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COMMUTATIVE MONOID OF THE SET OF k-ISOMORPHISM CLASSES OF SIMPLE CLOSED k-SURFACES IN $\mathbf{Z}^3$

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Abstract. In this paper we prove that with some hypothesis the set of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$ forms a commutative monoid with an operation derived from a digital connected sum, $k \in \{18, 26\}$. Besides, with some hypothesis the set of k-homotopy equivalence classes of closed k-surfaces in $\mathbf{Z}^3$ is also proved to be a commutative monoid with the above operation, $k \in \{18, 26\}$.

1. Introduction

In order to study some properties of the set of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$, we need to recall some notions, as follows. In algebra, a monoid is defined to be a set X with a binary operation $* : X \times X \to X$, obeying the following axioms:
• $(X, *)$ has the associative law,
• there is an element $e \in X$ such that for any element $x \in X$ $x * e = e * x = x$ and further,
• if $x * y = y * x$ for any elements $x, y \in X$, then we say that $(X, *)$ is a commutative monoid.

Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers, respectively. Let $\mathbf{Z}^n$ be the set of lattice points in Euclidean n-dimensional space, $n \in \mathbf{N}$. In [27] a closed k-surface was studied in $\mathbf{Z}^3$, $k \in \{6, 26\}$ and in [1] a closed 18-surface was introduced in $\mathbf{Z}^3$. Besides, the study of

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various properties of a closed k-surface in $\mathbf{Z}^3$ and digital space includes the papers [1, 9, 10, 14, 25].

The connected sum in geometric topology cannot be available in discrete (or digital) geometry. Thus we need its digital version to study a digital k-surface. Motivated by the notion of connected sum in geometric topology, its digital version was established in [9] (see also [7, 14]). Thus, the notion of digital connected sum of two simple closed k-surfaces was introduced and further, its digital topological properties were partially studied [9, 15]. In [5] a geometric realization of a digital space $X \subset \mathbf{Z}^3$ has been introduced. Moreover, in [15] the Euler characteristic of a digital space was studied in relation with a digital connected sum in [9] (see also [15]). In [14] two types of simple, closed 18-surfaces in $\mathbf{Z}^3$ were introduced. One is 18-contractible, denoted by MSS'_{18} (see 3.1) and the other is not 18-contractible, denoted by MSS_{18} (see 3.1). Especially, MSS'_{18} plays an important role in establishing the monoid structure of the set of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$.

In this paper we prove that with some hypothesis the set of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$ forms a commutative monoid with an operation derived from a digital connected sum in [9], $k \in \{18, 26\}$. Besides, we prove that both MSS'_{18} and MSS_{18} are 26-surfaces and further, MSS'_{18} is proved 26-contractible. Moreover, k-contractibility of MSS'_k allows us to establish a commutative monoid of the set of k-isomorphism classes of simple closed k-surfaces with an operation derived from a digital connected sum, $k \in \{18, 26\}$. In other words, the k-isomorphism class of MSS'_k, denoted by $[MSS'_k]$, is proved to be the identity element for the above-mentioned monoid, $k \in \{18, 26\}$. Similarly, with some hypothesis we also form another commutative monoid of the set of k-homotopy classes of closed k-surfaces in $\mathbf{Z}^3$, $k \in \{18, 26\}$. This kinds of two monoids of the sets of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$ and k-homotopy equivalence classes of closed k-surfaces can be used in classifying simple closed k-surfaces in $\mathbf{Z}^3$

This paper is organized as follows. Section 2 provides basic notions. Section 3 investigates some properties of a closed k-surface and a relative k-homotopy, $k \in \{18, 26\}$. Section 4 establishes a commutative monoid of the set of k-isomorphism classes of closed k-surfaces with an operation derived from a digital connected sum, $k \in \{18, 26\}$. Section 5 shows that with some hypothesis the set of k-homotopy equivalence classes of closed k-surfaces with an operation forms a commutative monoid, $k \in \{18, 26\}$.

2. Preliminaries

In order to make this paper self-contained, we recall some necessary terminology from earlier literature in [1, 3, 25, 28]. Since a closed k-surface in $\mathbf{Z}^3$ can be studied with a digital k-graph structure in $\mathbf{Z}^3$, we now use the $k(m,n)$ (or k_m)-adjacency relations of $\mathbf{Z}^n, n \in \mathbf{N}$ [8] (see also [12]):
Let m be a positive integer with $1 \leq m \leq n$. Then we say that two distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n$ are $k(m,n)$-*adjacent* according to m if
(1) there are at most m distinct indices i such that $|p_i - q_i| = 1$; and
(2) for all indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In terms of this operator the number m determines one of the $k(m,n)$-adjacency relations of $\mathbf{Z}^n$, we may use $k := k(m,n)$. Precisely, by $N_k^*(p)$ we denote the set of the points $q \in \mathbf{Z}^n$ which are k_m-adjacent to a given point p and the number $k := k(m,n)$ is the cardinal number of $N_k^*(p)$. Consequently, we obtain the following k-adjacency relations of $\mathbf{Z}^n$ [8] (see also [9, 15]).

Proposition 2.1. [19] $k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n$, *where* $C_i^n = \frac{n!}{(n-i)!\ i!}$.

In general, for a subset $X \subset \mathbf{Z}^n$ with k-adjacency, $n \in \mathbf{N}$, we call it *a digital space with k-adjacency*, denoted by (X,k), and further, (X,k) is usually considered in a digital picture $(\mathbf{Z}^n, k, \bar{k}, X)$ [27, 28], k and $\bar{k}$ are related to the adjacencies of X and $\mathbf{Z}^n - X$, respectively. In this paper, we assume $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$. Hereafter, we call briefly (X,k) a *space* if not confused. Owing to the *digital k-connectivity paradox* in [26], we commonly assume that $k \neq \bar{k}$ except for the case $(\mathbf{Z}, 2, 2, X)$. For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \leq n \leq b\}$ is called a *digital interval* [3].

A digital space (X,k) is a digital graph G_k [13] (see also [15, 16, 18]). To be specific, the vertex set of G_k can be considered as the set of points of X. Besides, two points $x_1, x_2 \in X$ determine a k-edge of G_k if and only if x_1 and x_2 are k-adjacent in X.

A k-path from x to y in X is a sequence $(x = x_0, x_1, x_2, \cdots, x_{m-1}, x_m = y)$ in X such that each point x_i is k-adjacent to x_{i+1} for $m \geq 1$ and $i \in [0, m-1]_{\mathbf{Z}}$. Then, the number m is called the *length* of this path [26]. If $x_0 = x_m$, then the k-path is said to be *closed* [26]. A set of lattice points is *k-connected* if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [25]. Thus a singleton set with k-adjacency is k-connected. For a digital space (X, k), two distinct points $x, y \in X$ are k-connected [22] if there is a k-path from x to y in X. For an adjacency relation k of $\mathbf{Z}^n$, a *simple k-path* with m elements in $\mathbf{Z}^n$ is assumed to be a sequence $(x_i)_{i\in[0,m-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [25]. Furthermore, a *simple closed k-curve* with l elements in $\mathbf{Z}^n$ is a sequence $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ derived from a simple k-curve $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k-adjacent if and only if $j = i+1 (mod\, l)$ or $i = j+1 (mod\, l)$ [25]. By $SC_k^{n,l}$ we denote a simple closed k-curve with l elements in $\mathbf{Z}^n, n \in \mathbf{N} - \{1\}$ [12].

Motivated by both the digital continuity of [28] and the (k_0, k_1)-continuity of [2], we say that a function $f : X \to Y$ is (k_0, k_1)-continuous at a point $x_0 \in X$.
Let (X, k_0) and (Y, k_1) be spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous at a point $x_0 \in X$ if and only if $f(N_{k_0}(x_0, 1)) \subset N_{k_1}(f(x_0), 1)$, where $N_{k_0}(x_0, 1) \subset X$ and $N_{k_1}(f(x_0), 1)$ $\subset Y$.

Unlike the pasting property of classical continuity in topology, the (k_0, k_1)-continuity has some intrinsic features [24]: (k_0, k_1)-continuity has *the almost pasting property* instead of *the pasting property* of classical topology.
For a k-adjacency relation of $\mathbf{Z}^n$, we recall that a *simple closed k-curve* with l elements in $X \subset \mathbf{Z}^n$ is the image of a $(2, k)$-continuous function $f : [0, l-1]_{\mathbf{Z}} \to X$ such that $f(i)$ and $f(j)$ are k-adjacent if and only if either $j = i+1 (mod\, l)$ or $i = j+1 (mod\, l)$ [26]. Thus, we may use the notation $SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ with $f(i) = c_i$ [12].

Recently, digital graph versions of (k_0, k_1)-continuity, (k_0, k_1)-homeomorphism, (k_0, k_1)-covering, and (k_0, k_1)-homotopy in digital topology were established in [13]. Consequently, we may use the term a (k_0, k_1)-*isomorphism* as in [4, 13] rather than a (k_0, k_1)-*homeomorphism* as in [3]:

Definition 1. [13] *(see also* [4]*) For two spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-isomorphism and use the notation $X \approx_{k_0} Y$ or $X \approx Y$ if not confused.*

3. Some properties of a simple closed k-surface in $\mathbf{Z}^3$, $k \in \{18, 26\}$

For a space (X, k) and its subset A, we call $((X, A), k)$ a *digital space pair* with k-adjacency. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a *pointed space* [3]. Motivated by the k-homotopy of [3], the *homotopy relative to a subset* $A \subset X$ was established in [9] and has been used in studying digital spaces in relation with a strong k-deformation retract, a k-homotopic thinning [10] (see also [16]), and a k-contractibility [17]. As special case of the (k_0, k_1)-homotopy in [3], we use the following k-homotopy in this paper.

Definition 2. [9] *(see also* [16]*) Let (X, k) and (Y, k) be spaces in $\mathbf{Z}^n$, and $A \subset X$. Let $f, g : X \to Y$ be (k, k) (briefly, k)-continuous functions. Suppose the existence of both $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that*

- *for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, k)$-continuous for all $t \in [0, m]_{\mathbf{Z}}$;*
- *for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is k-continuous for all $x \in X$.*

Then, F is called a k-homotopy between f and g, and f and g are k-homotopic in Y.

- *Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, then suppose the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$.*

Then, we call F a k-homotopy relative to A between f and g, and we say that f and g are k-homotopic relative to A in Y denoted by $f \simeq_{k \cdot rel.A} g$.

In Definition 2, if $A = \{x_0\} \subset X$, then we say that F is a pointed k-homotopy at $\{x_0\}$ in [3].

Definition 3. [3] *If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map with space x_0 relative to $\{x_0\}$, then we say that (X, x_0) is pointed k-contractible.*

Indeed, the notion of k-contractibility is slightly different from both the contractibility in Euclidean topology [3, 12] and the contractibility of [3].

In classical topology, the notions of *interior* and *exterior* have been essentially used in studying a topological space. By analogy, we obtain the following from the view point of digital topology.

Definition 4. [9] *Let* $c^* = (x_0, x_1, \cdots, x_n)$ *be a closed* k*-curve in* $\mathbf{Z}^2$. *Let* $\bar{c}^*$ *be the complement of* c^* *in* $\mathbf{Z}^2$. *A point* x *of* $\bar{c}^*$ *is said to be interior to* c^* *if it belongs to the bounded* $\bar{k}$*-connected component of* $\bar{c}^*$. *Otherwise, it is called exterior to* c^*. *The set of all interior(respectively exterior) points to* c^* *is denoted by* $Int(c^*)$*(respectively* $Ext(c^*)$*).*

We now recall the terminology for the study of a digital k-surface in $\mathbf{Z}^3$. A point $x \in X \subset \mathbf{Z}^3$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [1]. The k-corner x is called *simple* if y and z are not k-corners and if x is the only point k-adjacent to both y, z. X is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve [1]. For a k-connected space (X, k) in $\mathbf{Z}^3$, we recall $|X|^x = N_{26}^*(x) \cap X$, $N_{26}^*(x) = \{x' | x$ and x' are 26-adjacent$\}$. In other words, $|X|^x = N_{26}(x, 1) - \{x\}$ [9, 10, 14].

By using the above terminology, the notion of closed k-surface was introduced:

Definition 5. [1] *Let* (X, k) *be a space in* $\mathbf{Z}^3$, *and* $\bar{X} = \mathbf{Z}^3 - X$. *Then,* X *is called a closed* k*-surface if it satisfies the following:*
(1) In case $(k, \bar{k}) \in \{(26, 6), (6, 26)\}$, *then*
(a) for each point $x \in X$, $|X|^x$ *has exactly one* k*-component* k*-adjacent to* x*;*

(b) $|\bar{X}|^x$ *has exactly two* $\bar{k}$*-components which are* $\bar{k}$*-adjacent to* x*; we denote by* C^{xx} *and* D^{xx} *these two components; and*
(c) for any point $y \in N_k(x) \cap X$, $N_{\bar{k}}(y) \cap C^{xx} \neq \emptyset$ *and* $N_{\bar{k}}(y) \cap D^{xx} \neq \emptyset$, *where* $N_k(x) = N_k^*(x) \cup \{x\}$ *and* $N_k^*(x) = \{x' | x$ *and* x' *are* $k-$*adjacent*$\}$.
(2) In case $(k, \bar{k}) = (18, 6)$, *then*
(a) X *is* k*-connected,*
(b) for each point $x \in X$, $|X|^x$ *is a generalized simple closed* k*-curve.*

In (1) and (2), for $k \in \{18, 26\}$ *if the image* $|X|^x$ *is a simple closed* k*-curve, then* X *is called simple.*

Obviously, we observe that each closed 6-surface is simple (see MSS_6 in Figure 1). Furthermore, in this paper we will not consider the *orientability* of a closed k-surface in [27].

The paper [14] establishes the following:

$$\left\{ \begin{array}{l} MSS_{18} \approx_{18} (MSC_8 \times \{1\}) \cup (\mathrm{Int}(MSC_8) \times \{0, 2\}); \\ MSS'_{18} \approx_{18} (MSC'_8 \times \{1\}) \cup (\mathrm{Int}(MSC'_8) \times \{0, 2\}), \end{array} \right\} \quad (3.1)$$

where '×' means the Cartesian product (or digital product) and
$MSC_8 := ((0,0),(1,-1),(2,-1),(3,0),(2,1),(1,1))$ and,
$MSC'_8 := ((0,0),(1,1),(0,2),(-1,1))$.

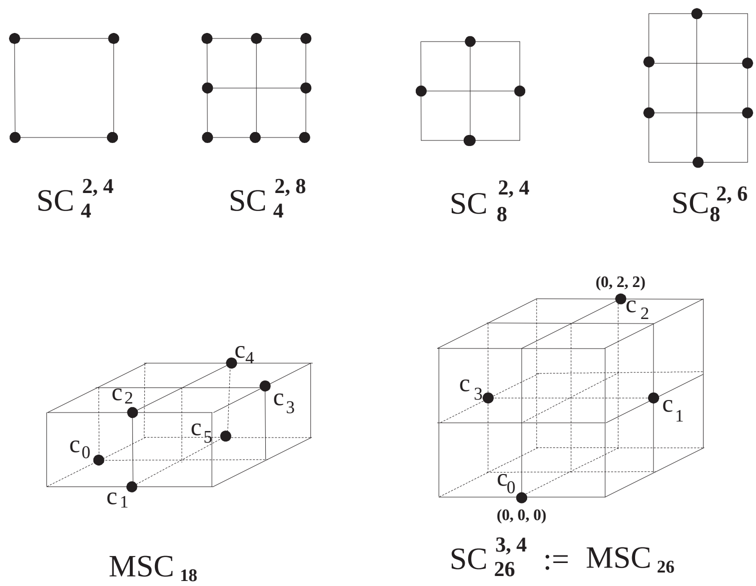

FIGURE 1. Simple closed 6-, 18-, and 26-surfaces from [9, 14, 15] with $(6,26)$-and $(k,6)$-structures, $k \in \{18,26\}$

Remark 3.1. *The space MSS'_{18} in (3.1) can be represented as the set*
$\{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$ *in* $(\mathbf{Z}^3, 18, 6, MSS'_{18})$.

In [14], it turns out that MSS_{18} is a simple closed 18-surface not 18-contractible (see Figure 1) and further, MSS'_{18} is a simple closed 18-surface which is 18-contractible. In this paper each of MSS_{18} and MSC'_{18} is considered with an $(18,6)$ or a $(26,6)$-structure instead of the others in [6].

Both 18-surfaces MSS_{18} and MSS'_{18} have some useful properties, as follows.

Lemma 3.2. *(1) MSS_k is unique up to k-isomorphism, $k \in \{18,26\}$.*
(2) MSS_{18} is also a simple closed 26-surface not 26-contractible.
(3) MSS'_{18} is also a simple closed 26-surface which is 26-contractible.

Proof: (1) Trivial.

(2) Since MSS_{18} is obviously a simple closed 26-surface with a $(26, 6)$-structure and further, there is no 26-homotopy on MSS_{18} making $1_{MSS_{18}}$ 26-homotopic to a constant map $c_{\{p_i\}}$, where p_i is an arbitrary point in MSS_{18}. Thus, MSS_{18} cannot be 26-homotopy equivalent to a singleton in MSS_{18}, the proof is completed.

(3) MSS'_{18} is obviously a simple closed 26-surface with a $(26, 6)$-structure. Furthermore, MSS'_{18} is 26-contractible due to the 18-contractibility of MSS'_{18} in [14]. □

Hereafter, by Lemma 3.2, we may use MSS_{18} and MSS'_{18} as MSS_{26} and MSS'_{26}, respectively. Namely, we may use $MSS_{18} := MSS_{26}$ and $MSS'_{18} := MSS'_{26}$ in this paper.

4. Commutative monoid of the set of k-isomorphism classes of simple closed k-surfaces in $\mathbf{Z}^3$

In relation with the establishment of a digital version of a connected sum, we have used the following spaces
$MSC'^*_8 := MSC'_8 \cup \{q\}$ and $MSC^*_8 := MSC_8 \cup \{x_1, x_2\}$, come from MSC'_8 and MSC_8 in $\mathbf{Z}^2$ [9, 14]. MSC'^*_8 has been used in establishing a digital connected sum. In this section we denote by SC_k the set of all simple closed k-surface $X \subset \mathbf{Z}^3$ in which each point $x \in X$ has a subset $N_k(x, 1) \subset X$ satisfying $N_k(x, 1) \approx_{(k,8)} MSC'^*_8$, $k \in \{18, 26\}$.

In addition, we obtain the following:

(1) $MSC'^*_8 := MSC'_8 \cup Int(MSC'_8)$,
where $MSC'_8 \approx_8 \{w_0 = (0, 0), w_1 = (-1, 1), w_2 = (-2, 0), w_3 = (-1, -1)\}$.

(2) $MSC^*_8 := MSC_8 \cup Int(MSC_8) \approx_{(8,4)} N_4(p, 1) \subset \mathbf{Z}^2, p \in \mathbf{Z}^2$,
where $MSC_8 \approx_8 \{c_0 = (0, 0), c_1 = (1, 1), c_2 = (1, 2), c_3 = (0, 3), c_4 = (-1, 2),$
$c_5 = (-1, 1)\}$.

Since a simple closed k-surface in SC_k has a subset $A \subset X$ satisfying $A \approx_{(k,8)} MSC'^*_8$, $k \in \{18, 26\}$, hereafter, we may take a subset $A \approx_{(k,8)} MSC'^*_8$ for the digital connected sum of Definition 6 below. Thus we can establish a commutative monoid structure of the set of k-isomorphism classes of simple closed k-surfaces in SC_k with an operation derived from the digital connected sum, $k \in \{18, 26\}$. As a special case of the digital connected sum in [9], we introduce the following which is suitable for an establishment of a monoid of the set of k-isomorphism classes of simple closed k-surfaces in SC_k.

Definition 6. *Let X and Y be simple closed k-surfaces in $SC_k, k \in \{18, 26\}$. Consider $A' \subset A \subset X$ and take $A - A' \subset X$, where $A \approx_{(k,8)} MSC_8'^{*}$ and $A' \approx_{(k,8)} Int(MSC_8')$. Let $f : A \to f(A) \subset Y$ be a k-isomorphism. Remove A' and $f(A')$ from X and Y, respectively. Then, the disjoint union of X' and Y' induced from the identification x with $f(x) \in Y'$ for all $x \in A - A'$ is taken, denoted by $X \sharp Y$, where $X' = X - A'$, $Y' = Y - f(A')$ and any two points $p \in X' \subset X \sharp Y$ and $q \in Y' \subset X \sharp Y$ with $p, q \notin X \sharp Y - f(A - A')$ are not 26-adjacent in $X \sharp Y$.*

Remark 4.1. *In relation with the conditions (a) and (b) of (1), and (b) of (2) in Definition 5, we need the statement that any two points $p \in X' \subset X \sharp Y$ and $q \in Y' \subset X \sharp Y$ with $p, q \notin X \sharp Y - f(A - A')$ are not 26-adjacent in $X \sharp Y$ of Definition 6.*

In order to show that a digital connected sum is essentially used in establishing a monoid structure of the set of k-isomorphism classes of simple closed k-surfaces in $SC_k, k \in \{18, 26\}$, we use the following:

Example 4.2. *(1) $MSS_{26} \sharp MSS'_{26} \approx_{26} MSS_{26}$.*
(2) $MSS'_{26} \sharp MSS'_{26} \approx_{26} MSS'_{26}$.

Proof: (1) We can consider $MSS_{26} \sharp MSS'_{26}$ with 26-adjacency in $(\mathbf{Z}^3, 26, 6, MSS_{26} \sharp MSS'_{26})$ so that $MSS_{26} \sharp MSS'_{26} \approx_{26} MSS_{26}$ [9].
Precisely, take two subsets, $\{p_0, p_1, p_9, p_5, p_7\} := A \subset MSS_{26}$ (see Figure 1) and $\{c_0, c_1, c_2, c_3, c_4\} := B \subset MSS'_{26}$ (see Figure 1) which are 26-isomorphic to each other. Then, consider a 26-isomorphism $f : A \to B$ such that

$$f(p_0) = c_0, f(p_1) = c_1, f(p_9) = c_2, f(p_5) = c_3, f(p_7) = c_4$$

and remove the two points $p_0 \in MSS_{26}$ and $c_0 = f(p_0) \in MSS'_{26}$.
Gluing the two remaining sets $MSS_{26} - \{p_0\}$ and $MSS'_{26} - \{c_0\}$, we obtain $MSS_{26} \sharp MSS'_{26}$ by using the map f so that $MSS_{26} \sharp MSS'_{26}$ is still 26-isomorphic to the space MSS_{26}.
By the same method as above, we obtain $MSS_{18} \sharp MSS'_{18} \approx_{18} MSS_{18}$ is also established with 18-adjacency in $(\mathbf{Z}^3, 18, 6, MSS_{18} \sharp MSS'_{18})$.
(2) By the same method as Example 4.2(1), the proof is completed. □

By the same method as above, we obtain that $MSS_{26} \sharp MSS_{26}$ is another simple closed 26-surface. While there are many types of MSS_{26} $\sharp MSS_{26}$, those are 26-isomorphic to each other.

Consequently, we obtain the following:

Theorem 4.3. *Let X and Y be simple closed k-surfaces in SC_k, $k \in \{18, 26\}$. Then $X \sharp Y$ is a simple closed k-surface in SC_k.*

Let X, Y, and Z be simple closed k-surfaces in $SC_k, k \in \{18, 26\}$. Even though $X\sharp Y$ and $(X\sharp Y)\sharp Z$ need not be equal to $Y\sharp X$ and $X\sharp(Y\sharp Z)$, respectively, $X\sharp Y$ and $(X\sharp Y)\sharp Z$ are k-isomorphic to $Y\sharp X$ and $X\sharp(Y\sharp Z)$, respectively. Thus we observe that the set of k-isomorphism classes of simple closed k-surfaces in SC_k forms a commutative monoid with an operation induced from the digital connected sum of Definition 6. For a simple closed k-surface X in SC_k, consider the k-isomorphism class of X, $k \in \{18, 26\}$, *i.e.,*

$$[X] := \{X' | X \approx_k X'\}.$$

Lemma 4.4. *Let $X, Y, Z,$ and W be simple closed k-surfaces in $SC_k, k \in \{18, 26\}$. If $X \approx_k Y$ and $Z \approx_k W$, then $X\sharp Z \approx_k Y\sharp W$ in SC_k.*

Proof: Let $h_1 : X \to Y$ be a k-isomorphism and let $h_2 : Z \to W$ be a k-isomorphism. Since each of $X, Y, Z,$ and W has a subset $A \approx_{(k,8)} MSC_8'^{*}$, we obtain both $X\sharp Z$ and $Y\sharp W$ with k-adjacency, $k \in \{18, 26\}$. For any k-isomorphism, its restriction map on any subset of the domain of the given k-isomorphism is also a k-isomorphism [14].

For $A \subset X$, consider $f : A \to f(A) \subset Z$ which is a k-isomorphism of Definition 6 related to $X\sharp Z$, and
$i_1 : X - A' \to X\sharp Z$ which is an inclusion map, where $A' \approx_{(k,8)} Int(MSC_8')$, and $A' \subset A$ and further,
$i_2 : Z - f(A') \to X\sharp Z$ which is an inclusion map.

Besides, for $A \subset Y$ consider $g : A \to g(A) \subset W$
which is a k-isomorphism of Definition 6 related to $Y\sharp W$ and further,
$j_1 : Y - A' \to Y\sharp W$ which is an inclusion map, and
$j_2 : W - g(A') \to Y\sharp W$ which is an inclusion map.

Then we have a map $h : X\sharp Z \to Y\sharp W$ defined by

$$h(t) = \begin{cases} j_1 \circ h_1|_{X-A'} \circ i_1^{-1}(t) & \text{if} \quad t \in X - A' \subset X\sharp Z; \\ j_2 \circ h_2|_{Z-f(A')} \circ i_2^{-1}(t) & \text{if} \quad t \in Z - f(A') \subset X\sharp Z, \end{cases}$$

where $A' \approx_{(k,8)} Int(MSC_8')$ and $A' \subset A$. Then h is a k-isomorphism, which means that $X\sharp Z \approx_k Y\sharp W$. □

By Lemma 4.4 we obtain the following:

Definition 7. *Let X and Y be simple closed k-surfaces in $SC_k, k \in \{18, 26\}$. Then we define $[X] \cdot [Y] = [X\sharp Y]$.*

By Definition 7, Remark 3.1, and Lemma 4.4, we obtain the following:

Theorem 4.5. *The set of k-isomorphism classes of simple closed k-surfaces in SC_k is a commutative monoid with the '$\cdot$' operation in Definition 7, $k \in \{18, 26\}$.*

Proof: Let us prove the following: Let X, Y, and Z be simple closed k-surfaces in $SC_k, k \in \{18, 26\}$, then we suffice to prove the following:
(1) $([X] \cdot [Y]) \cdot [Z] = [X] \cdot ([Y] \cdot [Z])$.
(2) $[MSS'_k] \cdot [X] = [X]$ and $[X] \cdot [MSS'_k] = [X]$.
(3) $[X] \cdot [Y] = [Y] \cdot [X]$.
Let us now prove (1). We suffice to prove that $(X \sharp Y) \sharp Z \approx_k X \sharp (Y \sharp Z)$, $k \in \{18, 26\}$. By Definition 6, consider a subset $A \subset X, Y$, and Z such that $A \approx_{(k,8)} MSC'^{*}_8$. While $(X \sharp Y) \sharp Z$ need not be equal to $X \sharp (Y \sharp Z)$, they are k-isomorphic to each other by the similar method as that of Lemma 4.4, which proves the assertion (1).

(2) Since $MSS'_k \sharp X \approx_k X \approx_k X \sharp MSS'_k$ via $A(\subset MSS'_k) \approx_{(k,8)} MSC'^{*}_8 \subset X$, $k \in \{18, 26\}$, which proves the assertion (2).

(3) Obviously, by Definition 6, consider two k-isomorphisms $f : A \to f(A)$ and $f^{-1} : f(A) \to A$. Then, $X \sharp Y$ is k-isomorphism to $Y \sharp X$, which proves the assertion (3). □

By Theorem 4.5 (2), it turns out that $[MSS'_k]$ acts the identity element under the operation '$\cdot$' of Definition 6, $k \in \{18, 26\}$.

5. Commutative monoid of the set of k-homotopy classes of closed k-surfaces

The notion of k-homotopy equivalence has been introduced in [11] and has been used in classifying discrete objects with a k-homotopy equivalence; however there are insufficient presentations of some topics in [11]. Thus, the paper [23] contains the corrected one.

Definition 8. [11] *(see also* [23]*) For two discrete topological spaces with k-adjacency (X, k) and (Y, k) in $\mathbf{Z}^n$, if there are k-continuous maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h \simeq_k 1_X$ and $h \circ l \simeq_k 1_Y$, then the map $h : X \to Y$ is called a (digital) k-homotopy equivalence. And we use the notation $X \simeq_{k \cdot h \cdot e} Y$.*

In Section 5, we still need to take the subset $A \approx_{(k,8)} MSC'^{*}_8$ to establish a commutative monoid of the set of k-homotopy equivalence classes of closed k-surfaces in $\mathbf{Z}^3$ with an operation derived from a digital connected sum of Definition 6.

Unlike the digital connected sum of Definition 6, there are some difficulties in establishing a digital connected sum of two closed k-surfaces X

and Y which are not simple because there may not be subsets A in both X and Y such that A is $(k,8)$-isomorphism to $MSC_8'^*$. Furthermore, we may also meet an obstacle to the establishment of $X\sharp Y\sharp Z$ for some closed k-surfaces X,Y, and Z in $\mathbf{Z}^3$. Thus, in this section we consider the set of only closed k-surfaces $X \subset \mathbf{Z}^3$ having a subset $A \subset X$ such that $A := N_k(x,1) \approx_{(k,8)} MSC_8'^*$ and establishing the associativity of the commutative monoid of the set of k-homotopy equivalence classes of closed k-surfaces in $\mathbf{Z}^3$, $k \in \{18,26\}$. Then we denote by CS_k the above set. Some k-homotopic properties of $X \in CS_k$ are now investigated in relation with the digital connected sum of Definition 6.

In CS_k, for a closed k-surface X, consider the k-homotopy equivalence class of X as follows.

$$[X] := \{X' | X \approx_{k \cdot h \cdot e} X'\}.$$

Using both an argument similar to that given for the proof of Lemma 4.4, Remark 4.1, and a k-homotopy equivalence instead of a k-isomorphism of Lemma 4.4, we obtain the following:

Lemma 5.1. *In $\mathbf{Z}^3$, let X, Y, Z, and W be spaces in CS_k. If $X \approx_{k\cdot h\cdot e} Y$ and $Z \approx_{k\cdot h\cdot e} W$, then $X\sharp Z \approx_{k\cdot h\cdot e} Y\sharp W$.*

By Lemma 5.1 and Definitions 6 and 7, we obtain that for $X,Y \in CS_k$, we define $[X]\cdot[Y]$ to be $[X\sharp Y]$.

Obviously, for X,Y, and Z in CS_k, $X\sharp Y$ and $(X\sharp Y)\sharp Z$ need not be equal to $Y\sharp X$ and $X\sharp(Y\sharp Z)$, respectively. Meanwhile, we obtain the following:

Theorem 5.2. *Let X, Y, and Z be closed k-surfaces in $CS_k, k \in \{18,26\}$. Then we obtain the following:*

(1) $([X]\cdot[Y])\cdot[Z] = [X]\cdot([Y]\cdot[Z])$, $k \in \{18,26\}$.

(2) $[MSS_k']\cdot[X] = [X]$ *and* $[X]\cdot[MSS_k'] = [X]$.

(3) $[X]\cdot[Y] = [Y]\cdot[X]$.

Proof: (1) Since $(X\sharp Y)\sharp Z$ is k-homotopy equivalent to $X\sharp(Y\sharp Z)$, $k \in \{18,26\}$, the proof is completed.

(2) Since $MSS_k'\sharp X \approx_{k\cdot h\cdot e} X \approx_{k\cdot h\cdot e} X\sharp MSS_k'$ by using $A(\subset MSS_k') \approx_{(k,8)} MSC_8'^* \subset X$, $k \in \{18,26\}$, the proof is completed.

(3) Obviously, $X\sharp Y$ and $Y\sharp X$ are k-homotopy equivalent to each other, the proof is completed. □

By Theorem 5.2(2), it turns out that $[MSS_k']$ is the identity element under the operation '$\cdot$' in Definition 7, $k \in \{18,26\}$.

Remark 5.3. *(Correcting) In* [21]*, since the two objects U_1 of Figure 6 and U_1 of Figure 7 are misprinted at the point $(0,0) \in \mathbf{Z}^2$. Thus they can be corrected, as follows (see Figure 2). With the same criterion, the objects E_1 of Figure 1 in* [20] *should be corrected at the point $(0,0) \in \mathbf{Z}^2$ (motivated from Figure 4 of* [12]*).*

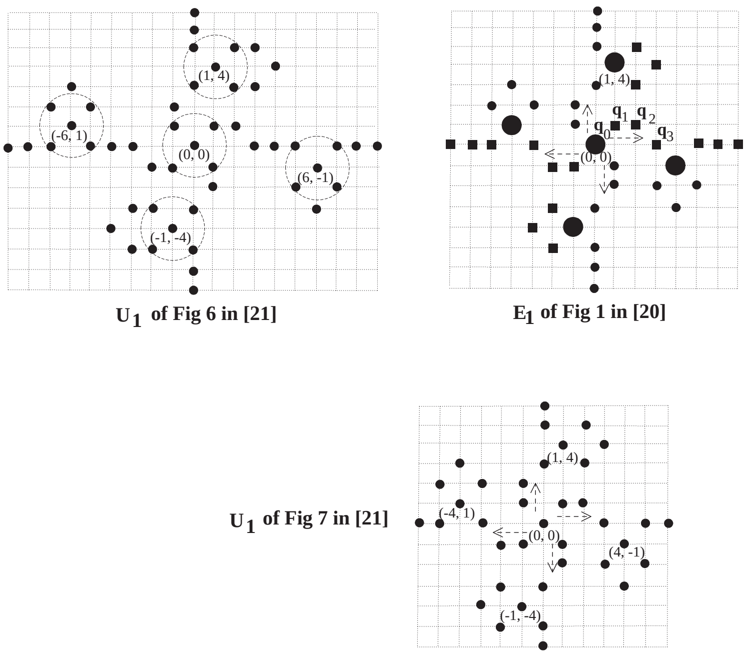

FIGURE 2. Correction of objects in [20, 21]

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