbb{N})$ and $U:C[0,1]\to c_0(L_\infty[0,1])$

$$U(f)=\left(Average_k\left(\alpha_{ni}r_i\int_0^1 f(t)\,r_{n+i}(t)\,dt\mid 1\leq i\leq n;\,L_\infty[0,1]\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\alpha_{\mathrm{mod}}\in c_0(l_\infty^n\mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $\alpha_{\mathrm{mod}}\in l_\infty(l_2^n\mid n\in\mathbb{N})$.

(iv) U is nuclear if and only if $\alpha_{\mathrm{mod}}\in c_0(l_2^n\mid n\in\mathbb{N})$.

The next example is a natural extension of Example 10(b), (bb). The proof follows from Corollary 7 and Proposition 8(viii). We remark that in the next example, if:

1) $S_n = [0,1]$, $\mu_n = \lambda$ and all $(g_{ni})_{1\le i\le n, n\in\mathbb{N}} \subset L_\infty(\mu_n)$ are continuous, then we must replace $L_\infty(\mu_n,\cdot)$ with $C([0,1],\cdot)$.

2) $S_n = [0,1]$, $\mu_n = \lambda$ and $g_{ni} = \alpha_{ni} r_i$ we get Example 10(b), (bb).

Example 11. Let $(S_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of finite measure spaces,

$$(g_{ni})_{1\le i\le n, n\in\mathbb{N}} \subset L_\infty(\mu_n),\ g_n : \Omega_n \to \mathbb{K}^n,\ g_n = (g_{n1},\dots,g_{nn}).$$

(a) Denote $g = (g_n)_{n\in\mathbb{N}}$ and suppose that $g \in l_\infty(L_\infty(\mu_n, l_2^n) \mid n\in\mathbb{N})$.

Let $U : C[0,1] \to c_0(L_\infty(\mu_n) \mid n\in\mathbb{N})$ be the operator defined by

$$U(f) = \left(Average\left(g_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1\le i\le n;\ L_\infty(\mu_n)\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $g \in c_0(L_\infty(\mu_n, l_2^n) \mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $g \in l_\infty(L_\infty(\mu_n, l_1^n) \mid n\in\mathbb{N})$.

(iv) U is nuclear if and only if $g \in c_0(L_\infty(\mu_n, l_1^n) \mid n\in\mathbb{N})$.

(aa) Let $k \ge 2$ be a natural number and denote $g_{\mathrm{mod}} = (b_{nk} g_n)_{n\in\mathbb{N}}$. Suppose that $g_{\mathrm{mod}} \in l_\infty(L_\infty(\mu_n, l_\infty^n) \mid n\in\mathbb{N})$ and let $U : C[0,1] \to c_0(L_\infty(\mu_n) \mid n\in\mathbb{N})$ be the operator defined by

$$U(f) = \left(Average_k\left(g_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1\le i\le n\right);\ L_\infty(\mu_n)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $g_{\mathrm{mod}} \in c_0(L_\infty(\mu_n, l_\infty^n) \mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $g_{\mathrm{mod}} \in l_\infty(L_\infty(\mu_n, l_2^n) \mid n\in\mathbb{N})$.

(iv) U is nuclear if and only if $g_{\mathrm{mod}} \in c_0(L_\infty(\mu_n, l_2^n) \mid n\in\mathbb{N})$.

The proof of the next example follows from Corollary 7 and Proposition 8(ix).

Example 12. Let $(S_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of finite measure spaces,

$$(g_{ni})_{1\le i\le n, n\in\mathbb{N}} \subset L_1(\mu_n).$$

(a) Denote $\beta_n = \left(\int_S g_{n1} d\mu_n, \dots, \int_S g_{nn} d\mu_n\right)$, $\beta = (\beta_n)_{n\in\mathbb{N}}$, suppose that each g_{ni} takes positive values and $\beta \in l_\infty(l_2^n \mid n\in\mathbb{N})$.

Let $U : C[0,1] \to c_0(L_1(\mu_n) \mid n\in\mathbb{N})$ be the operator defined by

$$U(f) = \left(Average\left(g_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1\le i\le n\right);\ L_1(\mu_n)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta \in c_0(l_2^n \mid n\in\mathbb{N})$.

(iii) U is absolutely summing if and only if $\beta \in l_\infty (l_1^n \mid n \in \mathbb{N})$.

(iv) U is nuclear if and only if $\beta \in c_0 (l_1^n \mid n \in \mathbb{N})$.

(aa) Let $k \geq 2$ be a natural number. Denote

$$\beta_n = \left(\int_S |g_{n1}| \, d\mu_n, \ldots, \int_S |g_{nn}| \, d\mu_n \right),$$

$\beta_{\text{mod}} = (b_{nk}\beta_n)_{n\in\mathbb{N}}$ and suppose that $\beta_{\text{mod}} \in l_\infty (l_\infty^n \mid n \in \mathbb{N})$.

Let $U : C[0,1] \to c_0 (L_1 (\mu_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average_k \left(g_{ni} \int_0^1 f(t) \, r_{n+i}(t) \, dt \mid 1 \leq i \leq n \right); L_1(\mu_n) \right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta_{\text{mod}} \in c_0 (l_\infty^n \mid n \in \mathbb{N})$.

(iii) If, in addition, each g_{ni} takes positive values, U is absolutely summing if and only if $\beta_{\text{mod}} \in l_\infty (l_2^n \mid n \in \mathbb{N})$.

(iv) If, in addition, each g_{ni} takes positive values, U is nuclear if and only if $\beta_{\text{mod}} \in c_0 (l_2^n \mid n \in \mathbb{N})$.

In the rest of the paper, if $1 \leq p < 2$ define r by $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. The proof of the next example follows from Corollary 7 and Proposition 8(iv) and (v).

Example 13. (a) Let $(\alpha_{ni})_{i,n}$ be such that $\alpha \in l_\infty (l_r^n \mid n \in \mathbb{N})$ if $1 \leq p < 2$, or $\alpha \in l_\infty (l_\infty^n \mid n \in \mathbb{N})$ if $2 \leq p$ and $U : C[0,1] \to c_0 \left(l_p^n \mid n \in \mathbb{N}\right)$

$$U(f) = \left(Average \left(\alpha_{ni} e_{ni} \int_0^1 f(t) \, r_{n+i}(t) \, dt \mid 1 \leq i \leq n; \, l_p^n \right) \right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\alpha \in c_0 (l_r^n \mid n \in \mathbb{N})$ for $1 \leq p < 2$, or $\alpha \in c_0 (l_\infty^n \mid n \in \mathbb{N})$ for $2 \leq p$.

(iii) U is absolutely summing if and only if $\alpha \in l_\infty \left(l_p^n \mid n \in \mathbb{N}\right)$.

(iv) U is nuclear if and only if $\alpha \in c_0 \left(l_p^n \mid n \in \mathbb{N}\right)$.

(b) Let $(\alpha_{ni})_{i,n}$ be such that $\alpha \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$ and $U : C[0,1] \to c_0 (l_\infty^n \mid n \in \mathbb{N})$ the operator defined by

$$U(f) = \left(Average \left(\alpha_{ni} e_{ni} \int_0^1 f(t) \, r_{n+i}(t) \, dt \mid 1 \leq i \leq n; \, l_\infty^n \right) \right)_{n\in\mathbb{N}}.$$

Then U is absolutely summing; U is compact if and only if U is nuclear if and only if $\alpha \in c_0 (l_\infty^n \mid n \in \mathbb{N})$.

(c) Let $(\alpha_{nij})_{i,j,n}$ be such that $\beta \in l_\infty (l_\infty^n (l_2^n) \mid n \in \mathbb{N})$ and $U : C[0,1] \to c_0 (l_\infty^n \mid n \in \mathbb{N})$ the operator defined by

$$U(f) = \left(Average \left((\alpha_{ni1} e_{n1} + \cdots + \alpha_{nin} e_{nn}) \int_0^1 f(t) \, r_{n+i}(t) \, dt \mid 1 \leq i \leq n; \, l_\infty^n \right) \right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta \in c_0\left(l_\infty^n\left(l_2^n\right) \mid n \in \mathbb{N}\right)$.

(iii) U is absolutely summing if and only if $\beta \in l_\infty\left(l_\infty^n\left(l_1^n\right) \mid n \in \mathbb{N}\right)$.

(iv) U is nuclear if and only if $\beta \in c_0\left(l_\infty^n\left(l_1^n\right) \mid n \in \mathbb{N}\right)$.

The next example is a natural completion of Example 13. The proof follows from Corollary 7 and Proposition 8(iv) and (v).

Example 14. (a) Let $k \geq 2$ be a natural number, $1 \leq p < \infty$, $(\alpha_{ni})_{i,n}$ such that $\alpha_{\text{mod}} \in l_\infty\left(l_\infty^n \mid n \in \mathbb{N}\right)$ and $U : C[0,1] \to c_0\left(l_p^n \mid n \in \mathbb{N}\right)$

$$U(f) = \left(Average_k\left(\alpha_{ni}e_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n;\, l_p^n\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\alpha_{\text{mod}} \in c_0\left(l_\infty^n \mid n \in \mathbb{N}\right)$.

(iii) for $p \geq 2$, U is absolutely summing; for $1 \leq p < 2$, U is absolutely summing if and only if $\alpha_{\text{mod}} \in l_\infty\left(l_r^n \mid n \in \mathbb{N}\right)$.

(iv) for $p \geq 2$, U is nuclear if and only if U is compact if and only if $\alpha_{\text{mod}} \in c_0\left(l_\infty^n \mid n \in \mathbb{N}\right)$; for $1 \leq p < 2$, U is nuclear if and only if $\alpha_{\text{mod}} \in c_0\left(l_r^n \mid n \in \mathbb{N}\right)$.

(b) Let $k \geq 2$ be a natural number, $(\alpha_{ni})_{i,n}$ such that $\alpha_{\text{mod}} \in l_\infty\left(l_\infty^n \mid n \in \mathbb{N}\right)$ and $U : C[0,1] \to c_0\left(l_\infty^n \mid n \in \mathbb{N}\right)$

$$U(f) = \left(Average_k\left(\alpha_{ni}e_{ni}\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n;\, l_\infty^n\right)\right)_{n\in\mathbb{N}}.$$

Then U is absolutely summing; U is compact if and only if U is nuclear if and only if $\alpha_{\text{mod}} \in c_0\left(l_\infty^n \mid n \in \mathbb{N}\right)$.

(c) Let $k \geq 2$ be a natural number, $(\alpha_{nij})_{i,j,n}$ such that $\beta_{\text{mod}} \in l_\infty(l_\infty^n\left(l_\infty^n\right) \mid n \in \mathbb{N})$ and $U : C[0,1] \to c_0\left(l_\infty^n \mid n \in \mathbb{N}\right)$

$$U(f) = \left(Average_k\left((\alpha_{ni1}e_{n1} + \cdots + \alpha_{nin}e_{nn})\int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n;\, l_\infty^n\right)\right)_{n\in\mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta_{\text{mod}} \in c_0\left(l_\infty^n\left(l_\infty^n\right) \mid n \in \mathbb{N}\right)$.

(iii) U is absolutely summing if and only if $\beta_{\text{mod}} \in l_\infty\left(l_\infty^n\left(l_2^n\right) \mid n \in \mathbb{N}\right)$.

(iv) U is nuclear if and only if $\beta_{\text{mod}} \in c_0\left(l_\infty^n\left(l_2^n\right) \mid n \in \mathbb{N}\right)$.

The Examples 13 and 14 can be extended to a more general situation. For this we recall, that if $1 \leq p \leq \infty$, a Banach space X contains l_p^n's uniformly if and only if **there exists** $\lambda \geq 1$ such that for each $n \in \mathbb{N}$ there exists a bounded linear operator $J : l_p^n \to X$ such that

$$(*) \qquad \|\xi\|_p \leq \|J(\xi)\|_X \leq \lambda \|\xi\|_p,\ \forall \xi \in l_p^n.$$

A deep result of Krivine's, see [7, p. 233], asserts that a Banach space X contains l_p^n's uniformly if and only if **for all** $\lambda > 1$, all $n \in \mathbb{N}$ there exists a bounded linear operator $J : l_p^n \to X$ such that

$$\|\xi\|_p \le \|J(\xi)\|_X \le \lambda \|\xi\|_p, \forall \xi \in l_p^n.$$

For example, from Khinchin's inequality it follows that for each $1 \le p < \infty$, $L_p[0,1]$ contains l_2^n's uniformly, thus Example 10(a), (aa) is a particular case of next example.

From [1, Exercise 8.18(a), p. 107] it follows that $L_\infty[0,1]$ contains l_1^n's uniformly, thus Example 10(b), (bb) is also a particular case of next example.

In the statement of the next example we will use the operator J which occur in $(*)$.

Example 15. (a) Let $1 \le p < \infty$, X be a Banach space which contains l_p^n's uniformly, $(\alpha_{ni})_{i,n}$ such that $\alpha \in l_\infty(l_r^n \mid n \in \mathbb{N})$ if $1 \le p < 2$, or $\alpha \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$ if $2 \le p$. Let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average \left(\alpha_{ni} J(e_{ni}) \int_0^1 f(t) r_{n+i}(t)\, dt \right) \mid 1 \le i \le n \right)_{n \in \mathbb{N}}.$$

Then

(i) U_α is weakly compact.

(ii) U_α is compact if and only if $\alpha \in c_0(l_r^n \mid n \in \mathbb{N})$ if $1 \le p < 2$, or $\alpha \in c_0(l_\infty^n \mid n \in \mathbb{N})$ if $2 \le p$.

(iii) U_α is absolutely summing if and only if $\alpha \in l_\infty\left(l_p^n \mid n \in \mathbb{N}\right)$.

(iv) U_α is nuclear if and only if $\alpha \in c_0\left(l_p^n \mid n \in \mathbb{N}\right)$.

(b) Let X be a Banach space which contains l_∞^n's uniformly, $(\alpha_{ni})_{i,n}$ such that $\alpha \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$. Let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average \left(\alpha_{ni} J(e_{ni}) \int_0^1 f(t) r_{n+i}(t)\, dt \right) \mid 1 \le i \le n \right)_{n \in \mathbb{N}}.$$

Then

(i) U_α is absolutely summing.

(ii) U_α is compact if and only if U_α is nuclear if and only if $\alpha \in c_0(l_\infty^n \mid n \in \mathbb{N})$.

(c) Let X be a Banach space which contains l_∞^n's uniformly, $(\alpha_{nij})_{i,j,n}$ such that $\beta \in l_\infty(l_\infty^n(l_2^n) \mid n \in \mathbb{N})$ and for each natural number n denote

$$x_{ni} = \alpha_{ni1} J(e_{n1}) + \alpha_{ni2} J(e_{n2}) + \cdots + \alpha_{nin} J(e_{nn}).$$

Let $U_\beta : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\beta(f) = \left(Average \left(x_{ni} \int_0^1 f(t) r_{n+i}(t)\, dt) \mid 1 \le i \le n \right) \right)_{n \in \mathbb{N}}.$$

Then

(i) U_β is weakly compact.

(ii) U_β is compact if and only if $\beta \in c_0(l_\infty^n(l_2^n) \mid n \in \mathbb{N})$.

(iii) U_β is absolutely summing if and only if $\beta \in l_\infty(l_\infty^n(l_1^n) \mid n \in \mathbb{N})$.

(iv) U_β is nuclear if and only if $\beta \in c_0 \left(l_\infty^n \left(l_1^n \right) \mid n \in \mathbb{N} \right)$.

(aa) Let $k \geq 2$ be a natural number, $1 \leq p < \infty$, X a Banach space which contains l_p^n's uniformly, $\left(\alpha_{ni} \right)_{i,n}$ such that $\alpha_{\mathrm{mod}} \in l_\infty \left(l_\infty^n \mid n \in \mathbb{N} \right)$.

Let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average_k \left(\alpha_{ni} J\left(e_{ni}\right) \int_0^1 f(t)\, r_{n+i}(t)\, dt \right) \mid 1 \leq i \leq n \right)_{n \in \mathbb{N}}.$$

Then

(i) U_α is weakly compact.

(ii) U_α is compact if and only if $\alpha_{\mathrm{mod}} \in c_0 \left(l_\infty^n \mid n \in \mathbb{N} \right)$.

(iii) for $p \geq 2$, U_α is absolutely summing; for $1 \leq p < 2$, U is absolutely summing if and only if $\alpha_{\mathrm{mod}} \in l_\infty \left(l_r^n \mid n \in \mathbb{N} \right)$.

(iv) for $p \geq 2$, U_α is nuclear if and only if U_α is compact if and only if $\alpha_{\mathrm{mod}} \in c_0 \left(l_\infty^n \mid n \in \mathbb{N} \right)$; for $1 \leq p < 2$, U_α is nuclear if and only if $\alpha_{\mathrm{mod}} \in c_0 \left(l_r^n \mid n \in \mathbb{N} \right)$.

(bb) Let $k \geq 2$ be a natural number, X a Banach space which contains l_∞^n's uniformly, $\left(\alpha_{ni} \right)_{i,n}$ such that $\alpha_{\mathrm{mod}} \in l_\infty \left(l_\infty^n \mid n \in \mathbb{N} \right)$. Let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average_k \left(\alpha_{ni} J\left(e_{ni}\right) \int_0^1 f(t)\, r_{n+i}(t)\, dt \right) \mid 1 \leq i \leq n \right)_{n \in \mathbb{N}}.$$

Then

(i) U_α is absolutely summing.

(ii) U_α is compact if and only if U_α is nuclear if and only if $\alpha_{\mathrm{mod}} \in c_0(l_\infty^n \mid n \in \mathbb{N})$.

(cc) Let $k \geq 2$ be a natural number, X a Banach space which contains l_∞^n's uniformly, $\left(\alpha_{nij} \right)_{i,j,n}$ such that $\beta_{\mathrm{mod}} \in l_\infty \left(l_\infty^n \left(l_\infty^n \right) \mid n \in \mathbb{N} \right)$ and for each natural number n denote

$$x_{ni} = \alpha_{ni1} J\left(e_{n1}\right) + \alpha_{ni2} J\left(e_{n2}\right) + \cdots + \alpha_{nin} J\left(e_{nn}\right).$$

Let $U_\beta : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\beta(f) = \left(Average_k \left(x_{ni} \int_0^1 f(t)\, r_{n+i}(t)\, dt \right) \mid 1 \leq i \leq n \right)_{n \in \mathbb{N}}.$$

Then

(i) U_β is weakly compact.

(ii) U_β is compact if and only if $\beta_{\mathrm{mod}} \in c_0 \left(l_\infty^n \left(l_\infty^n \right) \mid n \in \mathbb{N} \right)$.

(iii) U_β is absolutely summing if and only if $\beta_{\mathrm{mod}} \in l_\infty \left(l_\infty^n \left(l_2^n \right) \mid n \in \mathbb{N} \right)$.

(iv) U_β is nuclear if and only if $\beta_{\mathrm{mod}} \in c_0 \left(l_\infty^n \left(l_2^n \right) \mid n \in \mathbb{N} \right)$.

Proof. (a) From Corollary 7(a), U_α is weakly compact and further U_α is compact if and only if $w_2 \left(\alpha_{ni} J\left(e_{ni}\right) \mid 1 \leq i \leq n \right) \to 0$;

U_α is absolutely summing if and only if $\sup\limits_{n \in \mathbb{N}} w_1 \left(\alpha_{ni} J\left(e_{ni}\right) \mid 1 \leq i \leq n \right) < \infty$;

U_α is nuclear if and only if $w_1 \left(\alpha_{ni} J\left(e_{ni}\right) \mid 1 \leq i \leq n \right) \to 0$.

From $(*)$ and Proposition 8(iii) and (iv) the statement follows.

(aa) The proof is similar to that of (a) and use Corollary 7(b). We omit the details. The proofs of (b), (bb) and (c), (cc) are also similar to that of (a). We prove now (c).

From Corollary 7(a) U_β is weakly compact and further

U_β is compact if and only if $w_2(x_{ni} \mid 1 \le i \le n) \to 0$;

U_β is absolutely summing if and only if $\sup_{n\in\mathbb{N}} w_1(x_{ni} \mid 1 \le i \le n) < \infty$;

U_β is nuclear if and only if $w_1(x_{ni} \mid 1 \le i \le n) \to 0$.

From $(*)$ an Proposition 8(iii) and (v) we get the statement.

Since, by the famous Dvoretzky theorem, see [2, Chapter 19], each infinite dimensional Banach space contains l_2^n's uniformly, i.e., $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$ there exists a bounded linear operator $J : l_2^n \to X$ such that

$$(**) \qquad \|\xi\|_2 \le \|J(\xi)\|_X \le (1+\varepsilon)\|\xi\|_2, \forall \xi \in l_2^n$$

from Example 15(a), (aa) we get the next example; we use in the statement of this example the operator J from $(**)$. □

Example 16. Let X be an infinite dimensional Banach space and $(\alpha_{ni})_{i,n}$.

(a) If $\alpha \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$, let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average\left(\alpha_{ni} J(e_{ni}) \int_0^1 f(t) r_{n+i}(t)\, dt \right) \mid 1 \le i \le n \right)_{n\in\mathbb{N}}.$$

Then

(i) U_α is weakly compact.

(ii) U_α is compact if and only if $\alpha \in c_0(l_\infty^n \mid n \in \mathbb{N})$.

(iii) U_α is absolutely summing if and only if $\alpha \in l_\infty(l_2^n \mid n \in \mathbb{N})$.

(iv) U_α is nuclear if and only if $\alpha \in c_0(l_2^n \mid n \in \mathbb{N})$.

(b) If $k \ge 2$ is a natural number and $\alpha_{\mathrm{mod}} \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$, let $U_\alpha : C[0,1] \to c_0(X)$ be the operator defined by

$$U_\alpha(f) = \left(Average_k\left(\alpha_{ni} J(e_{ni}) \int_0^1 f(t) r_{n+i}(t)\, dt \right) \mid 1 \le i \le n \right)_{n\in\mathbb{N}}.$$

Then

(i) U_α is absolutely summing.

(ii) U_α is nuclear if and only if U_α is compact if and only if $\alpha_{\mathrm{mod}} \in c_0(l_\infty^n \mid n \in \mathbb{N})$.

The proof of the next example follows from Corollary 7 and Proposition 8(x), (xi).

Example 17. Let $(S_n, \Xi_n, \nu_n)_{n\in\mathbb{N}}$ be a sequence of finite measure spaces.

(a) Let $(g_{ni})_{1\le i\le n} \subset L_1(\nu_n)$, $g_n = (g_{n1}, \dots, g_{nn}) : S_n \to \mathbb{K}^n$ be such that

$$\sup_{n\in\mathbb{N}} \sup_{\beta\in l_2^n, \|\beta\|_2 \le 1} \int_{S_n} \|M_\beta(g_n(s_n))\|_2 \, d\nu_n(s_n) < \infty.$$

Let $U : C[0,1] \to c_0(L_1(\lambda \otimes \nu_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average \left(r_i g_{ni} \int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n \right); L_1(\lambda \otimes \nu_n) \right)_{n \in \mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if

$$\sup_{\beta \in l_2^n, \|\beta\|_2 \leq 1} \int_{S_n} \|M_\beta(g_n(s_n))\|_2\, d\nu_n(s_n) \to 0.$$

(iii) U is absolutely summing if and only if $\sup\limits_{n \in \mathbb{N}} \int_{S_n} \|g_n(s_n)\|_2\, d\nu_n(s_n) < \infty$.

(iv) U is nuclear if and only if $\int_{S_n} \|g_n(s_n)\|_2\, d\nu_n(s_n) \to 0$.

(b) Let $(\alpha_{ni})_{i,n} \subset \mathbb{K}$, $(E_{ni})_{1 \leq i \leq n} \subset \Sigma_n$ be a partition for S_n,

$$\beta_n = (\alpha_{n1}\nu_n(E_{n1}), \ldots, \alpha_{nn}\nu_n(E_{nn})),\ \beta = (\beta_n)_{n \in \mathbb{N}}$$

such that $\beta \in l_\infty(l_2^n \mid n \in \mathbb{N})$.

Let $U : C[0,1] \to c_0(L_1(\lambda \otimes \nu_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average \left(\alpha_{ni} r_i \chi_{E_{ni}} \int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n \right); L_1(\lambda \otimes \nu_n) \right)_{n \in \mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta \in c_0(l_2^n \mid n \in \mathbb{N})$.

(iii) U is absolutely summing if and only if $\beta \in l_\infty(l_1^n \mid n \in \mathbb{N})$.

(iv) U is nuclear if and only if $\beta \in c_0(l_1^n \mid n \in \mathbb{N})$.

(bb) Let $k \geq 2$ be a natural number, $(\alpha_{ni})_{i,n} \subset \mathbb{K}$, $(E_{ni})_{1 \leq i \leq n} \subset \Sigma_n$ be a partition for S_n, $\beta_n = (\alpha_{n1}\nu_n(E_{n1}), \ldots, \alpha_{nn}\nu_n(E_{nn}))$, $\beta_{\mathrm{mod}} = (b_{nk}\beta_n)_{n \in \mathbb{N}}$ such that $\beta_{\mathrm{mod}} \in l_\infty(l_\infty^n \mid n \in \mathbb{N})$.

Let $U : C[0,1] \to c_0(L_1(\mu \otimes \nu_n) \mid n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(Average_k \left(\alpha_{ni} r_i \chi_{E_{ni}} \int_0^1 f(t)\, r_{n+i}(t)\, dt \mid 1 \leq i \leq n \right); L_1(\lambda \otimes \nu_n) \right)_{n \in \mathbb{N}}.$$

Then

(i) U is weakly compact.

(ii) U is compact if and only if $\beta_{\mathrm{mod}} \in c_0(l_\infty^n \mid n \in \mathbb{N})$.

(iii) U is absolutely summing if and only if $\beta_{\mathrm{mod}} \in l_\infty(l_2^n \mid n \in \mathbb{N})$.

(iv) U is nuclear if and only if $\beta_{\mathrm{mod}} \in c_0(l_2^n \mid n \in \mathbb{N})$.

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References

[1] A. Defant and K. Floret, *Tensor Norms and Operators Ideals*, North-Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993.
[2] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.
[3] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
[4] L. Drewnowski and Z. Lipecki, *On vector measures which have everywhere infinite variation or noncompact range*, Dissertationes Math. (Rozprawy Mat.) **339** (1995), 39 pp.
[5] R. Latała and K. Oleszkiewicz, *On the best constant in the Khinchin-Kahane inequality*, Studia Math. **109** (1994), no. 1, 101–104.
[6] A. Pietsch, *Eigenvalues and s-numbers*, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987.
[7] A. Pietsch and J. Wenzel, *Orthonormal Systems and Banach Space Geometry*, Encyclopedia of Mathematics and its Applications, 70. Cambridge University Press, Cambridge, 1998.
[8] D. Popa, *Examples of operators on* $C[0,1]$ *distinguishing certain operator ideals*, Arch. Math. (Basel) **88** (2007), no. 4, 349–357.
[9] N. Tomczak-Jagermann, *Banach-Mazur distances and finite dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, Inc., 1989.

Department of Mathematics
University of Constanta
Bd. Mamaia 124, 8700 Constanta, Romania
E-mail address: dpopa@univ-ovidius.ro