J. Korean Math. Soc. **47** (2010), No. 5, pp. 879–897
DOI 10.4134/JKMS.2010.47.5.879

SKEW POLYNOMIAL RINGS OVER SEMIPRIME RINGS

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Abstract. Y. Hirano introduced the concept of a quasi-Armendariz ring which extends both Armendariz rings and semiprime rings. A ring R is called *quasi-Armendariz* if $a_iRb_j = 0$ for each i, j whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. In this paper, we first extend the quasi-Armendariz property of semiprime rings to the skew polynomial rings, that is, we show that if R is a semiprime ring with an epimorphism σ, then $f(x)R[x;\sigma]g(x) = 0$ implies $a_iR\sigma^{i+k}(b_j) = 0$ for any integer $k \geq 0$ and i, j, where $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\sigma]$. Moreover, we extend this property to the skew monoid rings, the Ore extensions of several types, and skew power series ring, etc. Next we define σ-skew quasi-Armendariz rings for an endomorphism σ of a ring R. Then we study several extensions of σ-skew quasi-Armendariz rings which extend known results for quasi-Armendariz rings and σ-skew Armendariz rings.

Throughout this paper R denotes an associative ring with identity. We denote by $R[x]$ the polynomial ring with an indeterminate x over R. Rege and Chhawchharia [18] introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz ring" was chosen from the fact that Armendariz [2, Lemma 1] had showed that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Many properties of Armendariz rings have been studied by several authors [1, 8, 10, 11, 12]. Hirano [5] introduced a quasi-Armendariz ring which is generalizing an Armendariz ring. A ring R is called *quasi-Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each i, j. Hirano [5, Corollary 3.8] proved that semiprime rings are quasi-Armendariz rings. Moreover, he showed that the class of quasi-Armendariz rings is Morita stable [4, Theorem 3.12 and Proposition 3.13], and that if R is a quasi-Armendariz ring, then some extensions of R (e.g., the n-by-n upper triangular matrix ring, the polynomial ring) are also quasi-Armendariz rings. But most of these properties are not stable in Armendariz rings (for example, [10, Examples 1 and 3, etc.]).

Received June 18, 2008; Revised August 26, 2008.
2000 *Mathematics Subject Classification.* 16N60, 16S36.
Key words and phrases. semiprime ring, quasi-Armendariz ring, skew polynomial ring.

For a ring R with a ring endomorphism σ and an σ-derivation δ, the *Ore extension* $R[x;\sigma,\delta]$ of R is the ring of polynomials in x over R with usual addition and with multiplication subject to the rule $xa=\sigma(a)x+\delta(a)$ for any $a\in R$. If $\delta=0$, then $R[x;\sigma,\delta]=R[x;\sigma]$ is called the *skew polynomial ring.*

On the other hand, Hong, Kim, and Kwak [6] introduced σ-skew Armendariz for an endomorphism σ of a ring R. A ring R is called a *σ-skew Armendariz* if for $f(x)=\sum_{i=0}^m a_ix^i$ and $g(x)=\sum_{j=0}^n b_jx^j$ in $R[x;\sigma]$, $f(x)g(x)=0$ implies $a_i\sigma^i(b_j)=0$ for all $0\le i\le m$, and $0\le j\le n$. They proved that σ-rigid rings are σ-skew Armendariz, where a ring R is *σ-rigid* if for an endomorphism σ of R, $a\sigma(a)=0$ implies $a=0$. It can be easily shown that σ-rigid rings are reduced. But by [6, Example 2], reduced rings are not σ-skew Armendariz in general, even if σ is an automorphism of R. We also can find more results for skew Armendariz rings in [3, 14].

Even though reduced rings are not σ-skew Armendariz, in Section 1, we show that if R is a semiprime ring with an epimorphism σ, then $f(x)R[x;\sigma]g(x)=0$ implies $a_iR\sigma^{i+k}(b_j)=0$ for any integer $k\ge 0$ and i,j, where $f(x)=\sum_{i=0}^m a_ix^i, g(x)=\sum_{j=0}^n b_jx^j\in R[x;\sigma]$. Moreover, we extend the quasi-Armendariz property of semiprime rings to the skew monoid rings, the Ore extensions of several types, and skew power series ring, etc.

Based on results in Section 1, we define σ-skew quasi-Armendariz rings for an endomorphism σ of a ring R in Section 2. Then we study several extensions of σ-skew quasi-Armendariz rings which extend known results for quasi-Armendariz rings and σ-skew Armendariz rings.

1. Polynomial extensions of semiprime rings

Recall that a monoid G is called a *unique product monoid* (simply, *u.p.-monoid*) if for any two nonempty finite subsets $A,B\subseteq G$ there exists $c\in G$ uniquely presented in the form ab where $a\in A$ and $b\in B$. The class of u.p.-monoids is quite large and important (see [15] and [16] for details). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups.

Let R be a ring and G a u.p.-monoid. Assume that there is a monoid homomorphism into the epimorphism monoid of R via the acting of G on R. We denote by $\sigma_g(r)$ the image of $r\in R$ under $g\in G$. The skew monoid ring $R*G$ is a ring which as a left R-module is free with basis G and multiplication defined by the rule $gr=\sigma_g(r)g$.

Theorem 1.1. *Let R be a semiprime ring and G a u.p.-monoid. Then $(a_0g_0+\cdots+a_mg_m)R*G(b_0h_0+\cdots+b_nh_n)=0$ with $a_i,b_j\in R$, $g_i,h_j\in G$ if and only if $a_iR\sigma_{g_i}(\sigma_g(b_j))=0$ for any $g\in G$ and $0\le i\le m$ and $0\le j\le n$.*

Proof. Suppose that $(a_0g_0+\cdots+a_mg_m)R*G(b_0h_0+\cdots+b_nh_n)=0$ with $a_i,b_j\in R$, $g_i,h_j\in G$. Then for any $r\in R$ and $g\in G$, we have the following equation:

$$(*) \qquad (a_0g_0 + \cdots + a_mg_m)gr(b_0h_0 + \cdots + b_nh_n) = 0.$$

We will show that $a_iR\sigma_{g_i}(\sigma_g(b_j)) = 0$ for any $g \in G$ and $0 \le i \le m$ and $0 \le j \le n$ by using induction on m. If $m = 0$, then

$$\begin{aligned} 0 &= (a_0g_0)gr(b_0h_0 + \cdots + b_nh_n) \\ &= a_0\sigma_{g_0}(\sigma_g(rb_0))g_0gh_0 + \cdots + a_0\sigma_{g_0}(\sigma_g(rb_n))g_0gh_n. \end{aligned}$$

By [15, Lemma 1, p.119], $g_igh_u \ne g_0gh_v$ if $u \ne v$. Thus $a_0\sigma_{g_0}(\sigma_g(rb_j)) = 0$ for all $0 \le j \le n$ and hence $a_0R\sigma_{g_0}(\sigma_g(b_j)) = 0$ since $\sigma_{g_0} \cdot \sigma_g$ is surjective. Suppose that $m \ge 1$. Since G is a u.p.-monoid, there exist p, q such that g_pgh_q is uniquely presented by considering two subsets $A = \{g_0g, g_1g, \ldots, g_mg\}$ and $B = \{h_0, h_1, \ldots, h_n\}$ of G. After reordering if necessary, we may assume that $p = 0$ and $q = 0$. Then from Eq.$(*)$, we have $a_0\sigma_{g_0}(\sigma_g(rb_0)) = 0$. Moreover, since $\sigma_{g_0} \cdot \sigma_g$ is surjective, $a_0R\sigma_{g_0}(\sigma_g(b_0)) = 0$. Thus for any $s \in R$, we have

$$\begin{aligned} 0 &= (a_0g_0 + \cdots + a_mg_m)grb_0s(b_0h_0 + \cdots + b_nh_n) \\ &= (a_1g_1 + \cdots + a_mg_m)gr(b_0sb_0h_0 + \cdots + b_0sb_nh_n). \end{aligned}$$

By the induction hypothesis, $a_i\sigma_{g_i}(\sigma_g(rb_0sb_j)) = 0$ for any $1 \le i \le m$ and $0 \le j \le n$. Then

$$0 = a_i\sigma_{g_i}(\sigma_g(rb_0sb_0)) = a_i\sigma_{g_i}(\sigma_g(r))\sigma_{g_i}(\sigma_g(b_0))\sigma_{g_i}(\sigma_g(s))\sigma_{g_i}(\sigma_g(b_0)).$$

Since $\sigma_{g_i} \cdot \sigma_g$ is surjective for any $1 \le i \le m$, $a_iR\sigma_{g_i}(\sigma_g(b_0))R\sigma_{g_i}(\sigma_g(b_0)) = 0$. Since R is semiprime, $a_iR\sigma_{g_i}(\sigma_g(b_0)) = 0$ for any $1 \le i \le m$. Consequently, we have $a_iR\sigma_{g_i}(\sigma_g(b_0)) = 0$ for any $0 \le i \le m$. Thus Eq.$(*)$ becomes

$$(a_0g_0 + \cdots + a_mg_m)gr(b_1h_1 + \cdots + b_nh_n) = 0.$$

Continuing the process as above, we can get

$$a_i\sigma_{g_i}(\sigma_g(rb_j)) = 0,$$

and so

$$a_iR\sigma_{g_i}(\sigma_g(b_j)) = 0$$

for any $g \in G$ and $0 \le i \le m$ and $0 \le j \le n$. □

A skew (Laurent) polynomial ring $R[x;\sigma]$ ($R[x, x^{-1};\sigma]$) with an epimorphism (an automorphism) σ over R is a skew monoid ring $R * G$ with $G = \{1, x, x^2, \ldots\}$ ($G = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$) and $\sigma_x(r) = \sigma(r)$ for $r \in R$. We denote by $\mathbb{Z}$ the ring of integers.

Corollary 1.2. *Let R be a semiprime ring with an epimorphism σ and $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\sigma]$. Then $f(x)R[x;\sigma]g(x) = 0$ if and only if $a_iR\sigma^{i+k}(b_j) = 0$ for any integer $k \ge 0$, $0 \le i \le m$ and $0 \le j \le n$.*

Corollary 1.3. *Let R be a semiprime ring with an automorphism σ. Then for $f(x) = \sum_{i=m}^{n} a_i x^i, g(x) = \sum_{j=s}^{l} b_j x^j \in R[x, x^{-1}; \sigma]$, where $n, m, s, l \in \mathbb{Z}$, $f(x)R[x, x^{-1}; \sigma]g(x) = 0$ if and only if $a_i R\sigma^{i+t}(b_j) = 0$ for any i, j and integer t.*

From Corollary 1.2, we may conjecture that the condition "σ is an epimorphism of R" can be replaced by "σ is a monomorphism of R". But the following example erases the possibility.

Example 1.4. We refer the example of [13, Example 3.7]. Let R be a subset of $\mathbb{N} \times \mathbb{N}$ matrices over a field K defined as follows

$$R = \{M \mid M = \sum_{i,j=1}^{n} a_{ij}e_{ij} + a \sum_{i=n+1}^{\infty} e_{ii} \text{ for some } n \in \mathbb{N} \text{ and } a_{ij}, a \in K\},$$

where $\{e_{ij}\}_{i,j \in \mathbb{N}}$ denotes the set of matrix units. Then R is a prime ring. The map $\sigma : R \to R$ defined by

$$\sigma(\sum_{i,j=1}^{n} a_{ij}e_{ij} + a \sum_{i=n+1}^{\infty} e_{ii}) = ae_{11} + \sum_{i,j=1}^{n} a_{ij}e_{(i+1)(j+1)} + a \sum_{i=n+2}^{\infty} e_{ii}$$

is a monomorphism of R. Note that $e_{11}\sigma(R) = Ke_{11}$. Therefore, for any integer $t \geq 0$, we have $e_{11}xRx^t e_{11} = Ke_{11}e_{(2+t)(2+t)}x^{t+1} = 0$, and so $e_{11}xR[x; \sigma]e_{11} = 0$. But $e_{11}R\sigma(e_{11}) \neq 0$.

However, we have the following on a reduced ring (i.e., a ring has no nonzero nilpotent elements) with an endomorphism.

Remark 1. Let R be a reduced ring with an endomorphism σ and $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$. Then $f(x)R[x; \sigma]g(x) = 0$ if and only if $a_i R\sigma^{i+t}(b_j) = 0$ for any integer $t \geq 0$ and $0 \leq i \leq m$, $0 \leq j \leq n$.

Proof. Suppose that $f(x)R[x; \sigma]g(x) = 0$, where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x; \sigma]$. Equivalently, for any $r \in R$ and integer $t \geq 0$,

$$(1) \qquad (a_0 + a_1 x + \cdots + a_m x^m)x^t r(b_0 + b_1 x + \cdots + b_n x^n) = 0.$$

We claim that $a_i R\sigma^{i+t}(b_j) = 0$ for any $0 \leq i \leq m$, $0 \leq j \leq n$. We proceed by induction on $i + j$. If $i + j = 0$, then $a_0\sigma^t(b_0) = 0$ and so $a_0 R\sigma^t(b_0) = 0$ since R is reduced. Suppose that our claim is true for $i + j = k - 1$, where $1 \leq k \leq m + n$. This implies that $a_i R\sigma^{i+t}(b_j) = 0$ for $i + j = 0, 1, \ldots, k - 1$. Then we have

$$(2) \qquad a_0\sigma^t(rb_k) + a_1\sigma^{1+t}(rb_{k-1}) + \cdots + a_k\sigma^{k+t}(rb_0) = 0.$$

We first replace r by b_0 in Eq.(2). Then from Eq.(2), $0 = a_0\sigma^t(b_0 b_k) + a_1\sigma^{1+t}(b_0 b_{k-1}) + \cdots + a_k\sigma^{k+t}(b_0 b_0) = a_k\sigma^{k+t}(b_0 b_0)$. Thus $a_k\sigma^{k+t}(b_0)\sigma^{k+t}(b_0) = 0$. Since R is reduced, $a_k\sigma^{k+t}(b_0) = 0$ and moreover $a_k R\sigma^{k+t}(b_0) = 0$. Thus Eq.(2) becomes

$$(3) \qquad a_0\sigma^t(rb_k) + a_1\sigma^{1+t}(rb_{k-1}) + \cdots + a_{k-1}\sigma^{k-1+t}(rb_1) = 0.$$

We next replace r by b_1 in Eq.(3). Then from Eq.(3), we have $a_{k-1}\sigma^{k-1+t}(b_1b_1) = 0$ and so $a_{k-1}R\sigma^{k-1+t}(b_1) = 0$ by the same method as above. Continuing this process, we have $a_iR\sigma^{i+t}(b_j) = 0$ for any $i+j = k$. Consequently we have $a_iR\sigma^{i+t}(b_j) = 0$ for any integer $t \geq 0$ and $0 \leq i \leq m$, $0 \leq j \leq n$. □

Now we extend Corollary 1.2 and Remark 1 to the Ore extension $R[x;\sigma,\delta]$ over a semiprime ring R.

Lemma 1.5. *Let R be a semiprime ring and consider $R[x;\sigma,\delta]$ with an automorphism σ and σ-derivation δ over R. Then we have the following assertions*:

(1) *If $aR\sigma^n(b) = 0$ for some $a, b \in R$ and all integer $n \geq 0$, then $aR\delta^m(b) = 0$ for all integer $m \geq 0$.*

(2) *If $aR\sigma^n(b) = 0$ for some $a, b \in R$ and all integer $n \geq 0$, then $aR\sigma^{n_1}\delta^{m_1} \cdots \sigma^{n_t}\delta^{m_t}(b) = 0$ for all integers $m_i, n_j \geq 0$.*

Proof. (1) Suppose that $aR\sigma^n(b) = 0$ for some $a, b \in R$ and all integer $n \geq 0$. We will proceed by induction on m to show $aR\delta^m(b) = 0$ for all integer $m \geq 0$. For $m = 0$, it is trivial. We now suppose $m \geq 1$. Since σ is an automorphism of R, $a = \sigma(a')$ for some $a' \in R$ and so $a'R\sigma^n(b) = 0$ for all $n \geq 0$ from $\sigma(a'R\sigma^n(b)) = aR\sigma^{n+1}(b) = 0$. Thus we obtain $a'R\delta^{m-1}(b) = 0$ by induction hypothesis. From $\delta(a'R\delta^{m-1}(b)) = 0$, we have $\sigma(a')R\delta^m(b) = -\delta(a'R)\delta^{m-1}(b)$. Note that by the induction hypothesis, $aR\delta^{m-1}(b) = 0$ and so $\delta^{m-1}(b)Ra = 0$ since R is semiprime. Then $(aR\delta^m(b)R)^2 = \sigma(a')R\delta^m(b)RaR\delta^m(b)R = -\delta(a'R)(\delta^{m-1}(b)Ra)R\delta^n(b)R = 0$. Since R is semiprime, $aR\delta^m(b) = 0$.

(2) Suppose that $aR\sigma^n(b) = 0$ for some $a, b \in R$ and all integer $n \geq 0$. Equivalently, $aR\sigma^i(\sigma^{n_t}(b)) = 0$ for all integers $i, n_t \geq 0$. Then by (1), we have $aR\delta^{m_{t-1}}(\sigma^{n_t}(b)) = 0$ for all $m_{t-1} \geq 0$. Moreover, since $a'R\sigma^n(b) = 0$ for all $n \geq 0$ as in the proof of (1), $a'R\delta^m(b) = 0$ by (1) and so $aR\sigma(\delta^m(b)) = 0$ for all $m \geq 0$, where $\sigma(a') = a$. Also since $a''R\sigma^n(b) = 0$ for all $n \geq 0$ similarly (where $\sigma^2(a'') = a$), $a''R\delta^m(b) = 0$ by (1) and so $aR\sigma^2(\delta^m(b)) = 0$ for all $m \geq 0$. Continuing this process, we have $aR\sigma^{n_1}\delta^{m_1} \cdots \sigma^{n_t}\delta^{m_t}(b) = 0$ for all integers $m_i, n_j \geq 0$. □

Theorem 1.6. *Let R be a semiprime ring with an automorphism σ of finite order. Then for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\sigma,\delta]$,*

$$f(x)R[x;\sigma,\delta]g(x) = 0 \text{ if and only if } a_iR\sigma^{n_1}\delta^{m_1} \cdots \sigma^{n_t}\delta^{m_t}(b_j) = 0$$

for all integers $m_u, n_v \geq 0$ and $0 \leq i \leq m$, $0 \leq j \leq n$.

Proof. It is enough to show the necessity. Suppose that $f(x)R[x;\sigma,\delta]g(x) = 0$. Then for any $r \in R$ and integer $t \geq 0$, we have

$$(*) \qquad (a_0 + a_1x + \cdots + a_mx^m)rx^t(b_0 + b_1x + \cdots + b_nx^n) = 0.$$

By Lemma 1.5, it suffices to show that $a_iR\sigma^l(b_j) = 0$ for any integer $l \geq 0$ and $0 \leq i \leq m$, $0 \leq j \leq n$. We proceed by induction on $i+j$. If $i+j = 0$, then $a_0rx^tb_0 = 0$ and so $a_0R\sigma^t(b_0) = 0$ for any integer $t \geq 0$. Suppose that $i+j \geq 1$. From Eq.$(*)$, we have $a_m\sigma^m(r)\sigma^{m+t}(b_n) = 0$. Since σ has a finite order, $a_mR\sigma^l(b_n) = 0$ for any integer $l \geq 0$. Hence from $f(x)R[x;\sigma,\delta]g(x) = 0$, for any $r, s \in R$, we have

$$\begin{aligned} 0 &= (a_0 + \cdots + a_mx^m)rx^t\sigma^{-(m+t)}(a_m)s(b_0 + \cdots + b_nx^n) \\ &= (a_0 + \cdots + a_mx^m)rx^t(\sigma^{-(m+t)}(a_m)sb_0 + \cdots + \sigma^{-(m+t)}(a_m)sb_{n-1}x^{n-1}). \end{aligned}$$

Then $a_m\sigma^m(r)\sigma^{m+t}(\sigma^{-(m+t)}(a_m)sb_{n-1}) = 0$ and so $a_mRa_mR\sigma^{m+t}(b_{n-1}) = 0$. Since R is semiprime, we have $a_mR\sigma^{m+t}(b_{n-1}) = 0$ and hence $a_mR\sigma^l(b_{n-1}) = 0$ for any integer $l \geq 0$. Continuing this process, we have $a_mR\sigma^l(b_j) = 0$ for any integer $l \geq 0$ and $0 \leq j \leq n$. Thus by Lemma 1.5, Eq.(∗) becomes

$$(a_0 + a_1x + \cdots + a_{m-1}x^{m-1})rx^t(b_0 + b_1x + \cdots + b_nx^n) = 0.$$

By the induction hypothesis, we have $a_iR\sigma^l(b_j) = 0$ for any integer $l \geq 0$, $0 \leq i \leq m-1$ and $0 \leq j \leq n$. In the above, $a_mR\sigma^l(b_j) = 0$ for any integer $l \geq 0$ and $0 \leq j \leq n$. Therefore $a_iR\sigma^l(b_j) = 0$ for any integer $l \geq 0$, $0 \leq i \leq m$ and $0 \leq j \leq n$. □

Corollary 1.7. *Let R be a semiprime ring. Then $f(x)R[x;\delta]g(x) = 0$ for $f(x) = \sum_{i=0}^m, g(x) = \sum_{j=0}^n \in R[x;\delta]$ if and only if $a_iR\delta^l(b_j) = 0$ for any integer $l \geq 0$, $0 \leq i \leq m$ and $0 \leq j \leq n$.*

The following example shows that the condition "σ has a finite order" is essential in Theorem 1.6.

Example 1.8. We refer the example of [9, Example 4.3]. Let F be a field and $F_i = F$ for $i \in \mathbb{Z}$. Let R be a F-subalgebra of $\prod_{i\in\mathbb{Z}} F_i$ generated by $\oplus_{i\in\mathbb{Z}}F_i$ and $1_{\prod_{i\in\mathbb{Z}} F_i}$. Then

$$R = \{(a_i) \in \prod_{i\in\mathbb{Z}} F_i \mid a_i \text{ is eventually constant}\}.$$

Let σ be an automorphism of R defined by $\sigma((a_i)) = (a_{i+1})$. Then σ does not have a finite order. Let $e_1 = (a_i) \in R$ with $a_1 = 1$ and $a_i = 0$ for all $i \neq 1$. Then $e_1xR[x;\sigma]e_1x = 0$, but $e_1Re_1 \neq 0$. In spite of this fact, since R is semiprime, by Corollary 1.2, $f(x)R[x;\sigma]g(x) = 0$ if and only if $a_iR\sigma^{i+k}\sigma(b_j) = 0$ for any integer $t \geq 0$ and i, j, where $f(x) = \sum_{i=0}^m a_ix^i, g(x) = \sum_{j=0}^n b_jx^j \in R[x;\sigma]$.

For skew power series rings, we already obtained the following result using a similar method as in the proof of Remark 1.

Remark 2 ([7, Lemma 4]). Let R be a semiprime ring with an epimorphism σ. Then for $f(x) = \sum_{i=0}^\infty a_ix^i, g(x) = \sum_{j=0}^\infty b_jx^j \in R[[x;\sigma]]$, $f(x)R[[x;\sigma]]g(x) = 0$ if and only if $a_iR\sigma^{i+t}(b_j) = 0$ for all $t, i, j \geq 0$.

2. Skew quasi-Armendariz rings

Based on Corollary 1.2, σ-skew Armendariz rings in [6] and quasi-Armendariz rings in [5], we define the following.

Definition 2.1. Let σ be an endomorphism of a ring R. A ring R is called a σ-skew quasi-Armendariz ring if for $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\sigma]$, $f(x)R[x;\sigma]g(x) = 0$ implies $a_i R\sigma^i(b_j) = 0$ for all $0 \leq i \leq m$, and $0 \leq j \leq n$.

Remark 3. Let R be a σ-skew quasi-Armendariz ring with $f(x)R[x;\sigma]g(x) = 0$. Then $f(x)x^t R[x;\sigma]g(x) = 0$ for any integer $t \geq 0$ and so $a_i R\sigma^{i+t}(b_j) = 0$. Therefore, comparing with Corollary 1.2, Definition 2.1 makes sense.

By Remark 1, if R is a reduced ring, then R is σ-skew quasi-Armendariz when σ is an endomorphism of R. But reduced rings are not σ-skew Armendariz even if σ is an automorphism (see Example 2.2(1) below). We also note that if R is a σ-skew Armendariz ring, then R is σ-skew quasi-Armendariz when σ is an epimorphism of R. Moreover, by Corollary 1.2, semiprime rings are also σ-skew quasi-Armendariz when σ is an epimorphism of R. However, semiprime rings are not σ-skew quasi-Armendariz when σ is a monomorphism of R by Example 1.4. Therefore σ-skew quasi-Armendariz rings extend both σ-skew Armendariz rings and semiprime rings when σ is an epimorphism of R. We note that the semiprimenesses of R and $R[x;\sigma]$ do not depend on each other by [9, Example 4.3] and [17, Theorem 2.2].

The following examples show that σ-skew quasi-Armendariz rings strictly contain σ-skew Armendariz rings and semiprime (so reduced) rings in spite of σ being bijective.

Example 2.2. (1) Let $R = F \oplus F$, where F is a field, and let $\sigma : R \to R$ be an automorphism of R defined by $\sigma((a,b)) = (b,a)$. Then by [6, Example 2], R is not σ-skew Armendariz. By Remark 1, reduced rings with any endomorphism σ are always σ-skew quasi-Armendariz.

(2) We consider the ring

$$R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\},$$

where $\mathbb{Q}$ is the set of all rational numbers, respectively. Let $\sigma : R \to R$ be an automorphism defined by $\sigma\left(\left(\begin{smallmatrix} a & t \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & t/2 \\ 0 & a \end{smallmatrix}\right)$. Then R is σ-skew Armendariz by [6, Example 1], and so R is σ-skew quasi-Armendariz. But R is not semiprime (so not reduced) obviously.

We thereafter investigate the extensions, that is, matrix rings, polynomial rings, homomorphic images and classical quotient rings over a σ-skew quasi-Armendariz ring.

We first study several types of matrix rings over σ-skew quasi-Armendariz rings. The $n \times n$ full (or upper triangular) matrix ring over quasi-Armendariz ring is quasi-Armendariz [5, Theorem 3.12]. We extends these results to σ-skew quasi-Armendariz rings. We denote the $n \times n$ full matrix ring over R by $\mathbb{M}_n(R)$. Recall that if σ is an endomorphism of a ring R, then the map $\bar{\sigma} : \mathbb{M}_n(R) \to \mathbb{M}_n(R)$ defined by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ is an endomorphism of $\mathbb{M}_n(R)$ and clearly this map extends σ.

Theorem 2.3. *Let σ be an endomorphism of a ring R and fix $n \geq 2$. Then R is σ-skew quasi-Armendariz if and only if $\mathbb{M}_n(R)$ is $\bar{\sigma}$-skew quasi-Armendariz.*

Proof. We present only the case when $n = 2$, the general case can be proved by the same method. Note that $\mathbb{M}_n(R)[x;\bar{\sigma}] \cong \mathbb{M}_n(R[x;\sigma])$. Then $f, g \in \mathbb{M}_n(R)[x;\bar{\sigma}]$ can be expressed by the following forms:

$$f(x) = \sum_{i=0}^{p} \begin{pmatrix} a_{i_{11}} & a_{i_{12}} \\ a_{i_{21}} & a_{i_{22}} \end{pmatrix} x^i = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

$$g(x) = \sum_{j=0}^{q} \begin{pmatrix} b_{j_{11}} & b_{j_{12}} \\ b_{j_{21}} & b_{j_{22}} \end{pmatrix} x^j = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where $f_{st} = \sum_{i=0}^{p} a_{i_{st}} x^i$ and $g_{uv} = \sum_{j=0}^{q} b_{j_{uv}} x^j$. Suppose that

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mathbb{M}_2(R[x;\sigma]) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = 0.$$

Then for any $r_{ij} \in R$ and integers $w_{ij} \geq 0$,

$$(*) \qquad \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} r_{11}x^{w_{11}} & r_{12}x^{w_{12}} \\ r_{21}x^{w_{21}} & r_{22}x^{w_{22}} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = 0.$$

If we take r_{ij}'s are zero when $i \neq t$ or $j \neq u$ in Eq.$(*)$, then we have $f_{st} r_{tu} x^{w_{tu}} g_{uv} = 0$ for each $1 \leq t, u \leq 2$. Consequently, $f_{st} R[x;\sigma] g_{uv} = 0$ for any $1 \leq s, t, u, v \leq 2$. Since R is σ-skew quasi-Armendariz, we have $a_{i_{st}} R \sigma^i(b_{j_{uv}}) = 0$ for any $0 \leq i \leq p$ and $0 \leq j \leq q$. Therefore, from the fact

$$\bar{\sigma}^i \left(\begin{pmatrix} b_{j_{11}} & b_{j_{12}} \\ b_{j_{21}} & b_{j_{22}} \end{pmatrix} \right) = \begin{pmatrix} \sigma^i(b_{j_{11}}) & \sigma^i(b_{j_{12}}) \\ \sigma^i(b_{j_{21}}) & \sigma^i(b_{j_{22}}) \end{pmatrix},$$

we have

$$\begin{pmatrix} a_{i_{11}} & a_{i12} \\ a_{i_{21}} & a_{i_{22}} \end{pmatrix} \mathbb{M}_2(R) \bar{\sigma}^i \left(\begin{pmatrix} b_{j_{11}} & b_{j_{12}} \\ b_{j_{21}} & b_{j_{22}} \end{pmatrix} \right) = 0$$

for any $0 \leq s \leq p$ and $0 \leq t \leq q$. Therefore $\mathbb{M}_2(R)$ is $\bar{\sigma}$-skew quasi-Armendariz.

The converse can be easily checked using diagonal matrices. □

The class of quasi-Armendariz rings is Morita stable by [5, Theorem 3.12 and Proposition 3.13]. By the same way as in [5, Proposition 3.13], we also have the following result. Let σ be an endomorphism of a ring R and e an idempotent of R such that $\sigma(e) = e$. Then we have an endomorphism $\bar{\sigma} : eRe \to eRe$ defined by $\bar{\sigma}(ere) = e\sigma(r)e$. We note that there exists a σ-skew quasi-Armendariz ring with idempotent which is not fixed by σ (see Example 2.2(1)).

Proposition 2.4. *Let σ be an endomorphism of a ring R and $e^2 = e \in R$ with $\sigma(e) = e$. If R is σ-skew quasi-Armendariz, then eRe is $\bar{\sigma}$-skew quasi-Armendariz.*

We denote the $n \times n$ upper triangular matrix ring over a ring R by $\mathbb{UM}_n(R)$. By the same method as in the proof of Theorem 2.3, we obtain the following.

Theorem 2.5. *Let σ be an endomorphism of a ring R and fix $n \geq 2$. Then R is σ-skew quasi-Armendariz if and only if $\mathbb{UM}_n(R)$ is $\bar{\sigma}$-skew quasi-Armendariz.*

For a ring R, let

$$R_w = \{M \in \mathbb{UM}_w(R) \mid M = \sum_{i=1}^{w} ae_{ii} + \sum_{1 \leq i < j \leq w} a_{ij}e_{ij}\},$$

where e_{st}'s are matrix units in $\mathbb{UM}_w(R)$. For an endomorphism σ of R, if R is σ-rigid (equivalently, $R[x;\sigma]$ is reduced by [6, Proposition 3]), then R_2 and R_3 are $\bar{\sigma}$-skew Armendariz rings by [7, Proposition 17]. But R_w are not $\bar{\sigma}$-skew Armendariz for $w \geq 4$ even if σ is an automorphism by [6, Example 18]. However, we show that if R is semiprime, then R_w is a $\bar{\sigma}$-skew quasi-Armendariz ring for any integer $w \geq 2$ when σ is an epimorphism of R.

Theorem 2.6. *Let σ be an epimorphism of a ring R. If R is semiprime, then R_w is $\bar{\sigma}$-skew quasi-Armendariz for any integer $w \geq 2$.*

Proof. Suppose that $fR_w[x;\bar{\sigma}]g = 0$ for $f, g \in R_w[x;\bar{\sigma}]$. Let $S = R[x;\sigma]$ and note that $R_w[x;\bar{\sigma}] \cong R[x;\sigma]_w = S_w$ for any integer $w \geq 2$. Then f and g can be expressed by the following forms:

$$f = \sum_{u=0}^{n} A_u x^u = \sum_{i=1}^{w} f_{11}e_{ii} + \sum_{1 \leq i < j \leq w} f_{ij}e_{ij},$$
$$g = \sum_{v=0}^{m} B_v x^v = \sum_{s=1}^{w} g_{11}e_{ss} + \sum_{1 \leq s < t \leq w} g_{st}e_{st},$$

where $A_u = (a_{ij}^u), B_v = (b_{st}^v) \in R_w$ and $f_{ij}, g_{st} \in R[x;\sigma]$. We will show that $f_{ij}Sg_{st} = 0$ for all $1 \leq i, j, s, t \leq w$. We will proceed by induction on w. Let

$$f = \begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{11} \end{pmatrix}, g = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{11} \end{pmatrix}$$

such that $fS_2g = 0$. Then

$$\begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{11} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{11} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{11} \end{pmatrix} = 0$$

for any $\left(\begin{smallmatrix} h_{11} & h_{12} \\ 0 & h_{11} \end{smallmatrix}\right) \in S_2$. Then we have the following:

$$f_{11}h_{11}g_{11} = 0; \tag{1}$$

$$f_{11}h_{11}g_{12} + f_{11}h_{12}g_{11} + f_{12}h_{11}g_{11} = 0. \tag{2}$$

By Eq.(1), $f_{11}Sg_{11} = 0$ and so Eq.(2) becomes $f_{11}rx^pg_{12} + f_{12}rx^pg_{11} = 0$ for any $r \in R$ and integer $p \geq 0$. Then we have

(3) $$a_{11}^0 r\sigma^p(b_{12}^0) + a_{12}^0 r\sigma^p(b_{11}^0) = 0;$$

(4) $$a_{11}^0 r\sigma^p(b_{12}^1) + a_{11}^1\sigma(r)\sigma^{1+p}(b_{12}^0) + a_{12}^0 r\sigma^p(b_{11}^1) + a_{12}^1\sigma(r)\sigma^{1+p}(b_{11}^0) = 0;$$

$$\vdots$$

(5) $$a_{11}^n\sigma^n(r)\sigma^{n+p}(b_{12}^m) + a_{12}^n\sigma^n(r)\sigma^{n+p}(b_{11}^m) = 0.$$

Since $f_{11}Sg_{11} = 0$, by Corollary 1.2, we have

(6) $$a_{11}^u R\sigma^{u+p}(b_{11}^v) = 0$$

for any integer $p \geq 0$, $0 \leq u \leq n$ and $0 \leq v \leq m$. If we multiply $s\sigma^p(b_{11}^0)$ (where $s \in R$) on the right side of Eq.(3), then $a_{12}^0 r\sigma^p(b_{11}^0)s\sigma^p(b_{11}^0) = 0$ and so $a_{12}^0 R\sigma^p(b_{11}^0)R\sigma^p(b_{11}^0) = 0$. Since R is semiprime, we have

(7) $$a_{12}^0 R\sigma^p(b_{11}^0) = 0 \text{ and so } a_{11}^0 R\sigma^p(b_{12}^0) = 0$$

for any integer $p \geq 0$. If we multiply $s\sigma^{1+p}(b_{11}^0)$ (where $s \in R$) on the right side of Eq.(4), using Eqs.(6), (7) and the fact that p is any integer, then $a_{12}^1 R\sigma^{1+p}(b_{11}^0)R\sigma^{1+p}(b_{11}^0) = 0$. Since R is semiprime, $a_{12}^1 R\sigma^{1+p}(b_{11}^0) = 0$ for any integer $p \geq 0$. Then Eq.(4) becomes

(4') $$a_{11}^0 r\sigma^p(b_{12}^1) + a_{11}^1\sigma(r)\sigma^{1+p}(b_{12}^0) + a_{12}^0 r\sigma^p(b_{11}^1) = 0.$$

If we replace r in Eq.($4'$) by $r\sigma^p(b_{11}^1)s$ (where $s \in R$), then $a_{12}^0 r\sigma^p(b_{11}^1)s\sigma^p(b_{11}^1) = 0$ and so $a_{12}^0 R\sigma^p(b_{11}^1) = 0$ for any integer $p \geq 0$. Continuing the above processes, from Eq.($4'$) we have $a_{11}^1 R\sigma^{1+p}(b_{12}^0) = 0$ and $a_{11}^0 R\sigma^p(b_{12}^1) = 0$. Inductively, we have

(8) $$a_{11}^u R\sigma^{u+p}(b_{12}^v) = 0 \text{ and } a_{12}^u R\sigma^{u+p}(b_{11}^v) = 0$$

for any integer $p \geq 0$, $0 \leq u \leq n$ and $0 \leq v \leq m$. Consequently, from Eq.(8) we have

$$f_{11}Sg_{11} = 0, f_{11}Sg_{12} = 0, f_{12}Sg_{11} = 0.$$

Assume that our claim is true for $w = k - 1$. Let $f = (f_{ij})$, $g = (g_{st}) \in S_k$ with $fS_kg = 0$. Note that we can imbed S_{k-1} into S_k via

$$\sum_{i=1}^{k-1}\alpha e_{ii} + \sum_{1\leq i<j\leq k-1}\alpha_{ij}e_{ij} \mapsto \sum_{i=1}^{k}\alpha e_{ii} + \sum_{1\leq i<j\leq k}\alpha_{ij}e_{ij},$$

where $\alpha_{ik} = 0$ for any $1 \leq i \leq k-1$. Since $fS_kg = 0$, we have $fS_{k-1}g = 0$. By the induction hypothesis, we have

(9) $$f_{ij}Sg_{st} = 0$$

for any $1 \leq i, j, s, t \leq k - 1$. Now from the fact that $f(h_{ij})g = 0$ for any $(h_{kl}) \in S_k$, the $(k-1, k)$-entry,

$$f_{11}h_{11}g_{(k-1)k} + (f_{11}h_{(k-1)k} + f_{(k-1)k}h_{11})g_{11} = 0.$$

Since $f_{11}Sg_{11} = 0$ by Eq.(9), we have

$$f_{11}rx^t g_{(k-1)k} + f_{(k-1)k}rx^t g_{11} = 0 \tag{10}$$

for any $r \in R$ and integer $t \geq 0$. Repeating the computation from (3) to (8) on Eq.(10), we have

$$f_{(k-1)k}Sg_{11} = 0, f_{11}Sg_{(k-1)k} = 0. \tag{11}$$

From Eqs.(9), (11) and the $(k-2, k)$-entry is zero, we have

$$f_{11}h_{11}g_{(k-2)k} + f_{(k-2)(k-1)}h_{11}g_{(k-1)k} + f_{(k-2)k}h_{11}g_{11} = 0.$$

After we repeat the similar computation as above, we have

$$f_{11}Sg_{(k-2)k} = 0, f_{(k-2)(k-1)}Sg_{(k-1)k} = 0, f_{(k-2)k}Sg_{11} = 0.$$

Continuing this process, we have $f_{ij}Sg_{st} = 0$ for any $1 \leq i, j, s, t \leq k$. Therefore R_w is $\bar{\sigma}$-skew quasi-Armendariz for any $w \geq 2$. □

We also consider the following subring of R_n. Let

$$V_n(R) = \{M \in R_n \mid M = \sum_{1\leq i\leq j\leq n} a_{ij}e_{ij}, \text{ where } a_{ij} = a_{(i+1)(j+1)}\}.$$

By the same method as in the proof of Theorem 2.6, we have the following.

Theorem 2.7. *Let σ be an epimorphism of a ring R. If R is semiprime, then V_n is $\bar{\sigma}$-skew quasi-Armendariz for any integer $n \geq 2$.*

For an integer $n \geq 2$, let $RA = \{rA \mid r \in R\}$ for any $A \in \mathbb{M}_n(R)$ and $V = \sum_{i=1}^{n-1} e_{i(i+1)}$. Then by [11], $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$, where I_n is the $n \times n$ identity matrix, and the map $\rho : V_n(R) \to R[x]/\langle x^n\rangle$ defined by $\rho(a_0I_n + a_1V + \cdots + a_{n-1}V^{n-1}) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n\rangle$ is a ring isomorphism. So we have the following.

Corollary 2.8. *Let σ be an epimorphism of a ring R. If R is semiprime, then $R[x]/\langle x^n\rangle$ is $\bar{\sigma}$-skew quasi-Armendariz for any integer $n \geq 2$.*

From Theorem 2.6, one may suspect that R_w may be also a $\bar{\sigma}$-skew quasi-Armendariz ring for any integer $w \geq 2$ when R is σ-skew quasi-Armendariz for an epimorphism σ of R. But the following example erases the possibility.

Example 2.9. Let S be any semiprime ring and $R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in S\}$. Then R is an $\bar{I}_S$-skew quasi-Armendariz ring by Theorem 2.6, where I_S is the identity map of S. Let $R_2 = \{\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right) \mid A, B \in R\}$ and

$$f(x) = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{12} \end{pmatrix} + \begin{pmatrix} e_{12} & -(e_{11} + e_{22}) \\ 0 & e_{12} \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{12} \end{pmatrix} + \begin{pmatrix} e_{12} & e_{11} + e_{22} \\ 0 & e_{12} \end{pmatrix} x$$

in $R_2[x; \bar{\bar{I}}_S]$, where e_{ij}'s are the matrix units in $\mathbb{M}_2(S)$. Then $f(x)R_2[x; \bar{\bar{I}}_S]g(x) = 0$, but $\left(\begin{smallmatrix} e_{12} & 0 \\ 0 & e_{12} \end{smallmatrix}\right) R_2 \left(\begin{smallmatrix} e_{12} & e_{11}+e_{22} \\ 0 & e_{12} \end{smallmatrix}\right) \neq 0$.

Remark 4. Recently, Hashemi [4] defined M-quasi-Armendariz rings as follows: for a monoid M, a ring R is called *M-quasi-Armendariz* if whenever $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha R[M]\beta = 0$, then $a_iRb_j = 0$ for each i, j. Then he asserted that if a ring R is reduced and M-Armendariz, then R is M-quasi-Armendariz. However, we note that M-Armendariz rings are M-quasi-Armendariz. For, suppose that $\alpha R[M]\beta = 0$. Then $\alpha r\beta = 0$ for any $r \in R$ and so $\alpha\beta' = 0$, where $\beta' = rb_1h_1 + \cdots + rb_mh_m$. Since R is M-Armendariz, $a_irb_j = 0$ for each i, j, and therefore $a_iRb_j = 0$.

Moreover, in [4, Proposition 1.2], he proved that if R is a M-Armendariz and reduced ring, then R_n is M-quasi-Armendariz for each $n \geq 2$. However, using the same method as in the proof of Theorem 2.6, we can show that if R is a M-quasi-Armendariz and semiprime ring, then R_n is M-quasi-Armendariz for each $n \geq 2$.

We next study the polynomial ring and the Laurent polynomial ring over a σ-skew quasi-Armendariz ring. If R is quasi-Armendariz, then the polynomial ring $R[x]$ is quasi-Armendariz [4, Theorem 3.16]. We extend this result to σ-skew quasi-Armendariz rings. Recall that if σ is an endomorphism of a ring R, then the map $\bar{\sigma} : R[x] \to R[x]$ defined by $\bar{\sigma}(\sum_{i=0}^{m} a_ix^i) = \sum_{i=0}^{m} \sigma(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends σ. The *Laurent polynomial ring* $R[x, x^{-1}]$ with an indeterminate x, consists of all formal sums $\sum_{i=k}^{n} a_ix^i$, where $a_i \in R$ and k, n are (possibly negative) integers. The map $\bar{\sigma} : R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $\bar{\sigma}(\sum_{i=k}^{n} a_ix^i) = \sum_{i=k}^{n} \sigma(a_i)x^i$ extends σ and is also an endomorphism of $R[x, x^{-1}]$.

Theorem 2.10. *Let σ be an endomorphism of a ring R and $\sigma^t = I_R$ for some positive integer t. Then the following statements are equivalent*:

(1) *R is σ-skew quasi-Armendariz.*
(2) *$R[x]$ is $\bar{\sigma}$-skew quasi-Armendariz.*
(3) *$R[x, x^{-1}]$ is $\bar{\sigma}$-skew quasi-Armendariz.*

Proof. We only give the proof of (1)⇔(3) since (1)⇔(2) can be proved by the same method.

(1)⇒(3): We refer the proof of [3, Proposition 7].

Suppose $f(y)R[x, x^{-1}][y; \bar{\sigma}]g(y) = 0$, where $f(y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m, g(y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n \in R[x, x^{-1}][y; \bar{\sigma}]$. We also let $f_i(x) = \sum_{u=s_i}^{p_i} a_ux^u$, $g_j(x) = \sum_{v=k_j}^{q_j} b_vx^v$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$, where $a_{s_i}, \ldots, a_{p_i}, b_{k_j}, \ldots, b_{q_j} \in R$ and $s_i, p_i, k_j, q_j \in \mathbb{Z}$. Take positive integers s, k such that $s = \max\{|s_i| \mid i = 0, 1, \ldots, m\}$ and $k = \max\{|k_j| \mid j = 0, 1, \ldots, n\}$. Let $f'(y) = x^sf(y) = f_0'(x) + f_1'(x)y + \cdots + f_m'(x)y^m$ and $g'(y) = x^kg(y) = g_0'(x) + g_1'(x)y + \cdots + g_n'(x)y^n$, where $f_i'(x) = f_i(x)x^s$ and $g_j'(x) = g_j(x)x^k$. Now we take a positive integer l such that $l > \sum_{i=0}^{m} \deg(f_i'(x)) + \sum_{j=0}^{n} \deg(g_j'(x))$. Let $f'(x) = f_0'(x^t) + f_1'(x^t)x^{tl+1} + \cdots + f_m'(x^t)x^{mtl+m}$ and $g'(x) = g_0'(x^t) + g_1'(x^t)x^{tl+1} + \cdots + g_n'(x^t)x^{ntl+n}$. Then we claim that

$$f'(x)R[x; \sigma]g'(x) = 0,$$

equivalently, $f'(x)rx^w g'(x) = 0$ for any integer $w \geq 0$. Since

$$f(y)R[x,x^{-1}][y;\bar{\sigma}]g(y) = 0,$$

$f'(y)R[x,x^{-1}][y;\bar{\sigma}]g'(y) = 0$ and so $f'(y)ry^w g'(y) = 0$ for any integer $w \geq 0$. Thus

$$\begin{aligned} f'_0(x)r\bar{\sigma}^w(g'_0(x)) &= 0; \\ f'_0(x)r\bar{\sigma}^w(g'_1(x)) + f'_1(x)\bar{\sigma}(r)\bar{\sigma}^{w+1}(g'_0(x))) &= 0; \\ &\vdots \\ f'_m(x)\bar{\sigma}^m(r)\bar{\sigma}^{m+w}(g'_n(x)) &= 0. \end{aligned}$$

Using these equations, we have $f'(x)rx^w g'(x) = 0$ for any integer $w \geq 0$. Thus

$$\begin{aligned} &(a_{s_0}x^{t(s_0+s)} + \cdots + a_{p_0}x^{t(p_0+s)} + \cdots + a_{s_m}x^{t(s_m+s+ml)+m} + \cdots + a_{p_m}x^{t(p_m+s+ml)+m}) \\ &rx^w(b_{k_0}x^{t(k_0+k)} + \cdots + b_{q_0}x^{t(q_0+k)} + \cdots + b_{k_n}x^{t(k_n+k+nl)+n} + \cdots + b_{q_n}x^{t(q_n+k+nl)+n}) \\ &= 0. \end{aligned}$$

Since R is σ-skew quasi-Armendariz and σ^t is the identity map, we have $a_{\alpha_i}R\sigma^i(b_{\beta_j}) = a_{\alpha_i}R\sigma^{t(\alpha_i+s+il)+i}(b_{\beta_j}) = 0$ for any $\alpha_i \in \{s_i, \ldots, p_i\}$ and $\beta_j \in \{k_j, \ldots, q_j\}$, where $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore

$$f_i(x)R[x,x^{-1}]\bar{\sigma}^i(g_j(x)) = 0.$$

(3)⇒(1): Let $f(y) = a_0 + a_1y + \cdots + a_my^m, g(y) = b_0 + b_1y + \cdots + b_ny^n \in R[y;\sigma]$ such that $f(y)R[y;\sigma]g(y) = 0$. Now let $f(u) = a_0 + a_1u + \cdots + a_mu^m$ and $g(u) = b_0 + b_1u + \cdots + b_nu^n \in R[x,x^{-1}][u;\bar{\sigma}]$. We claim that $f(u)R[x,x^{-1}][u;\bar{\sigma}]g(u) = 0$, equivalently, $f(u)rx^ku^sg(u) = 0$ for any $r \in R$ and $k, s \in \mathbb{Z}$ with $s \geq 0$. Since $f(y)R[y;\sigma]g(y) = 0$, $f(y)ry^sg(y) = 0$. Then we have

$$\begin{aligned} &f(u)rx^ku^sg(u) \\ &= (a_0rx^k + a_1\bar{\sigma}(rx^k)u + \cdots + a_m\bar{\sigma}^m(rx^k)u^m)u^s(b_0 + b_1u + \cdots + b_nu^n) \\ &= (a_0rx^k + a_1\sigma(r)x^ku + \cdots + a_m\sigma^m(r)x^ku^m)u^s(b_0 + b_1u + \cdots + b_nu^n) \\ &= x^k(a_0 + a_1u + \cdots + a_mu^m)ru^s(b_0 + b_1u + \cdots + b_nu^n) = 0. \end{aligned}$$

Since $R[x,x^{-1}]$ is $\bar{\sigma}$-skew quasi-Armendariz, we have $a_iR[x,x^{-1}]\bar{\sigma}^i(b_j) = 0$ for all i, j and so $a_iR\sigma^i(b_j) = 0$. Therefore R is σ-skew quasi-Armendariz. □

We now consider the homomorphic images of σ-skew quasi-Armendariz rings. For an ideal I of R, if $\sigma(I) \subseteq I$, then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(a+I) = \sigma(a) + I$ is an endomorphism of a factor ring R/I. We now note that the homomorphic image of σ-skew quasi-Armendariz rings need not to be so in general.

Example 2.11. We use the argument in [6, Example 7]. Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider the ring

$$R = \left\{ \begin{pmatrix} a & \bar{b} \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, \bar{b} \in \mathbb{Z}_4 \right\}.$$

Let $\sigma : R \to R$ be an automorphism defined by $\sigma\left(\left(\begin{smallmatrix} a & \bar{b} \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -\bar{b} \\ 0 & a \end{smallmatrix}\right)$. Then R is σ-skew Armendariz by [6, Example 7], and so R is σ-skew quasi-Armendariz because R is commutative. Let $I = \left\{\left(\begin{smallmatrix} a & \bar{0} \\ 0 & a \end{smallmatrix}\right) \mid a \in 4\mathbb{Z}\right\}$. Then $\sigma(I) = I$ and the factor ring $R/I \cong \left\{\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right) \mid \bar{a}, \bar{b} \in \mathbb{Z}_4\right\}$ is not σ-skew quasi-Armendariz. In fact,

$$\left(\begin{pmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{pmatrix} + \begin{pmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{pmatrix} x\right)(R/I)[x;\bar{\sigma}]\left(\begin{pmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{pmatrix} + \begin{pmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{pmatrix} x\right) = 0.$$

But $\left(\begin{smallmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{smallmatrix}\right)(R/I)\,\bar{\sigma}\left(\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix}\right)\right) \neq 0$.

However, we obtain the following result referring the method in the proof of [9, Lemma 3.6].

Proposition 2.12. *Let σ be an endomorphism of a ring R and I an ideal of R with $\sigma(I) = I$. If R is σ-skew quasi-Armendariz, then $R/r_R(I)$ is $\bar{\sigma}$-skew quasi-Armendariz.*

Moreover, we may ask that R is an σ-skew quasi-Armendariz ring if for a nonzero proper ideal I of R with $\sigma(I) = I$, R/I is $\bar{\sigma}$-skew quasi-Armendariz and I is σ-skew quasi-Armendariz as a ring. However, we also have a counterexample to this situation as in the following.

Example 2.13. Consider the ring

$$R = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{pmatrix} \mid \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}.$$

Let $\sigma : R \to R$ be an automorphism defined by $\sigma\left(\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \bar{a} & -\bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right)$. By the argument of Example 2.11, R is not σ-skew quasi-Armendariz. Let $I = \left\{\left(\begin{smallmatrix} \bar{0} & \bar{b} \\ 0 & \bar{0} \end{smallmatrix}\right) \mid \bar{b} \in \mathbb{Z}_4\right\}$. Then $\sigma(I) = I$ and the factor ring $R/I \cong \mathbb{Z}_4$ is $\bar{\sigma}$-skew quasi-Armendariz. Moreover, I is σ-skew quasi-Armendariz as a ring.

Proposition 2.14. *For an endomorphism σ of a ring R, suppose that R/I is a $\bar{\sigma}$-skew quasi-Armendariz ring for an ideal I of R. If I is semiprime as a ring, then R is σ-skew quasi-Armendariz.*

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$ such that $f(x)R[x;\sigma]g(x) = 0$. Then $\bar{f}(x)(R/I)[x;\bar{\sigma}]\bar{g}(x) = \bar{0}$, where $\bar{a} = a + I$ and $\bar{f}(x) = \bar{a}_0 + \bar{a}_1x + \cdots + \bar{a}_nx^n, \bar{g}(x) = \bar{b}_0 + \bar{b}_1x + \cdots + \bar{b}_mx^m \in (R/I)[x;\bar{\sigma}]$. Since R/I is $\bar{\sigma}$-skew quasi-Armendariz, $a_iR\sigma^i(b_j) \subseteq I$ for all i, j. Moreover, we can get

$$a_iR\sigma^{i+s}(b_j) \subseteq I \tag{1}$$

for any integer $s \geq 0$. We proceed by the induction on $\deg f(x) = n$ with $n \geq 0$. If $n = 0$, then we are done. Suppose that $n \geq 1$. We first claim that

$a_0R\sigma^t(b_j) = 0$ for all integer $t \geq 0$ and $0 \leq j \leq m$. Assume that there exists b_j such that $a_0R\sigma^{t_1}(b_j) \neq 0$ for some t_1. Then we can take k in $\{1, 2, \ldots, m\}$ such that k is the smallest one with respect to the property $a_0R\sigma^{t_2}(b_k) \neq 0$ for some t_2. So for $j \in \{0, 1, \ldots, k-1\}$, $a_0R\sigma^t(b_j) = 0$ for any t. Note that $\sigma^t(b_j)Ia_0 = 0$. Indeed, $(\sigma^t(b_j)Ia_0R)^2 = \sigma^t(b_j)I(a_0R\sigma^t(b_j))Ia_0R = 0$. Since $\sigma^t(b_j)Ia_0R \subseteq I$ and I is semiprime as a ring, $\sigma^t(b_j)Ia_0R = 0$ and so $\sigma^t(b_j)Ia_0 = 0$. Now we note that

$$\begin{aligned}(a_{k-j}R\sigma^t(b_j))(Ra_0R\sigma^{t_2}(b_k))^2 &= (a_{k-j}R\sigma^t(b_j))(Ra_0R\sigma^{t_2}(b_k)R)(a_0R\sigma^{t_2}(b_k)) \\ &\subseteq (a_{k-j}R\sigma^t(b_j))I(a_0R\sigma^{t_2}(b_k)) \\ &= a_{k-j}R(\sigma^t(b_j)Ia_0)R\sigma^{t_2}(b_k) = 0\end{aligned}$$

by Eq.(1). The coefficient of the term x^{k+t_2} in $f(x)R[x;\sigma]g(x) = 0$ is

$$0 = a_0r\sigma^{t_2}(b_k) + a_1\sigma(r)\sigma^{t_2+1}(b_{k-1}) + \cdots + a_k\sigma^k(r)\sigma^{t_2+k}(b_0) \tag{2}$$

for any $r \in R$. Multiplying $(Ra_0R\sigma^{t_2}(b_k))^2$ to Eq.(2) on the right side, we have

$$\begin{aligned}0 &= (a_0r\sigma^{t_2}(b_k) + a_1\sigma(r)\sigma^{t_2+1}(b_{k-1}) + \cdots + a_k\sigma^k(r)\sigma^{t_2+k}(b_0))(Ra_0R\sigma^{t_2}(b_k))^2 \\ &= a_0r\sigma^{t_2}(b_k)(Ra_0R\sigma^{t_2}(b_k))^2\end{aligned}$$

and so $(Ra_0R\sigma^{t_2}(b_k))^3 = 0$. Since $Ra_0R\sigma^{t_2}(b_k) \subseteq I$ by Eq.(1) and I is semiprime as a ring, we have $a_0R\sigma^{t_2}(b_k) = 0$, which is a contradiction. Consequently, $a_0R\sigma^t(b_j) = 0$ for all $j \in \{0, 1, \ldots, m\}$ and thus we have that $f_1(x)R[x;\sigma]g(x) = 0$, where $f_1(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$. But the degree of $f_1(x)$ is less than n. By the induction hypothesis, we get $a_iR\sigma^i(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Therefore R is σ-skew quasi-Armendariz. □

We consider the classical left quotient ring $Q(R)$ of a σ-skew quasi-Armendariz ring R. Recall that a ring R is *left Ore* if there exists the classical left quotient ring $Q(R)$ of R. Let σ be an automorphism of a left Ore ring R. Then for any $b^{-1}a \in Q(R)$ where $a, b \in R$ with b regular, the induced map $\bar{\sigma} : Q(R) \to Q(R)$ defined by $\bar{\sigma}(b^{-1}a) = \sigma(b)^{-1}\sigma(a)$ extends to an automorphism of $Q(R)$.

Theorem 2.15. *Let R be a left Ore ring with an automorphism σ of R. If R is σ-skew quasi-Armendariz, then $Q(R)$ is $\bar{\sigma}$-skew quasi-Armendariz.*

Proof. Let $Q(R) = Q$ and $f(x) = \sum_{i=0}^{m} \alpha_i x^i$, $g(x) = \sum_{j=0}^{n} \beta_j x^j \in Q[x]$ such that $f(x)Q[x;\bar{\sigma}]g(x) = 0$. We may assume that $\alpha_i = u^{-1}a_i$, $\beta_j = v^{-1}b_j$ with $a_i, b_j \in R$ and regular elements $u, v \in R$. Since $f(x)Q[x;\bar{\sigma}]g(x) = 0$, we have $u^{-1}(a_0 + a_1x + \cdots + a_mx^m)Qx^kv^{-1}(b_0 + b_1x + \cdots + b_nx^n) = 0$ for any integer $k \geq 0$. For each $k \geq 0$, note that $Q\sigma^k(v)^{-1} = Q$ and also $Q = Qv^{-1}$. Thus we have

$$\begin{aligned}0 &= (a_0 + a_1x + \cdots + a_mx^m)Qx^k(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_0 + a_1x + \cdots + a_mx^m)Qv^{-1}Rx^k(b_0 + b_1x + \cdots + b_nx^n)\end{aligned}$$

for any $k \geq 0$. Let $t^{-1}s \in Q$, $sv^{-1} = v'^{-1}s'$ and $t^{-1}v'^{-1} = t'^{-1}$. Then

$$\begin{aligned}0 &= (a_0 + a_1x + \cdots + a_mx^m)t^{-1}sv^{-1}Rx^k(b_0 + b_1x + \cdots + b_nx^n)\\ &= (a_0 + a_1x + \cdots + a_mx^m)t'^{-1}s'Rx^k(b_0 + b_1x + \cdots + b_nx^n)\\ &= (a_0t'^{-1}s' + a_1\bar{\sigma}(t'^{-1}s')x + \cdots + a_m\bar{\sigma}^m(t'^{-1}s')x^m)Rx^k(b_0 + b_1x + \cdots + b_nx^n)\\ &= (a_0t'^{-1}s' + a_1\sigma(t')^{-1}\sigma(s')x + \cdots + a_m\sigma^m(t')^{-1}\sigma^m(s')x^m)\\ &\quad \times Rx^k(b_0 + b_1x + \cdots + b_nx^n).\end{aligned}$$

We now let $a_i\sigma^i(t')^{-1} = w^{-1}a_i'$. Then we have $w^{-1}(a_0's' + a_1'\sigma(s')x + \cdots + a_m\sigma^m(s')x^m)Rx^k(b_0 + b_1x + \cdots + b_nx^n) = 0$ and so $(a_0's' + a_1'\sigma(s')x + \cdots + a_m\sigma^m(s')x^m)Rx^k(b_0+b_1x+\cdots+b_nx^n)=0$. Since R is σ-skew quasi-Armendariz,

$$(*) \qquad a_i'\sigma^i(s')R\sigma^i(b_j) = 0 \text{ and so } w^{-1}a_i'\sigma^i(s')R\sigma^i(b_j) = 0$$

for any $1 \leq i \leq m$, $1 \leq j \leq n$. We now will show that $u^{-1}a_iQ\sigma^i(v^{-1}b_j) = 0$. From Eq.$(*)$ and the same argument as above, we have $(a_0 + a_1x + \cdots + a_mx^m)t^{-1}sv^{-1}b_j = 0$ for any $t^{-1}s \in Q$ and $1 \leq j \leq n$, and so $(u^{-1}a_0+u^{-1}a_1x+\cdots + u^{-1}a_mx^m)Qv^{-1}b_j = 0$ for any $1 \leq j \leq n$. Hence $u^{-1}a_iQ\sigma^i(v^{-1}b_j) = 0$ for any $1 \leq i \leq m$, $1 \leq j \leq n$. Therefore Q is $\bar{\sigma}$-skew quasi-Armendariz. □

Hirano [5, Proposition 3.4] proved that a ring R is quasi-Armendariz if and only if $\Phi : \Gamma \to \Delta$ is bijective with $\Phi(A) = AR[x]$, where $\Gamma = \{r_R(U) \mid U \text{ is an ideal of } R\}$ and $\Delta = \{r_R(V) \mid V \text{ is an ideal of } R[x]\}$.

Finally, we introduce a similar result for skew quasi-Armendariz rings. Let A be an ideal of a ring R and suppose that $i = i(A)$ is a nonnegative integer depending on A. Define

$$A' = \{ax^k \mid a \in A, k \geq i = i(A)\} \subseteq R[x;\sigma].$$

Note $A' = \cup_{t=0}^{\infty} Ax^{i+t}$. Moreover $r_{R[x;\sigma]}(A')$ and $r_R(A') = r_{R[x;\sigma]}(A') \cap R$ are ideals of $R[x;\sigma]$ and R, respectively. For, let $f(x) \in r_{R[x;\sigma]}(A')$ and $g(x) = \sum_{i=0}^n b_ix^i \in R[x;\sigma]$. For any $ax^k \in A'$, $ax^kg(x)f(x) = \sum_{i=0}^n a\sigma^k(b_i)x^{k+i}f(x) = 0$ since $a\sigma^k(b_i) \in A$ and $a\sigma^k(b_i)x^{k+i} \in A'$. Thus $g(x)f(x) \in r_{R[x;\sigma]}(A')$ and so $r_{R[x;\sigma]}(A')$ is an ideal of $R[x;\sigma]$, entailing that $r_R(A')$ is an ideal of R.

Given ideals A_j $(j \in I)$ of R, $r_{R[x;\sigma]}(\cup_jA_j') = \cap_jr_{R[x;\sigma]}(A_j')$; hence $r_R(\cup_jA_j') = r_{R[x;\sigma]}(\cup_jA_j') \cap R$ and $r_{R[x;\sigma]}(\cup_jA_j')$ are ideals of R and $R[x;\sigma]$ respectively, with the help of the preceding computation.

Let

$$\Gamma = \{r_R(\cup_jB_j') \mid B_j \text{ is an ideal of } R \text{ for } j \in I\}$$

and

$$\Delta = \{r_{R[x;\sigma]}(V) \mid V \text{ is an ideal of } R[x;\sigma]\}.$$

Then we obtain an injective map $\Phi : \Gamma \to \Delta$ defined by $\Phi(r_R(\cup_jB_j')) = r_R(\cup_jB_j')R[x;\sigma]$ as in the proof of Theorem 2.16 below.

Theorem 2.16. *Let σ be an epimorphism of R. Then the following statements are equivalent*:

(1) *R is σ-skew quasi-Armendariz.*
(2) *$\Phi : \Gamma \to \Delta$ is bijective with $\Phi(r_R(\cup_j B'_j)) = r_R(\cup_j B'_j)R[x;\sigma]$.*

Proof. We first claim that Φ is well-defined. For $r_R(\cup_j B'_j) \in \Gamma$, let $g(x) = b_0 + b_1x + \cdots + b_mx^m \in r_R(\cup_j B'_j)R[x;\sigma]$. Then $b_0, b_1, \ldots, b_m \in r_R(\cup_j B'_j)$ and so $b_\ell x^\ell \in r_{R[x;\sigma]}(\cup_j B'_j)$ for each ℓ, entailing $g(x) \in r_{R[x;\sigma]}(\cup_j B'_j)$. Conversely, let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in r_{R[x;\sigma]}(\cup_j B'_j)$. Then $0 = bx^k(a_0 + a_1x + \cdots + a_nx^n) = bx^ka_0 + bx^ka_1x + \cdots + bx^ka_nx^n$ for all $bx^k \in \cup_j B'_j$. If $bx^ka_t \neq 0$ for some t, then $b\sigma^k(a_t) \neq 0$ and so $bx^ka_tx^t = b\sigma^k(a_t)x^{k+t} \neq 0$; hence $bx^kf(x) \neq 0$, a contradiction. Thus $a_j \in r_R(\cup_j B'_j)$ and we get $f(x) \in r_R(\cup_j B'_j)R[x;\sigma]$. Consequently $r_R(\cup_j B'_j)R[x;\sigma] = r_{R[x;\sigma]}(\cup_j B'_j)$ and so we obtain

$$\begin{aligned} r_R(\cup_j B'_j)R[x;\sigma] &= r_{R[x;\sigma]}(\cup_j B'_j) = r_{R[x;\sigma]}((\cup_j B'_j)R[x;\sigma]) \\ &= r_{R[x;\sigma]}(R[x;\sigma](\cup_j B'_j)R[x;\sigma]), \end{aligned}$$

determining the map $\Phi : \Gamma \to \Delta$ with $\Phi(r_R(\cup_j B'_j)) = r_R(\cup_j B'_j)R[x;\sigma]$.

Next we show that Φ is injective. Put $\Phi(r_R(\cup_s A'_s)) = \Phi(r_R(\cup_t A'_t))$. Then

$$r_R(\cup_s A'_s)R[x;\sigma] = r_R(\cup_t A'_t)R[x;\sigma] \text{ and } r_{R[x;\sigma]}(\cup_s A'_s) = r_{R[x;\sigma]}(\cup_t A'_t)$$

by the result above. It then follows

$$r_R(\cup_s A'_s) = r_{R[x;\sigma]}(\cup_s A'_s) \cap R = r_{R[x;\sigma]}(\cup_t A'_t) \cap R = r_R(\cup_t A'_t),$$

proving that Φ is injective.

(1)$\Rightarrow$(2): It suffices to show that Φ is surjective. Let V be an ideal of $R[x;\sigma]$ and $f(x) = a_0 + a_1x + \cdots + a_nx^n \in V$. If $g(x) = b_0 + b_1x + \cdots + b_mx^m \in r_{R[x;\sigma]}(f(x)R[x;\sigma])$, then $f(x)R[x;\sigma]g(x) = 0$ and $f(x)x^tR[x;\sigma]g(x) = 0$ for all nonnegative integer t. Since R is σ-skew quasi-Armendariz, we have $a_iR\sigma^{i+t}(b_j) = 0$ for each $0 \leq i \leq n, 0 \leq j \leq m$. Then for any $0 \leq j \leq m$, we have $b_j \in r_R(a_iRx^{i+t}) = r_R(Ra_iRx^{i+t})$ for each $0 \leq i \leq n$; hence $b_j \in \cap_{i=0}^n r_R(Ra_iRx^{i+t}) = r_R(\cup_{i=0}^n Ra_iRx^{i+t})$. Set $A_i = Ra_iR$ for $i = 0, 1, \ldots, n$. Then $A'_i = \{dx^j \mid d \in A_i, j \geq i\} = \cup_{t=0}^\infty Ra_iRx^{i+t}$ with $i = i(A_i)$. So $g(x) \in r_R(\cup_{i=0}^n A'_i)R[x;\sigma]$ and hence $r_{R[x;\sigma]}(f(x)R[x;\sigma]) \subseteq r_R(M_f)R[x;\sigma]$, where $M_f = \cup_{i=0}^n A'_i$. Conversely, let $g(x) \in r_R(M_f)R[x;\sigma] = r_{R[x;\sigma]}(M_f)$. Since every term of polynomials in $f(x)R[x;\sigma]$ is a sum of monomials contained in M_f, we get $f(x)R[x;\sigma]g(x) = 0$ and thus $g(x) \in r_{R[x;\sigma]}(f(x)R[x;\sigma])$, concluding $r_{R[x;\sigma]}(f(x)R[x;\sigma]) = r_R(M_f)R[x;\sigma]$. Consequently

$$\begin{aligned} r_{R[x;\sigma]}(V) &= \bigcap_{f(x)\in V} r_{R[x;\sigma]}(f(x)R[x;\sigma]) = \bigcap_{f(x)\in V} r_{R[x;\sigma]}(M_f) \\ &= r_{R[x;\sigma]}(\bigcup_{f(x)\in V} M_f) = r_{R[x;\sigma]}(M_V) \\ &= r_{R[x;\sigma]}(\cup_j B'_j) = r_R(\cup_j B'_j)R[x;\sigma] = \Phi(r_R(\cup_j B'_j)), \end{aligned}$$

where $M_V = \cup_{ij}(Ra_{ij}R)'$ and a_{ij} runs over the set of all coefficients of polynomials in V. Thus Φ is surjective.

(2)⇒(1): Let $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$ with $f(x)R[x;\sigma]g(x) = 0$. Since Φ is surjective,

$$r_{R[x;\sigma]}(R[x;\sigma]f(x)R[x;\sigma]) = r_R(\cup_j B'_j)R[x;\sigma]$$

for some $r_R(\cup_j B'_j) \in \Gamma$. Note $r_R(\cup_j B'_j)R[x;\sigma] = r_{R[x;\sigma]}(\cup_j B'_j)$, so $(\cup_j B'_j)g(x) = 0$. Then for any $dx^k \in \cup_j B'_j$ we get $dx^k(b_0 + b_1x + \cdots + b_mx^m) = 0$; hence $dx^kb_j = 0$ for all $j = 0, 1, \ldots, m$ by the same computation as above. Consequently $b_j \in r_{R[x;\sigma]}(\cup_j B'_j) = r_{R[x;\sigma]}(R[x;\sigma]f(x)R[x;\sigma])$ for any $j = 0, 1, \ldots, m$. Especially $(a_0 + a_1x + \cdots + a_nx^n)Rb_j = 0$ for any $j = 0, 1, \ldots, m$. Now from the hypothesis that σ is surjective, we get $a_iR\sigma^i(b_j) = 0$ for all i, j. Therefore R is σ-skew quasi-Armendariz. □

Acknowledgements. The first named author was supported by the Kyung Hee University in the program of 2007. The second named author was supported by the National Research Foundation of Korea grant funded by the Korea Government (No. 2009-0074018), while the third named author was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-521-C00003).

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