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HOMOGENEOUS IDEAL $I(+)IM$ OF R(+)M

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Abstract. In this short paper, we show that properties of an ideal I of a ring R are related to those of the homogeneous ideal $I(+)IM$ of a ring $R(+)M$.

1. Introduction

Throughout this paper, all rings are commutative rings with unity and all modules are unital. R-module M is called *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = (N : M)M$. R-module M is said to be *divisible* if $M = rM$ whenever r is an element of R which is not a zero divisor.

R-module M is called *cancellation* if whenever $\mathcal{A}M = \mathcal{B}M$ for ideals $\mathcal{A}$ and $\mathcal{B}$ of R, then $\mathcal{A} = \mathcal{B}$.

Let M be an R-module. Consider $R(M) = \{(r, m) | r \in R, m \in M\}$ and let (r, m) and (s, n) be two elements of $R(M)$. Define:

1. $(r, m) = (s, n)$ if $r = s$ and $m = n$

2.$(r, m) + (s, n) = (r + s, m + n)$

3.$(r, m)(s, n) = (rs, rn + sm)$

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Under these definition, $R(M)$ becomes a commutative ring with unity and $R(M)$ is called the *idealization* of a ring R and an R-module M. Sometimes $R(M)$ is also denoted by $R(+)M$. We can find some basic results about $R(M)$ ([8]). An ideal H of $R(+)M$ is called *homogeneous* if $H = I(+)N$ where I is an ideal of R and N a submodule of M. In this case, $I(+)N = (R(+)M)(I(+)N) = I(+)(IM + N)$ gives $IM \subseteq N$. If $IM \subseteq N$, then M/N is an R/I-module and $g : R(+)M \to R/I(+)M/N$ defined by $g(r, m) = (r + I, m + N)$ is a ring homomorphism and $ker(g) = I(+)N$ and hence $I(+)N$ is an ideal of $R(+)M$. So we know that $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Every ideal of $R(+)M$ of the form $I(+)N$ is homogeneous. However, ideals of $R(+)M$ need not have the form $I(+)N$,that is, need not be homogeneous. $Z(+)2Z(2, 2)$ is not a homogeneous ideal of $Z(+)2Z([2])$. In this paper,we show that properties of an ideal I of a ring R are related to those of the homogeneous ideal $I(+)IM$ of a ring $R(+)M$.

2 . Ideals I and $I(+)IM$.

Compare the following Theorem with Proposition 11 in [3]

THEOREM 2.1. *Let R be a ring and M an R-module and I an ideal of R. If $I(+)IM$ is a cancellation ideal of $R(+)M$ then I is a cancellation ideal of R. The converse is true if M is a divisible multiplication module over a domain R with $ann(M \otimes R/\mathcal{A}) = \mathcal{A}$ for every ideal $\mathcal{A}$ of R.*

Proof. Suppose that $I(+)IM$ is a cancellation ideal of $R(+)M$ and let $\mathcal{A}I = \mathcal{B}I$ for ideals $\mathcal{A}, \mathcal{B}$ of R. $(\mathcal{A}(+)M)(I(+)IM) = \mathcal{A}I(+)(\mathcal{A}IM + IM) = \mathcal{A}I(+)IM = \mathcal{B}I(+)IM = \mathcal{B}I(+)(\mathcal{B}IM + IM) = (\mathcal{B}(+)M)(I(+)IM$. Since $I(+)IM$ is a cancellation ideal of $R(+)M$, $\mathcal{A}(+)M = \mathcal{B}(+)M$ and so $\mathcal{A} = \mathcal{B}$. Now we prove the converse.

If M is a divisible module over a domain R, then every ideal of $R(+)M$ is homogeneous ([4]-Theorem 3.3). So, for ideals H, H' of $R(+)M$ such that $H(I(+)IM) = H'(I(+)IM)$ we have $H = \mathcal{A}(+)N$ and $H' = \mathcal{B}(+)K$, where $\mathcal{A}$ and $\mathcal{B}$ are ideals of R and N, K are submodules of M

$H(I(+)IM) = (\mathcal{A}(+)N)(I(+)IM) = \mathcal{A}I(+)(\mathcal{A}IM + IN)$
$= \mathcal{A}I(+)IN$ since $\mathcal{A}M \subseteq N$.

Similarly $H'(I(+)IM) = (\mathcal{B}(+)K)(I(+)IM) = \mathcal{B}I(+)IK$.
Hence $\mathcal{A}I = \mathcal{B}I$ and $IN = IK$. Since I is cancellation, $\mathcal{A} = \mathcal{B}$ and from $N = (N : M)M$ and $K = (K : M)M$ we have $I(N : M)M = I(K : M)M$. On the other hand, $ann(M \otimes R/\mathcal{A}) = \mathcal{A}$ implies that $ann(M/\mathcal{A}M) = \mathcal{A}$ and hence $(\mathcal{A}M : M) = \mathcal{A}$ for any ideal $\mathcal{A}$ of R. Now, let $\mathcal{I}M = \mathcal{J}M$ for ideals $\mathcal{I}, \mathcal{J}$ of R. Then $\mathcal{I} = (\mathcal{I}M : M) = (\mathcal{J}M : M) = \mathcal{J}$. So M is cancellation and $I(N : M) = I(K : M)$. Again, since I is cancellation $(N : M) = (K : M)$. Therefore $N = (N : M)M = (K : M)M = K$ and $\mathcal{A}(+)N = \mathcal{B}(+)K$. i.e, $H = H'$ □

COROLLARY 2.2. *Let R be a ring and M an R-module. If every ideal of $R(+)M$ is cancellation then every ideal of R is cancellation.*

Proof. Suppose that every ideal of $R(+)M$ is cancellation and let $\mathcal{A}$ be any ideal of R. $\mathcal{A}(+)\mathcal{A}M$ is an ideal of $R(+)M$ and so cancellation . By Theorem 2.1 $\mathcal{A}$ is cancellation.

□

COROLLARY 2.3. *Let R be a ring and M a faithful multiplication R-module. If every faithful ideal of $R(+)M$ is cancellation then every faithful ideal of R is cancellation.*

Proof. Let $\mathcal{A}$ be any faithful ideal of R. Since M is a faithful multiplication R-module, we know that ann $(\mathcal{A}(+)\mathcal{A}M)$= $ann\mathcal{A}M$

$(+)(ann\mathcal{A})M$ ([5]-Remark 1) and $ann(\mathcal{A}M) \subseteq ann(\mathcal{A}) = 0$. Therefore ann $(\mathcal{A}(+)\mathcal{A}M)$ =0(+)0. So $(\mathcal{A}(+)\mathcal{A}M)$ is a faithful ideal of $R(+)M$ and by our assumption $(\mathcal{A}(+)\mathcal{A}M)$ is cancellation. Therefore $\mathcal{A}$ is cancellation by the above Theorem 2.1

PROPOSITION 2.4. *Let I be an ideal of R and M an $R-$ module. Then I is idempotent in R if and only if $I(+)IM$ is idempotent in $R(+)M$.*

Proof. Let $I^2 = I$. Then $(I(+)IM)^2 = (I(+)IM)(I(+)IM) = I^2(+)I^2M + I^2M) = (I^2(+)IM) = (I(+)IM)$. So $I(+)IM$ is idempotent. Conversely, if $(I(+)IM$ is idempotent then $I^2(+)IM$ $= I(+)IM$ and so $I^2 = I$. □

Ali([1]) defined idempotent submodule as follows. A submodule N of an R-module M is called *idempotent* if $N = (N : M)N$. If we put $N = I$ for an ideal I of R and $M = R$ then we know that this is a generalized concept of idempotent ideal.

PROPOSITION 2.5. *Let I be an ideal of R and M an $R-$ module. If $I(+)IM$ is an idempotent ideal of $R(+)M$ then IM is an idempotent submodule of M.*

Proof. Since $I(+)IM$ is idempotent, $(I(+)IM)^2 = I^2(+)I^2M = I(+)IM$. Then $I^2M = IM$ and $I^2M = I(IM) \subseteq (IM : M)IM \subseteq IM$. Hence $IM = (IM : M)IM$ and IM is idempotent.

□

PROPOSITION 2.6. *Let I be an ideal of R and M a finitely generated faithful multiplication $R-$ module. Then I is an idempotent ideal of R if and only if IM is an idempotent submodule of M.*

Proof. If I is an idempotent ideal of R then $I(+)IM$ is an idempotent ideal of $R(+)M$. Hence IM is an idempotent submodule of M (Proposition 2.5). Conversely, if IM is an idempotent submodule of M then $IM = (IM : M)IM$ and by our assumption $IM = I(IM : M)M = I^2M$. Since M is cancellation module([9]-Theorem 6.1) $I = I^2$ □

Ali([1]) introduced the concept of nilpotent submodule which is a generalized concept of nilpotent ideal. A submodule N of M is called a *nilpotent submodule* if $(N : M)^k N = 0$

PROPOSITION 2.7. *Let I be an ideal of R and M an $R-$ module. Then I is nilpotent in R if and only if $I(+)IM$ is nilpotent in $R(+)M$.*

Proof. Let $I^n = 0$ for some positive integer n. Then $(I(+)IM)^n = (I^n(+)I^nM) = 0(+)0$. Hence $I(+)IM$ is nilpotent. Conversely, if $I(+)IM$ is nilpotent then there exists a positive integer k such that $(I(+)IM)^k = 0(+)0$ and hence $I^k = 0$. □

PROPOSITION 2.8. *Let I be an ideal of R and M a finitely generated faithful multiplication $R-$ module. If $I(+)IM$ is a nilpotent ideal of $R(+)M$ then IM is a nilpotent submodule of M.*

Prof. By our assumption there exists a positive integer k such that $(I(+)IM)^k = 0(+)0$. So $I^kM = 0$. Since M is cancellation module([9]-Theorem 6.1) and faithful, $I = (IM : M)$([9]-Proposition 1.4) and hence $(IM : M)^k IM = I^k IM = I(I^kM) = 0$. Therefore IM is nilpotent. □

PROPOSITION 2.9. *Let I be an ideal of R and M a faithful multiplication $R-$ module. Then I is a nilpotent ideal of R if and only if IM is a nilpotent submodule of M.*

Proof. Suppose that IM is nilpotent. Then $(IM : M)^k IM = 0$ for some positive integer k. So $(IM : M)^{k-1}(IM : M)IM = 0$. Since M is a multiplication module, $(IM : M)^{k-1}I^2M = 0$.

$(IM : M)^{k-2}(IM : M)I^2M = 0$ and hence $(IM : M)^{k-2}I^3M = 0$. Continue this way. Then we arrive at $(IM : M)I^kM = I^k(IM : M)M = I^{k+1}M = 0$. Since M is faithful, $I^{k+1} = 0$. □

An ideal $\mathcal{A}$ of a ring R is said to be *regular* if it contains a nonzero divisor element.

THEOREM 2.10. *Let I be an ideal of a ring R and M an $R-$ module. If $I(+)IM$ is a regular ideal of $R(+)M$ then I is a regular ideal of R. The converse is true if M is torsion free.*

Proof. Suppose that $I(+)IM$ is regular and let $(i, m) \in I(+)IM$ be a regular element in $R(+)M$. To show that $i \in I$ is a regular element in R, let $ij = 0$ for $j \in R$. Then $(0, jm)(i, m) = (0, ijm) = (0, 0)$. Since (i, m) is regular, $jm = 0$ and $(j, 0)(i, m) = (0, 0)$. Thus $j = 0$. i.e, $i \in I$ is regular and I is regular. Conversely, suppose that M is torsion free and $i \in I$ is regular. Consider an element $(i, m') \in I(+)IM$ and let $(j, n)(i, m') = (0, 0)$ for an element $(j, n) \in R(+)M$. Then $ji = 0, jm' + in = 0$. Since i is regular, $j = 0$ and hence $0 = jm' + in = in$. Thus $n = 0$ because M is torsion free. So (i, m') is regular in $R(+)M$ and $I(+)IM$ is regular. □

THEOREM 2.11. *Let I be a nonzero ideal of a ring R and M an $R-$ module. If $I(+)IM$ is an invertible ideal of $R(+)M$ then I is an invertible ideal of R. The converse is true if M is faithful and multiplication.*

Proof. In a ring the concepts of invertible ideal and regular multiplication ideal coincide ([7]-Theorem 7.2). Therefore, if $I(+)IM$ is invertible, $I(+)IM$ is both regular and multiplication ideal. So, I is a multiplication ideal([2]-Theorem 7). Further I is regular by Theorem 2.10. Therefore I is invertible. Conversely, If I is invertible then I is regular and multiplication and so, $I(+)IM$

is a multiplication ideal([2]-Theorem 7). Since M is faithful multiplication, M is torsion free ([6]-Lemma 4.1) and since I is regular, $I(+)IM$ is regular by Theorem 2.10. Therefore $I(+)IM$ invertible. □

A ring R is *presimplifiable* if for $x, y \in R$, $xy = x$ implies $x = 0$ or y is a unit. $R-$ module M is *R-presimplifiable* if for $r \in R$ and $m \in M$, $rm = m$ implies r is a unit or $m = 0$. This generalizes the previous definition of R being presimplifiable.

THEOREM 2.12. *Let I be an ideal of a ring R and M an $R-$ module. Then, I and IM are R-presimplifiable if and only if $I(+)IM$ is $R(+)M$-presimplifible.*

Proof. Suppose that I and IM are R-simplifiable. Let (r, m) $(i, m') = (i, m')$, where $i \in I, m' \in IM, r \in R$ and $m \in M$. Then $ri = i$ and $rm' + im = m'$. Since I is R-presimplifiable $r \in U(R)$, the set of all unit elements in R or $i = 0$. However $U(R(+)M) = U(R)(+)M$([4]-Theorem 3.7) and hence if $r \in U(R)$ then $(r, m) \in U(R(+)M)$ and if $i = 0$ then $rm' = m'$. Since IM is $R-$presimplifiable $m' = 0$ and $(i, m') = (0, 0)$ or $(r, m) \in U(R)(+)M = U(R(+)M)$. In any case we have $(r, m) \in U(R(+)M)$ or $(i, m') = (0, 0)$. Conversely, Assume that $I(+)IM$ is $R(+)M$-presimplifiable. Let $ri = i$ for $r \in R$ and $i \in I$. Then $(r, 0)(i, 0) = (i, 0)$ and $(r, 0) \in U(R(+)M)$ or $(i, 0) = (0, 0)$. Therefore $r \in U(R)$ or $i = 0$. □

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