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SKELETONS OF HYPER MV-ALGEBRAS

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Abstract
Several types of skeletons in a hyper MV-algebra are introduced, and their relations are investigated.

1. Introduction

In order to provide an algebraic proof of completeness theorem of infinite valued Lukasewicz propositional calculus, Chang [1] introduced the notion of MV-algebras. The hyper structure theory was introduced by Marty [5] at 8th congress of Scandinavian Mathematicians in 1934. Since then many researchers have worked in these areas. Recently in [3], Ghorbani et al. applied the hyper structure to MV-algebras and introduced the concept of a hyper MV-algebra which is a generalization of an MV-algebra and investigated some related results. Torkzadeh and Ahadpanah [6] discussed hyper MV-ideals in hyper MV-algebras. Jun et al. [4] studied hyper MV-deductive systems in hyper MV-algebras. In this paper, we introduce several types of skeletons in hyper MV-algebras, and investigate their relations.

2. Preliminaries

Definition 2.1. [2] A *hyper MV-algebra* is a nonempty set M endowed with a hyper operation "$\oplus$", a unary operation "$*$" and a constant "0" satisfying the following axioms:

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(a2) $x \oplus y = y \oplus x$,
(a3) $(x^*)^* = x$,
(a4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(a5) $0^* \in x \oplus 0^*$,

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(a6) $0^* \in x \oplus x^*$,
(a7) $x \ll y,\ y \ll x \Rightarrow x = y$

for all $x, y, z \in M$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every subsets A and B of M, we define

$$A \ll B \Leftrightarrow (\exists\, a \in A)\,(\exists\, b \in B)\,(a \ll b),$$

$$A \oplus B = \bigcup_{a \in A, b \in B} a \oplus b.$$

We also define $0^* = 1$ and $A^* = \{a^* \mid a \in A\}$. The notation $x \ll y$ and $A \ll B$ will be also written as $y \gg x$ and $B \gg A$, respectively.

Proposition 2.2. [2] *Every hyper MV-algebra M satisfies the following assertions:*

(b1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(b2) $0 \ll x, \quad x \ll 1$,
(b3) $x \ll x$,
(b4) $x \ll y \Rightarrow y^* \ll x^*$,
(b5) $A \ll B \Rightarrow B^* \ll A^*$,
(b6) $A \ll A$,
(b7) $A \subseteq B \Rightarrow A \ll B$,
(b8) $x \ll x \oplus y, \quad A \ll A \oplus B$,
(b9) $(A^*)^* = A$,
(b10) $0 \oplus 0 = \{0\}$,
(b11) $x \in x \oplus 0$,
(b12) $y \in x \oplus 0 \Rightarrow y \ll x$,
(b13) $y \oplus 0 = x \oplus 0 \Rightarrow x = y$

for all $x, y, z \in M$ and subsets A, B and C of M.

Definition 2.3. [2] A nonempty subset S of a hyper MV-algebra M is called a *hyper MV-subalgebra* of M if S is a hyper MV-algebra under the hyper operation "$\oplus$" and the unary operation "$*$" on M.

Lemma 2.4. [2] *A nonempty subset S of a hyper MV-algebra M is hyper MV-subalgebra of M if and only if $x^* \in S$ and $x \oplus y \subseteq S$ for all $x, y \in S$.*

Definition 2.5. [4] A nonempty subset D of M is called a *weak hyper MV-deductive system* of M if it satisfies:

(c1) $0 \in D$,
(c2) $(\forall x, y \in M)((x^* \oplus y)^* \subseteq D,\ y \in D \Rightarrow x \in D)$.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$	$\{0,a,b,1\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
b	$\{0,a,b\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
1	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$

x	x^*
0	1
a	b
b	a
1	0

TABLE 1. $\oplus$-multiplication and unary operation

DEFINITION 2.6. [4] A non-empty subset D of M is called a *hyper MV-deductive system* of M if it satisfies (c1) and

(c3) $(\forall x, y \in M)((x^* \oplus y)^* \ll D,\, y \in D \;\Rightarrow\; x \in D)$.

3. Skeletons of hyper MV-algebras

In what follows let M denote a hyper MV-algebra unless otherwise specified.

DEFINITION 3.1. A nonempty subset K of a hyper MV-algebra M is called a $(\subseteq, \in)$-*skeleton* of M if it satisfies:

(d1) $1 \in K$,
(d2) $(\forall x \in K)(\forall y \in M)(x^* \oplus y \subseteq K \;\Rightarrow\; y \in K)$.

EXAMPLE 3.2. Let $M = \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 1. Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easy to check that $K_1 := \{1, a\}$, $K_2 := \{1, b\}$ and $K_3 := \{1, a, b\}$ are $(\subseteq, \in)$-skeletons of M.

THEOREM 3.3. *If D is a weak hyper MV-deductive system of M, then the set*

$$D^* := \{a^* \mid a \in D\}$$

is a $(\subseteq, \in)$-skeleton of M.

Proof. Obviously, $1 \in D^*$. Let $x \in D^*$ and $y \in M$ be such that $x^* \oplus y \subseteq D^*$. Using (a2) and (a3), we have

$$((y^*)^* \oplus x^*)^* = (y \oplus x^*)^* = (x^* \oplus y)^* \subseteq D.$$

It follows from (c2) that $y^* \in D$ so that $y \in D^*$. Hence D^* is a $(\subseteq, \in)$-skeleton of M. $\square$

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{0,b\}$	$\{0,a,b,1\}$
a	$\{0,a\}$	$\{a\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
b	$\{0,b\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
1	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$

x	x^*
0	1
a	b
b	a
1	0

TABLE 2. $\oplus$-multiplication and unary operation

THEOREM 3.4. *Let M be a hyper MV-algebra satisfying the following condition:*

$$(\forall x, y \in M)\,(y \in x \oplus y) \tag{3.1}$$

Then every subset of M containing 1 is a $(\subseteq, \in)$-skeleton of M.

Proof. Let K be a subset of M containing 1, and let $x, y \in M$ be such that $x^* \oplus y \subseteq K$ and $x \in K$. Assume that $y \notin K$. Then $y \in x^* \oplus y$ by (3.1), and so $x^* \oplus y \nsubseteq K$. This is a contradiction, and therefore $y \in K$, proving that K is a $(\subseteq, \in)$-skeleton of M. □

THEOREM 3.5. *If K_1 and K_2 are $(\subseteq, \in)$-skeletons of a hyper MV-algebra M,then so is $K_1 \cap K_2$.*

Proof. Straightforward. □

The following example shows that the union of $(\subseteq, \in)$-skeletons of a hyper MV-algebra M is not a $(\subseteq, \in)$-skeletons of a hyper MV-algebra M.

DEFINITION 3.6. A nonempty subset K of a hyper MV-algebra M is called a $(\gg, \in)$-*skeleton* of M if it satisfies (d1) and

(d3) $(\forall x \in K)(\forall y \in M)(x^* \oplus y \gg K \;\Rightarrow\; y \in K)$.

EXAMPLE 3.7. Let $M = \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 2. Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easy to check that $K_2 := \{1, b\}$ is a $(\gg, \in)$-skeleton of M, but $K_1 := \{1, a\}$ is not a $(\gg, \in)$-skeleton of M since $a^* \oplus b = b \oplus b = \{0, a, b, 1\} \gg \{1, a\}$.

PROPOSITION 3.8. *Every $(\gg, \in)$-skeleton K of M satisfies the following condition:*

$$(\forall x \in K)\;(\forall y \in M)\;(x \ll y \;\Rightarrow\; y \in K). \tag{3.2}$$

Proof. Let $x, y \in M$ be such that $x \ll y$ and $x \in K$. Then $1 \in x^* \oplus y$, and so $x^* \oplus y \gg K$ by (d1) and (b3). It follows from (d3) that $y \in K$. □

The following example shows that a ($\subseteq$, $\in$)-skeleton K of M may not satisfy the condition (3.2).

Example 3.9. Consider a hyper MV-algebra M in Example 3.2. Then $K_1 := \{1, a\}$ is a ($\subseteq$, $\in$)-skeleton of M which does not satisfy (3.2).

Let M satisfy the condition (3.1) and let K be a subset of M containing 1. Then K may not be a ($\gg$, $\in$)-skeleton of M as seen in Example 3.7.

Theorem 3.10. *Every ($\gg$, $\in$)-skeleton is a ($\subseteq$, $\in$)-skeleton.*

Proof. Let K be a ($\gg$, $\in$)-skeleton of M. Assume that $x^* \oplus y \subseteq K$ for all $x \in K$ and $y \in M$. Then $a \ll a$ for all $a \in x^* \oplus y$, and so $x^* \oplus y \gg K$. Using (d3), we have $y \in K$. Hence K is ($\subseteq$, $\in$)-skeleton of M. □

Theorem 3.11. *If D is a hyper MV-deductive system of M, then the set*

$$D^* := \{a^* \mid a \in D\}$$

is a ($\gg$, $\in$)-skeleton of M.

Proof. Obviously, $1 \in D^*$. Let $x \in D^*$ and $y \in M$ be such that $x^* \oplus y \gg D^*$. Then $((y^*)^* \oplus x^*)^* = (x^* \oplus y)^* \ll D$ by (a2), (a3), (b5) and (b9). It follows from (c3) that $y^* \in D$ so from (a3) that $y \in D^*$. Hence D^* is a ($\gg$, $\in$)-skeleton of M. □

Theorem 3.12. *If K_1 and K_2 are ($\gg$, $\in$)-skeletons of a hyper MV-algebra M,then so is $K_1 \cap K_2$.*

Proof. Straightforward. □

The following example shows that the union of ($\gg$, $\in$)-skeletons of a hyper MV-algebra M is not a ($\gg$, $\in$)-skeletons of a hyper MV-algebra M.

Definition 3.13. A nonempty subset K of a hyper MV-algebra M is called a ($\ll^*$, $\in$)-*skeleton* of M if it satisfies (d1) and

(d4) $(\forall x \in K)(\forall y \in M \setminus \{0\})(x^* \oplus y \ll K^* \ \Rightarrow \ y \in K)$.

Example 3.14. For a hyper MV-algebra M, the set $K := M \setminus \{0\}$ is a ($\ll^*$, $\in$)-skeleton of M.

Example 3.15. Let $M = \{0, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 3. Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easy to check that $K_1 := \{1\}$ and $K_2 := \{1, b\}$ are ($\ll^*$, $\in$)-skeletons of M. But $K := \{1, 0\}$ is not a ($\ll^*$, $\in$)-skeleton of M since $1^* \oplus b = \{b\} \ll K^*$ and $b \notin K^*$.

$\oplus$	0	b	1
0	$\{0\}$	$\{b\}$	$\{1\}$
b	$\{b\}$	$\{0, b, 1\}$	$\{0, b, 1\}$
1	$\{1\}$	$\{0, b, 1\}$	$\{1\}$

x	x^*
0	1
b	b
1	0

Table 3. $\oplus$-multiplication and unary operation

Remark 3.16. Let K be a proper subset of a hyper MV-algebra M. Then there exists $y \in M \setminus K$. If K contains 0, then $1 \in K^*$ and so $x^* \oplus y \ll K^*$ for all $x \in K$. This shows that every ($\ll^*$, $\in$)-skeleton does not contain 0.

Remark 3.17. There is no relation between a ($\subseteq$, $\in$)-skeleton and a ($\ll^*$, $\in$)-skeleton. In fact, $K_1 := \{1, a\}$ in Example 3.2 is a ($\subseteq$, $\in$)-skeleton of M. But it is not a ($\ll^*$, $\in$)-skeleton of M. Also, $K_2 := \{1, b\}$ in Example 3.15 is a ($\ll^*$, $\in$)-skeleton of M which is not a ($\subseteq$, $\in$)-skeleton of M.

Remark 3.18. A ($\gg$, $\in$)-skeleton may not be a ($\ll^*$, $\in$)-skeleton. For example, $K_1 := \{1, b\}$ in Example 3.7 is a ($\gg$, $\in$)-skeleton of M which is not a ($\ll^*$, $\in$)-skeleton of M.

Theorem 3.19. *If K_1 and K_2 are ($\ll^*$, $\in$)-skeletons of a hyper MV-algebra M,then so is $K_1 \cap K_2$.*

Proof. Straightforward. □

The following example shows that the union of ($\ll^*$, $\in$)-skeletons of a hyper MV-algebra M is not a ($\ll^*$, $\in$)-skeletons of a hyper MV-algebra M.

Definition 3.20. A nonempty subset K of a hyper MV-algebra M is called a *hyper MV-skeleton* of M if it satisfies

(d5) $(\forall x \in K)(\forall y \in M)(x^* \oplus y \ll K \ \Rightarrow \ y \in K)$.

Example 3.21. Let $M = \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 4. Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easy to check that $K_a := \{a\}$ is a hyper MV-skeleton of M. But $K_b := \{b\}$ is not a hyper MV-skeleton of M since $b^* \oplus a = a \oplus a = \{0, a\} \ll \{b\} = K_b$ and $a \notin K_b$.

Definition 3.22. A nonempty subset K of a hyper MV-algebra M is called a *weak hyper MV-skeleton* of M if it satisfies

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{b,1\}$
a	$\{0,a\}$	$\{0,a\}$	$\{b,1\}$	$\{b,1\}$
b	$\{b\}$	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$
1	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$

x	x^*
0	1
a	b
b	a
1	0

TABLE 4. $\oplus$-multiplication and unary operation

(d6) $(\forall x \in K)(\forall y \in M)((x^* \oplus y) \cap K \neq \emptyset \;\Rightarrow\; y \in K)$.

LEMMA 3.23. *For every subset K of a hyper MV-algebra M, we have*

$$(\forall x, y \in M)\,((x^* \oplus y) \cap K \neq \emptyset \;\Rightarrow\; x^* \oplus y \ll K). \tag{3.3}$$

Proof. Straightforward. □

THEOREM 3.24. *Every hyper MV-skeleton is a weak hyper MV-skeleton.*

Proof. It follows immediately from Lemma 3.23. □

The following example shows that the converse of Theorem 3.24 may not be true.

EXAMPLE 3.25. Consider a hyper MV-algebra M which is given in Example 3.21. Then $K := \{b, 1\}$ is a weak hyper MV-skeleton of M, but it is not a hyper MV-skeleton of M since $b^* \oplus a = \{0, a\} \ll K$ and $a \notin K$.

THEOREM 3.26. *If K_1 and K_2 are (weak) hyper MV-skeletons of a hyper MV-algebra M,then so is $K_1 \cap K_2$.*

Proof. Straightforward. □

The following example shows that the union of (weak) hyper MV-skeletons of a hyper MV-algebra M is not a (weak) hyper MV-skeleton of a hyper MV-algebra M.

DEFINITION 3.27. A nonempty subset K of a hyper MV-algebra M is called a *strong* ($\subseteq^*$, $\in$)-*skeleton* of M if it satisfies (d1) and

(d7) $(\forall x \in K)\ (\forall y \in M)\ (x^* \oplus y \subseteq K^* \;\Rightarrow\; y \in K)$.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a,b\}$	$\{0,b\}$	$\{0,a,b,1\}$
a	$\{0,a,b\}$	$\{0,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
b	$\{0,b\}$	$\{0,a,b,1\}$	$\{b\}$	$\{0,a,b,1\}$
1	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$

x	x^*
0	1
a	b
b	a
1	0

TABLE 5. $\oplus$-multiplication and unary operation

EXAMPLE 3.28. Let $M = \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 5. Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easy to check that $K_1 := \{0, 1\}$ is a strong ($\subseteq^*$, $\in$)-skeleton of M, but $K_2 := \{1\}$ is not a strong ($\subseteq^*$, $\in$)-skeleton of M since $1^* \oplus 0 = \{0\} \subseteq K_2^*$ and $0 \notin K_2$.

DEFINITION 3.29. A nonempty subset K of a hyper MV-algebra M is called a ($\subseteq^*$, $\in$)-*skeleton* of M if it satisfies (d1) and

(d8) $(\forall x \in K)\ (\forall y \in M \setminus \{0\})\ (x^* \oplus y \subseteq K^* \ \Rightarrow\ y \in K)$.

EXAMPLE 3.30. Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra which is given in Example 3.28. Then $K := \{a, b, 1\}$ is a ($\subseteq^*$, $\in$)-skeleton of M. But $K_a := \{a, 1\}$ is not a ($\subseteq^*$, $\in$)-skeleton of M since $a^* \oplus b = \{b\} \subseteq K_a^*$ but $b \notin K_a$.

Obviously, if M is a hyper MV-algebra, then $M \setminus \{0\}$ is a ($\subseteq^*$, $\in$)-skeleton of M. Note that every strong ($\subseteq^*$, $\in$)-skeleton is a ($\subseteq^*$, $\in$)-skeleton, but the converse is not true as seen in the following example.

EXAMPLE 3.31. Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra which is given in Example 3.28. Then $K := \{a, b, 1\}$ is a ($\subseteq^*$, $\in$)-skeleton of M, but it is not a strong ($\subseteq^*$, $\in$)-skeleton of M since $a^* \oplus 0 = \{0, b\} \subseteq K^*$ but $0 \notin K$.

Using (d7), we know that every ($\ll^*$, $\in$)-skeleton is a ($\subseteq^*$, $\in$)-skeleton. But the converse may not hold as seen in the following example.

EXAMPLE 3.32. Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra which is given in Example 3.28. Then $K := \{a, b, 1\}$ is a ($\subseteq^*$, $\in$)-skeleton of M. But it is not a ($\ll^*$, $\in$)-skeleton of M since $a^* \oplus 0 = \{0, b\} \ll K^*$ but $0 \notin K$.

THEOREM 3.33. *If K_1 and K_2 are (strong) ($\subseteq^*$, $\in$)-skeletons of a hyper MV-algebra M, then so is $K_1 \cap K_2$.*

Proof. Straightforward. $\square$

The following example shows that the union of (strong) ($\subseteq^*$, $\in$)-skeletons of a hyper MV-algebra M is not a (strong) ($\subseteq^*$, $\in$)-skeletons of M.

Remark 3.34. There is no relation between a ($\gg$, $\in$)-skeleton and a strong ($\subseteq^*$, $\in$)-skeleton as shown by the following examples.

Example 3.35. Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra which is given in Example 3.28. Then $K := \{0, 1\}$ is a strong ($\subseteq^*$, $\in$)-skeleton of M, but not a ($\gg$, $\in$)-skeleton of M since $1^* \oplus b = \{0, b\} \gg K$ but $b \notin K$.

Example 3.36. Let M be a hyper MV-algebra which is given in Example 3.7. Then $K := \{1, b\}$ is a ($\gg$, $\in$)-skeleton of M, but not a strong ($\subseteq^*$, $\in$)-skeleton of M since $b^* \oplus 0 = \{0, a\} \subseteq K^*$, but $0 \notin K$.

Remark 3.37. There is no relation between a ($\ll^*$, $\in$)-skeleton and a strong ($\subseteq^*$, $\in$)-skeleton as shown by the following examples.

Example 3.38. Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra which is given in Example 3.28. Then $K := \{0, 1\}$ is a strong ($\subseteq^*$, $\in$)-skeleton of M, but not a ($\ll^*$, $\in$)-skeleton of M since $0^* \oplus a = \{0, a, b, 1\} \ll K^*$ but $a \notin K$.

References

[1] C.C. Change, *Algebraic analysis of many valued logic,* Transactions American Mathematical Society, **88** (1958), 476–490.
[2] Sh. Ghorbani, *On hyper MV-algebras,* Ph. D. thesis, Shahbid Bahonar University of Kerman, 2007.
[3] Sh. Ghorbani, A. Hasankhani and E. Eslami, *Hyper MV-algerbas,* Set-Valued Math. Appl. (to appear).
[4] Y. B. Jun, M. S. Kang and H. S. Kim, *Hyper MV-deductive systems of hyper MV-algebras,* Internat. J. Fuzzy Systems (submitted).
[5] F. Marty, *Sur une generalization de la notion de groups,* 8th congress Math. Scandinaves, Stockholm, (1934), 45–49.
[6] L. Torkzadeh and A. Ahadpanah, *Hyper MV-ideals in hyper MV-algerbas,* Math. Logic. Quaterly (submitted).

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