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ON STABILITY OF EINSTEIN WARPED PRODUCT MANIFOLDS

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Abstract. Let $(B, \check{g})$ and $(N, \hat{g})$ be Einstein manifolds. Then, we get a complete (necessary and sufficient) condition for the warped product manifold $B \times_f N := (B \times N, \check{g} + f\hat{g})$ to be Einstein, and obtain a complete condition for the Einstein warped product manifold $B \times_f N$ to be weakly stable. Moreover, we get a complete condition for the map $i : (B, \check{g}) \times (N, \hat{g}) \to B \times_f N$, which is the identity map as a map, to be harmonic. Under the assumption that i is harmonic, we obtain a complete condition for $B \times_f N$ to be Einstein.

1. Introduction

A *harmonic map* ϕ from a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) is a critical point of the energy functional ([5, 9])

$$E(\phi) := \int_M e(\phi)\ v_g, \tag{1.1}$$

where $e(\phi) = \frac{1}{2}h(d\phi, d\phi)$. The second variation formula of the energy functional E for a harmonic map ϕ is given as follows ([6, 8, 9]):

$$H(E)_\phi(V, V) := \frac{d^2}{dt^2}E(\phi_t)|_{t=0} = \int_M h(V, J_\phi V)v_g, \tag{1.2}$$

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where J_ϕ is the Jacobi operator acting on $\Gamma(\phi^{-1}TN)$ and

$$V_p := (d\phi_t(p)/dt)|_{t=0}, \quad p \in M.$$

Then ϕ is said to be *stable* (resp. *weakly stable*) if $H(E)_\phi(V,V) > 0$ (resp. ≥ 0) for all $V \in \Gamma(\phi^{-1}TN)$, and otherwise, is said to be *unstable*.

To construct a harmonic map between two Riemannian manifolds and to show the stability of a given harmonic map are very important topics in the study on the theory of harmonic maps.

In this paper, let $(B, \check{g})$ and $(N, \hat{g})$ be two Riemannian manifolds. Then, $B \times_f N := (B \times N, \check{g} + f\hat{g})$, ($f$ being a positive C^∞-function on B), is said to be a *warped product manifold* ([1]) of $(B, \check{g})$ and $(N, \hat{g})$. We assume that $(B, \check{g})$ and $(N, \hat{g})$ are compact Einstein manifolds. Then, we obtain a necessary and sufficient condition for $B \times_f N$ to be Einstein (cf. Theorem 2.3). And then, using R. T. Smith's Stability Theorem, we get a necessary and sufficient condition for such an Einstein warped product manifold to be weakly stable (cf. Theorem 2.5), and get a sufficient condition for the Einstein warped product manifold to be stable (cf. Corollary 2.6). Moreover, we get a complete condition (cf. Proposition 2.1) for the map $i : (B, \check{g}) \times (N, \hat{g}) \to B \times_f N$, being the identity map as a map, to be harmonic. Under the assumption that i is harmonic, we obtain a complete condition (cf. Theorem 2.2) for $B \times_f N$ to be Einstein.

2. Main results

Let $(B^m, \check{g})$ and $(N^n, \hat{g})$ be two Riemannian manifolds. And let $B \times_f N := (B \times N, \check{g} + f\hat{g})$ be the warped product of B and N by the positive smooth function f on B. Let $\{\mathbf{b}_i\}_{i=1}^m$ and $\{\mathbf{n}_\alpha\}_{\alpha=1}^n$ be an (locally defined) orthonormal frames on $(B^m, \check{g})$ and $(N^n, \hat{g})$, respectively. We put $\mathbf{d}_\alpha := f^{-1/2}\, \mathbf{n}_\alpha$. Then

$$\{(\mathbf{b}_i, \mathbf{0}), (\mathbf{0}, \mathbf{d}_\alpha) \mid i = 1, 2, \cdots, m;\ \alpha = 1, 2, \cdots, n\} \tag{2.1}$$

is an (locally defined) orthonormal frame on $B \times_f N$. From now on, we simply denote $\mathbf{b}_i := (\mathbf{b}_i, \mathbf{0})$, $\mathbf{d}_\alpha := (\mathbf{0}, \mathbf{d}_\alpha)$. Let $\{\check{\theta}^i\}_{i=1}^m$, $\{\hat{\theta}^\alpha\}_{\alpha=1}^n$

and $\{\theta^i, \theta^\alpha\}_{i,\alpha}$ be the dual frames of $\{\mathbf{b}_i\}_{i=1}^m$, $\{\mathbf{n}_\alpha\}_{\alpha=1}^n$ and $\{\mathbf{b}_i, \mathbf{d}_\alpha\}_{i,\alpha}$, respectively.

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by ([2, 3, 4])

$$
\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
&\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])
\end{aligned} \tag{2.2}
$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Let $\check{\nabla}$, $\hat{\nabla}$ and ∇ be the Levi-Civita connections on $(B, \check{g})$, $(N, \hat{g})$ and $B \times_f N$, respectively. We introduce the notations $\check{\Gamma}^i_{jk}$ and $\hat{\Gamma}^\alpha_{\beta\gamma}$ such that $\check{\Gamma}^i_{jk} := \check{\theta}^i(\check{\nabla}_{\mathbf{b}_j}\mathbf{b}_k)$ and $\hat{\Gamma}^\alpha_{\beta\gamma} := \hat{\theta}^\alpha(\hat{\nabla}_{\mathbf{n}_\beta}\mathbf{n}_\gamma)$. We put

$$
\nabla_{\mathbf{b}_i}\mathbf{b}_j = \sum_{k=1}^m \Gamma^k_{ij}\mathbf{b}_k + \sum_{\alpha=1}^n \Gamma^\alpha_{ij}\mathbf{d}_\alpha, \quad \nabla_{\mathbf{b}_i}\mathbf{d}_\alpha = \sum_{k=1}^m \Gamma^k_{i\alpha}\mathbf{b}_k + \sum_{\gamma=1}^n \Gamma^\gamma_{i\alpha}\mathbf{d}_\gamma,
$$

$$
\nabla_{\mathbf{d}_\alpha}\mathbf{b}_i = \sum_{k=1}^m \Gamma^k_{\alpha i}\mathbf{b}_k + \sum_{\gamma=1}^n \Gamma^\gamma_{\alpha i}\mathbf{d}_\gamma, \quad \nabla_{\mathbf{d}_\alpha}\mathbf{d}_\beta = \sum_{k=1}^m \Gamma^k_{\alpha\beta}\mathbf{b}_k + \sum_{\gamma=1}^n \Gamma^\gamma_{\alpha\beta}\mathbf{d}_\gamma.
$$

Using (2.2) and the above equations, we get

$$
\begin{aligned}
&\Gamma^k_{ij} = \check{\Gamma}^k_{ij}, \quad \Gamma^\alpha_{ij} = \Gamma^j_{i\alpha} = \Gamma^\gamma_{i\alpha} = \Gamma^k_{\alpha i} = 0, \\
&\Gamma^\beta_{\alpha i} = -\Gamma^i_{\alpha\beta} = \frac{1}{2} f^{-1}\mathbf{b}_i(f)\ \delta_{\alpha\beta}, \quad \Gamma^\gamma_{\alpha\beta} = f^{-\frac{1}{2}}\ \hat{\Gamma}^\gamma_{\alpha\beta},
\end{aligned} \tag{2.3}
$$

that is

$$
\begin{aligned}
&\nabla_{\mathbf{b}_i}\mathbf{b}_j = \check{\nabla}_{\mathbf{b}_i}\mathbf{b}_j = \sum_{k=1}^m \check{\Gamma}^k_{ij}\ \mathbf{b}_k, \\
&\nabla_{\mathbf{b}_i}\mathbf{d}_\alpha = \mathbf{0}, \quad \nabla_{\mathbf{d}_\alpha}\mathbf{b}_i = -\frac{1}{2} f^{-1}\mathbf{b}_i(f)\ \mathbf{d}_\alpha, \\
&\nabla_{\mathbf{d}_\alpha}\mathbf{d}_\beta = -\frac{1}{2}\ \delta_{\alpha\beta}\ \sum_{i=1}^m \mathbf{b}_i(f)\ \mathbf{b}_i + f^{-\frac{1}{2}} \sum_{\gamma=1}^n \hat{\Gamma}^\gamma_{\alpha\beta}\ \mathbf{d}_\gamma \\
&\qquad\qquad = -\frac{1}{2}\ \delta_{\alpha\beta}\ \sum_{i=1}^m \mathbf{b}_i(f)\ \mathbf{b}_i + f^{-1}\hat{\nabla}_{\mathbf{n}_\alpha}\mathbf{n}_\beta.
\end{aligned} \tag{2.4}
$$

From (2.4) and $T^{\nabla} = \mathbf{0}$ (i.e. ∇ is torsion-free), we have

$$(2.5)\quad [\mathbf{b}_i, \mathbf{d}_\alpha] = -\frac{1}{2} f^{-1}\, \mathbf{b}_i(f)\, \mathbf{d}_\alpha,\ [\mathbf{d}_\alpha, \mathbf{d}_\beta] = f^{-\frac{1}{2}} \sum_{\gamma=1}^{n} \left(\hat{\Gamma}^{\gamma}_{\alpha\beta} - \hat{\Gamma}^{\gamma}_{\beta\alpha}\right) \mathbf{d}_\gamma.$$

Let (M, g), (N, h) be two Riemannian manifolds. Let $\phi : M \longrightarrow N$ be a smooth map. Let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N. We denote by $\Gamma(E)$, the space of all sections V of E. We denote by ∇, ${}^N\nabla$ the Levi-Civita connections of (M, g), (N, h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(\tilde{\nabla}_X V)_x := \frac{d}{dt}\ {}^N P_{\phi(\gamma(t))}{}^{-1} V_{\gamma(t)}|_{t=0}, \quad X \in \Gamma(TM),\ V \in \Gamma(E),$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \longrightarrow T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$, $0 \le s \le t$, given by the Levi-Civita connection ${}^N\nabla$ of (N, h).

For a C^∞-map ϕ of an m-dimensional compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), the following is well known (cf. [5, 7, 9]) : the map ϕ is harmonic if and only if $\tau(\phi) = 0$ on M, where

$$(2.6)\qquad \tau(\phi) := \sum_{i=1}^{m} \left\{ \tilde{\nabla}_{\mathbf{e}_i} \phi_* \mathbf{e}_i - \phi_*({}^M\nabla_{\mathbf{e}_i} \mathbf{e}_i) \right\}$$

for $\{\mathbf{e}_i\}_{i=1}^m$ an (locally defined) orthonormal frame on (M, g).

From (2.6), we obtain the fact that a necessary and sufficient condition for the identity map $i : (B, \breve{g}) \times (N, \hat{g}) \longrightarrow B \times_f N$ to be harmonic is

$$(2.7)\qquad \sum_{\alpha=1}^{n} \left(\nabla_{\mathbf{n}_\alpha} \mathbf{n}_\alpha - \hat{\nabla}_{\mathbf{n}_\alpha} \mathbf{n}_\alpha \right) = \mathbf{0}.$$

On the other hand, we get

$$(2.8)\qquad \nabla_{\mathbf{n}_\alpha} \mathbf{n}_\beta = f \nabla_{\mathbf{d}_\alpha} \mathbf{d}_\beta = f \sum_{\gamma=1}^{n} \Gamma^{\gamma}_{\alpha\beta} \mathbf{d}_\gamma.$$

Moreover, using (2.4), we have

$$\hat{\nabla}_{\mathbf{n}_\alpha}\mathbf{n}_\beta = f\left\{\nabla_{\mathbf{d}_\alpha}\mathbf{d}_\beta + \frac{1}{2}\delta_{\alpha\beta}\sum_{i=1}^{m}\mathbf{b}_i(f)\mathbf{b}_i\right\}. \tag{2.9}$$

By virtue of (2.3), (2.7), (2.8) and (2.9), we obtain

Proposition 2.1. *Let $B \times_f N$ be the warped product Riemannian manifold of $(B^m, \check{g})$ and $(N^n, \hat{g})$, respectively. Assume that $i : (B, \check{g}) \times (N, \hat{g}) \to B \times_f N$ is the identity map as a map. Then, the following statements are equivalent*:

(a) *i is harmonic*;
(b) $\sum_{\alpha=1}^{n}\nabla_{\mathbf{n}_\alpha}\mathbf{n}_\alpha = \sum_{\alpha=1}^{n}\hat{\nabla}_{\mathbf{n}_\alpha}\mathbf{n}_\alpha$;
(c) *f is constant on B*;
(d) $\sum_{\alpha=1}^{n}\Gamma^k_{\alpha\alpha} = 0$ *for each k ($k = 1, 2, \cdots, m$).*

From (2.3), (2.4), (2.5) and $R^{\nabla}(X,Y)Z := [\nabla_X, \nabla_Y](Z) - \nabla_{[X,Y]}Z$ for $X, Y, Z \in \mathfrak{X}(B \times N)$, we get

$$\begin{aligned}
&R^{\nabla}(\mathbf{b}_i,\mathbf{b}_j)\mathbf{b}_k = R^{\check{\nabla}}(\mathbf{b}_i,\mathbf{b}_j)\mathbf{b}_k, \quad R^{\nabla}(\mathbf{b}_i,\mathbf{b}_j)\mathbf{d}_\alpha = 0,\\
&R^{\nabla}(\mathbf{b}_i,\mathbf{d}_\alpha)\mathbf{b}_j = \frac{1}{4}\{2f^{-1}\mathbf{b}_i(\mathbf{b}_j(f)) - 2f^{-1}\sum_{k=1}^{m}\check{\Gamma}^k_{ij}\mathbf{b}_k(f)\\
&\qquad\qquad - f^{-2}\mathbf{b}_i(f)\ \mathbf{b}_j(f)\}\mathbf{d}_\alpha,\\
&R^{\nabla}(\mathbf{b}_i,\mathbf{d}_\alpha)\mathbf{d}_\beta = \frac{1}{4}\sum_{j=1}^{m}\{f^{-2}\mathbf{b}_i(f)\mathbf{b}_j(f) - 2f^{-1}\mathbf{b}_i(\mathbf{b}_j(f))\\
&\qquad\qquad - 2f^{-1}\sum_{k=1}^{m}\check{\Gamma}^j_{ik}\mathbf{b}_k(f)\}\delta_{\alpha\beta}\mathbf{b}_j,\\
&R^{\nabla}(\mathbf{d}_\alpha,\mathbf{d}_\beta)\mathbf{b}_i = 0,\\
&R^{\nabla}(\mathbf{d}_\alpha,\mathbf{d}_\beta)\mathbf{d}_\gamma = \hat{R}(\mathbf{d}_\alpha,\mathbf{d}_\beta)\mathbf{d}_\gamma\\
&\qquad\qquad + \frac{1}{4}f^{-2}\sum_{i=1}^{m}(\mathbf{b}_i(f))^2(\delta_{\alpha\gamma}\ \mathbf{d}_\beta - \delta_{\beta\gamma}\ \mathbf{d}_\alpha).
\end{aligned} \tag{2.10}$$

The *Ricci tensor field* Ric^{∇} of type (0,2) is defined by

$$Ric^{\nabla}(Y,Z) := trace\{X \to R^{\nabla}(X,Y)Z\} \quad (X,Y,Z \in \mathfrak{X}(B \times N)). \tag{2.11}$$

Now, we assume $(B, \check{g})$ and $(N, \hat{g})$ are Einstein manifolds such that

$$Ric^{\check{\nabla}} = \check{c}\check{g}, \quad Ric^{\hat{\nabla}} = \hat{c}\hat{g}. \tag{2.12}$$

From (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
Ric^{\nabla}(\mathbf{b}_i, \mathbf{b}_j) &= \check{c}\ \delta_{ij} + \frac{n}{4}\{f^{-2}\ \mathbf{b}_i(f)\ \mathbf{b}_j(f) \\
&\quad - 2f^{-1}\mathbf{b}_i(\mathbf{b}_j(f)) + 2f^{-1}\sum_{k=1}^{m}\check{\Gamma}_{ij}^{k}\mathbf{b}_k(f)\}, \\
Ric^{\nabla}(\mathbf{b}_i, \mathbf{d}_\alpha) &= 0, \\
Ric^{\nabla}(\mathbf{d}_\alpha, \mathbf{d}_\beta) &= \frac{1}{4}\{(2-n)f^{-2}\ \sum_{i=1}^{m}(\mathbf{b}_i(f))^2 \\
&\quad - 2f^{-1}\sum_{i=1}^{m}\mathbf{b}_i(\mathbf{b}_i(f)) + 2f^{-1}\sum_{i,k=1}^{n}\check{\Gamma}_{ii}^{k}\ \mathbf{b}_k(f)\}\delta_{\alpha\beta} \\
&\quad + f^{-1}\hat{c}\delta_{\alpha\beta}.
\end{aligned} \tag{2.13}$$

Then, we obtain the following

Theorem 2.2. *Let $(B^m, \check{g})$ and $(N^n, \hat{g})$ be Einstein manifolds such that $Ric^{\check{\nabla}} = \check{c}\check{g}$ and $Ric^{\hat{\nabla}} = \hat{c}\hat{g}$. Suppose that the identity map $i : (B \times N, \check{g} + \hat{g}) \to B \times_f N$ is harmonic. Then $B \times_f N$ be Einstein if and only if $\hat{c} = f\check{c}$.*

Proof. Assume i is harmonic. Then, from Proposition 2.1 we obtain the function f is constant on B.

Suppose that $B \times_f N$ is Einstein such that $Ric^{\nabla} = cg$ for a constant c. Then we get from the first formula of (2.13)

$$c\delta_{jk} = Ric^{\nabla}(\mathbf{b}_j, \mathbf{b}_k) = \check{c}\delta_{jk}. \tag{2.14}$$

Moreover, we have from the last formula of (2.13)

$$c\delta_{\alpha\beta} = Ric^{\nabla}(\mathbf{d}_\alpha, \mathbf{d}_\beta) = f^{-1}\ \hat{c}\ \delta_{\alpha\beta}. \tag{2.15}$$

By virtue of (2.14) and (2.15), we obtain

$$c = \check{c} = f^{-1}\ \hat{c}. \tag{2.16}$$

Conversely, let $\hat{c} = f\check{c}$. Then we get from (2.13)

$$\begin{cases} Ric^{\nabla}(\mathbf{b}_i, \mathbf{b}_j) = \check{c}\delta_{ij} = \check{c}g(\mathbf{b}_i, \mathbf{b}_j) \\ Ric^{\nabla}(\mathbf{b}_i, \mathbf{d}_\alpha) = 0 = cg(\mathbf{b}_i, \mathbf{d}_\alpha) \\ Ric^{\nabla}(\mathbf{d}_\alpha, \mathbf{d}_\beta) = f^{-1}\hat{c}\delta_{\alpha\beta} = \check{c}g(\mathbf{d}_\alpha, \mathbf{d}_\beta) \end{cases}$$

since f is constant. Hence $Ric^{\nabla} = \check{c}g$, and then $B \times_f N$ is Einstein. □

The Laplacian Δ_g of an n-dimensional Riemannian manifold (M, g) is given by $\Delta_g := -\sum_{i=1}^{n}({e_i}^2 - \nabla_{e_i}e_i)$, where $\{e_i\}_{i=1}^{n}$ is an (locally defined) orthonormal frame on (M, g)). We denote the spectrum $Spec(\Delta_g)$ of Δ_g of a compact Riemannian manifold (M, g) is denoted by ([8, 9])

$$Spec(\Delta_g) = \{\lambda_0(g) = 0 \leq \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq\uparrow \infty\}.$$

On the other hand, we get the following

Theorem 2.3. *Let $(B^m, \check{g})$ and $(N^n, \hat{g})$ be Einstein manifolds such that $Ric^{\check{\nabla}} = \check{c}\check{g}$ and $Ric^{\hat{\nabla}} = \hat{c}\hat{g}$. Then $B \times_f N$ is Einstein if and only if*

$$\text{(i)} \quad \begin{aligned} &4\check{c} + n\left\{f^{-2}(\mathbf{b}_j(f))^2 - 2f^{-1}\mathbf{b}_j(\mathbf{b}_j(f)) + 2f^{-1}\sum_{i=1}^{m}\check{\Gamma}^i_{jj}\mathbf{b}_i(f)\right\} \\ &\qquad = 2f^{-1}(2\hat{c} + \Delta_{\check{g}}f) + (2-n)f^{-2}||df||^2_{\check{g}} \end{aligned}$$

for each j, and

$$\text{(ii)} \quad \mathbf{b}_j(f)\mathbf{b}_k(f) - 2f\ \mathbf{b}_j(\mathbf{b}_k(f)) + 2f\sum_{i=1}^{m}\check{\Gamma}^i_{jk}\mathbf{b}_i(f) = 0$$

for $j, k \quad (j \neq k)$ are hold.

Proof. The warped product manifold $B \times_f N$ is Einstein if and only if

$$Ric^{\nabla} = cg \tag{2.17}$$

for some constant c. From (2.13) and (2.17), we get the fact that (2.17) holds if and only if

$$\begin{aligned} 4c\delta_{jk} &= 4\check{c}\delta_{jk} + n\{f^{-2}\mathbf{b}_j(f)\mathbf{b}_k(f) - 2f^{-1}\mathbf{b}_j(\mathbf{b}_k(f)) \\ &\qquad + 2f^{-1}\sum_{i=1}^{m}\check{\Gamma}^i_{jk}\mathbf{b}_i(f)\}, \\ 4c &= 2f^{-1}(2\hat{c} + \Delta_{\check{g}}f) + (2-n)f^{-2}||df||^2_{\check{g}}. \end{aligned} \tag{2.18}$$

□

In order to show the stability of Einstein manifolds, we introduce R.T. Smith's stability theorem :

Theorem 2.4[9]. *Let (M, g) be a compact Einstein Riemannian manifold such that the Ricci tensor ρ satisfies $\rho = cg$. Then, the identity map of (M, g) is weakly stable if and only if the first positive eigenvalue of the Laplacian Δ_g acting on $C^\infty(M)$, $\lambda_1(g)$, satisfies $\lambda_1(g) \geq 2c$.*

If, for a Riemannian manifold (M, g), the identity map of (M, g) is stable (resp. unstable) as a harmonic map, then the manifold (M, g) is said to be *stable* (resp. *unstable*).

Now, we obtain the following

Theorem 2.5. *Let $(B^m, \check{g})$ and $(N^n, \hat{g})$ be Einstein manifolds such that $m \neq n$, $Ric^{\check{\nabla}} = \check{c}\check{g}$ and $Ric^{\hat{\nabla}} = \hat{c}\hat{g}$. Suppose that $B \times_f N$ is Einstein. Then, the warped product manifold $B \times_f N$ is weakly stable if and only if*

$$\lambda_1(g) \geq \frac{2}{m-n}\left\{m\check{c} - nf^{-1}\hat{c} + \frac{1}{4}f^{-2}(n^2 - n)||df||_{\check{g}}{}^2\right\},$$

where $\lambda_1(g)$ is the first positive eigenvalue of the Laplacian Δ_g of the warped product manifold $B \times_f N$.

Proof. Assume $(B, \check{g})$, $(N, \hat{g})$ and $B \times_f N$ are Einstein. Then, summing over j which is appeared in the condition (i) of Theorem 2.3, we have

$$4m\check{c} + n\sum_{j=1}^{m}\left\{f^{-2}(\mathbf{b}_j(f))^2 - 2f^{-1}\mathbf{b}_j(\mathbf{b}_j(f)) + 2f^{-1}\sum_{k=1}^{m}\check{\Gamma}_{jj}^{k}\mathbf{b}_k(f)\right\}$$
$$= m\left\{2f^{-1}(2\hat{c} + \Delta_{\check{g}}f) - (n-2)f^{-2}||df||_{\check{g}}{}^2\right\}.$$

From the above equation, we obtain

$$4m\check{c} + n\left\{f^{-2}||df||_{\check{g}}^2 + 2f^{-1}\Delta_{\check{g}}f\right\}$$
$$= m\left\{2f^{-1}(2\hat{c} + \Delta_{\check{g}}f) - (n-2)f^{-2}||df||_{\check{g}}{}^2\right\},$$

and hence

$$2(m-n)f^{-1}\Delta_{\check{g}}f = 4m(\check{c} - f^{-1}\hat{c}) + \{m(n-2)+n\}f^{-2}||df||_{\check{g}}{}^2\ . \tag{2.19}$$

From the fact that $B \times_f N$ is Einstein, we get

$$Ric^{\nabla} = cg \text{ for some constant } c. \tag{2.20}$$

By the help of (2.19),(2.20) and the second formula of (2.18), we obtain

$$c = (m-n)^{-1}\left\{m\check{c} - nf^{-1}\hat{c} + \tfrac{1}{4}f^{-2}(n^2-n)||df||_{\check{g}}{}^2\right\}. \tag{2.21}$$

By virtue of (2.21) and Theorem 2.4, the proof of this theorem is completed. □

By the help of Theorems 2.4 and 2.5, we get

Corollary 2.6. *Let $(B^m, \check{g})$ and $(N^n, \hat{g})$ be Einstein manifolds such that $m \neq n$, $Ric^{\check{\nabla}} = \check{c}\check{g}$ and $Ric^{\hat{\nabla}} = \hat{c}\hat{g}$. Suppose that $B \times_f N$ is Einstein. Then, if*

$$\frac{1}{m-n}\left\{m\check{c} - nf^{-1}\hat{c} + \frac{1}{4}f^{-2}(n^2-n)\|\ df\ \|_{\check{g}}{}^2\right\} \leq 0,$$

the warped product manifold $B \times_f N$ is stable.

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