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INCLUSION AND EXCLUSION FOR FINITELY MANY TYPES OF PROPERTIES

Gab-Byoung Chae, Minseok Cheong, and Sang-Mok Kim

Abstract. Inclusion and exclusion is used in many papers to count certain objects exactly or asymptotically. Also it is used to derive the Bonferroni inequalities in probabilistic area [6]. Inclusion and exclusion on finitely many types of properties is first used in R. Meyer [7] in probability form and first used in the paper of McKay, Palmer, Read and Robinson [8] as a form of counting version of inclusion and exclusion on two types of properties. In this paper, we provide a proof for inclusion and exclusion on finitely many types of properties in counting version. As an example, the asymptotic number of general cubic graphs via inclusion and exclusion formula is given for this generalization.

1. Introduction

We begin with the definition of inclusion and exclusion (say I&E) with one property [9]. Let U be the universal set of S_0 elements, and suppose $A_1, \ldots, A_s$ are s subsets of U. For all integer $k > 0$, $[k]$ denotes the set $\{1, 2, \ldots, k\}$. The complement of a set C of U is denoted by $\overline{C}$. For $l = 0, \ldots s$, define

$$S_l = \sum |\bigcap_{i \in I} A_i|, \tag{1}$$

where the sum is over all l-subsets I of $[s]$. For $l = 0, \ldots s$, let N_l be the number of elements of U that belong to exactly l of the sets $\{A_1, \ldots, A_s\}$, that is,

$$N_l = \sum |\bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} \bar{A}_i|, \tag{2}$$

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where the sum is over all l-subsets I of $[s]$. By counting contribution of each element to N_u for $u = l, \ldots, s$, which is multi-counted for S_l, we have

$$S_l = \sum_{l \leq u \leq s} \binom{u}{l} N_u. \tag{3}$$

The numbers S_l and N_l are closely related and this relation is neatly expressed in terms of ordinary generating functions

$$S(x) = \sum_{i=0}^{s} S_i x^i \tag{4}$$

and

$$N(x) = \sum_{i=0}^{s} N_i x^i. \tag{5}$$

The following equation can be deduced from (3), (4), and (5).

Theorem 1.

$$N(x+1) = S(x) \tag{6}$$

If x is replaced by $x-1$ in the equality(6) and compare the coefficients, we have a relation:

$$N_l = \sum_{0 \leq i \leq s-l} (-1)^i \binom{l+i}{i} S_{l+i}. \tag{7}$$

It is important to study the upper and lower bounds for N_l. Therefore we consider truncation

$$\begin{aligned} &\sum_{0 \leq i \leq \alpha} (-1)^i \binom{l+i}{i} S_{l+i} \\ &\quad = \sum_{0 \leq i \leq \alpha} (-1)^i \binom{l+i}{i} \sum_{s \geq u \geq l+i} \binom{u}{l+i} N_u. \end{aligned} \tag{8}$$

where $0 < \alpha \leq s - l$. Now interchange the order of summation and obtain

$$\sum_{0 \leq i \leq \alpha} (-1)^i \binom{l+i}{i} S_{l+i} = \sum_{u \geq l} N_u \binom{u}{l} \sum_{0 \leq i \leq \alpha} (-1)^i \binom{u-l}{i} \tag{9}$$

It can be seen that:

$$\sum_{0 \leq i \leq \alpha} (-1)^i \binom{u-l}{i} = \begin{cases} (-1)^\alpha \binom{u-l-1}{\alpha}, & \text{if } u \geq l+1; \\ 1, & \text{if } u = l \end{cases} \tag{10}$$

so

$$\sum_{0\le i\le\alpha}(-1)^i\binom{l+i}{i}S_{l+i} = N_l + (-1)^\alpha\sum_{u\ge l+1}N_u\binom{u}{l}\binom{u-l-1}{\alpha} \tag{11}$$

Since the contribution of the sum on the right side of 11 is negative or positive according as s is odd or even, we can have the theorem below.

Theorem 2. *For $l=0,\dots s$ let $S_l=\sum|\bigcap_{i\in I}A_i|$, where the sum is over all l-subsets I of $[s]$, and N_l be the number of elements of U that belong to exactly l of the sets $\{A_1,\dots,A_s\}$. Then*

$$\sum_{0\le i\le 2\alpha-1}(-1)^i\binom{l+i}{i}S_{l+i} \le N_l \le \sum_{0\le i\le 2\alpha}(-1)^i\binom{l+i}{i}S_{l+i}. \tag{12}$$

I&E with two properties as a enumeration formula is introduced by McKay, Palmer, Read and Robinson [8] in which the following asymptotic estimate has been found using the equation which is similar to (12) with two properties : Let $g(2n,l_1,l_2)$ be the number of general cubic graphs on $2n$ labeled vertices with l_1 loops and l_2 double edges. Then for $l_1\ l_2=\mathrm{o}(\sqrt{n})$, they find

$$g(2n,l_1,l_2)=(1+\mathrm{o}(1))\frac{e^{-2}}{(3!)^{2n}}\cdot\frac{(6n)!}{2^{3n}\cdot(3n)!}\cdot\frac{2^{l_1}\cdot 2^{l_2}}{l_1!\cdot l_2!}. \tag{13}$$

It was used to find the total number of general cubic graphs with $2n$ vertices by summing up the values $g(2n,l_1,l_2)$ using equation (13) :

$$g(2n)=(1+\mathrm{o}(1))\frac{e^{2}}{(3!)^{2n}}\cdot\frac{(6n)!}{2^{3n}\cdot(3n)!}. \tag{14}$$

Wormald [10] first derived (14) by estimating the number of matrices with given row and column sums and it was also derived from matrix approximations of Bender and Canfield [1]. The formula (13) can be derived directly by the method of I&E on two types of properties in [8], but the proof of the inequality was omitted. The asymptotic numbers of general cubic graphs with given connectivity [3] are obtained as a result of (13) which may not be obtained from (14) alone without (13).

In this paper, we provide the proof for inclusion and exclusion on finitely many types of properties in Section 2. An inequality of I&E on three types of properties is obtained in Section 3, which is an application of I&E on three types of properties. We show that (13) and (14)can be obtained directly via I&E which gives a simpler method to handle triple edges than that of done in [3]. A generalization of the inequality of I&E on finitely many types of properties is used to find the asymptotic number of 4-regular graphs with given connectivity derived from I&E with five types of properties [4]. Most general graph theoretic terminologies and notations follow [5] and we assume the basic terminologies for I&E developed in [9] .

2. I&E for finitely many types of properties

Now we deal with k types of properties $k \geq 2$. Let U be the universal set of S_o elements. Let k be a positive integer where s_k is the number of subsets of U with property k. Suppose that $P_1^j, \ldots, P_{s_j}^j$ are subsets of U with property j $(j = 1, \ldots, k)$. For integers l_j with $0 \leq l_j \leq s_j$ $(j = 1, \ldots, k)$ define

$$S_{l_1,\ldots,l_k} = \sum \left| \bigcap_{i_1 \in I_1} P_{i_1}^1 \cap \cdots \cap \bigcap_{i_k \in I_k} P_{i_k}^k \right| \tag{15}$$

where the sum is over all l_j-subsets $I_j \subset [s_j]$ for $j = 1, \ldots, k$. For $0 \leq l_j \leq s_j (j = 1, \ldots, k)$, let $N_{l_1,\ldots,l_k}$ be the number of elements in U that belong to exactly l_j of the sets $\{P_i^j\}_{i=1}^{s_j}$ for $j = 1, \ldots, k$. That is

$$N_{l_1,\ldots,l_k} = \sum \left| \left(\bigcap_{i_1 \in I_1} P_{i_1}^1 \cap \bigcap_{i_1 \notin I_1} \overline{P_{i_1}^1} \right) \cap \cdots \cap \left(\bigcap_{i_k \in I_k} P_{i_k}^k \cap \bigcap_{i_k \notin I_k} \overline{P_{i_k}^k} \right) \right| \tag{16}$$

where the sum is again over all l_j-subsets $I_j \subset [s_j]$ for $j = 1, \ldots, k$. By counting the contribution to $S_{l_1,\ldots,l_k}$ of each element x of U that contributes to $N_{u_1,\ldots,u_k}$ for $u_1 \geq l_1, \ldots, u_k \geq l_k$, we have

$$S_{l_1,\ldots,l_k} = \sum_{\substack{l_1 \leq u_1 \leq s_1 \\ \vdots \\ l_k \leq u_k \leq s_k}} \prod_{i=1}^{k} \binom{u_i}{l_i} N_{u_1,\ldots,u_k}. \tag{17}$$

The numbers $S_{l_1,\dots,l_k}$ and $N_{l_1,\dots,l_k}$ are closely related and this relation is neatly expressed in terms of ordinary generating functions

$$S(x_1,\dots,x_k)=\sum_{l_1=0}^{s_1}\cdots\sum_{l_k=0}^{s_k}S_{l_1,\dots,l_k}x_1^{l_1}\dots x_k^{l_k} \tag{18}$$

and

$$N(x_1,\dots,x_k)=\sum_{l_1=0}^{s_1}\cdots\sum_{l_k=0}^{s_k}N_{l_1,\dots,l_k}x_1^{l_1}\dots x_k^{l_k}. \tag{19}$$

Then the following proposition can be obtained from (17), (18), and (19), respectively.

Proposition 3. *$S(x_1,\dots,x_k)$ and $N(x_1,\dots,x_k)$ are ordinary generating functions defined as above then*

$$N(x_1+1,\dots,x_k+1)=S(x_1,\dots,x_k). \tag{20}$$

If we set $x_1=\cdots=x_k=-1$ in (20), we obtain

$$N(0,\dots,0)=S(-1,\dots,-1). \tag{21}$$

This is the number of elements in U that belong to none of the sets $P_1^j,\cdots,P_{s_j}^j$ for $j=1,\dots,k$. Now if $x_1,\dots,x_k$ are replaced by $x_1-1,\dots,x_k-1$ in (20) respectively, we have, by comparing coefficients of $x^{l_1}\dots x^{l_k}$;

$$N_{l_1,\dots,l_k}=\sum_{\substack{0\le v_1\le s_1-l_1\\ \vdots\\ 0\le v_k\le s_k-l_k}}(-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i}S_{l_1+v_1,\dots,l_k+v_k}. \tag{22}$$

Then we consider the truncation
(23)

$$\sum_{\substack{0\le v_1\le \alpha_1\\ \vdots\\ 0\le v_k\le \alpha_k}}(-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i}S_{l_1+v_1,\dots,l_k+v_k}$$

$$=\sum_{\substack{0\le v_1\le \alpha_1\\ \vdots\\ 0\le v_k\le \alpha_k}}(-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i}\sum_{\substack{l_1+v_1\le u_1\\ \vdots\\ l_k+v_k\le u_k}}\prod_{i=1}^{k}\binom{u_i}{l_i+v_i}N_{u_1,\dots u_k},$$

where $0 < \alpha_i \le s_i - l_i$ and the right side has been obtained by substitution of (17). Now we interchange the order of summation and obtain
(24)

$$\sum_{\substack{0\le v_1\le\alpha_1\\ \vdots\\ 0\le v_k\le\alpha_k}} (-1)^{v_1+\cdots+v_k} \prod_{i=1}^{k} \binom{l_i+v_i}{v_i} S_{l_1+v_1,\ldots,l_k+v_k}$$

$$= \sum_{\substack{l_1\le u_1\\ \vdots\\ l_k\le u_k}} N_{u_1,\ldots u_k} \prod_{i=1}^{k} \binom{u_i}{l_i} \sum_{\substack{0\le v_1\le\alpha_1\\ \vdots\\ 0\le v_k\le\alpha_k}} (-1)^{v_1+\cdots+v_k} \prod_{i=1}^{k} \binom{u_i-l_i}{v_i}.$$

It can be seen that:

$$\sum_{\substack{0\le v_1\le\alpha_1\\ \vdots\\ 0\le v_k\le\alpha_k}} (-1)^{v_1+\cdots+v_k} \prod_{i=1}^{k} \binom{u_i-l_i}{v_i} \tag{25}$$

$$= \begin{cases} 1, & \text{if } u_i = l_i \text{ for all } i. \\ (-1)^{\alpha_1}\binom{u_1-l_1-1}{\alpha_1}, & \text{if } u_1 \geq l_1+1 \text{ and } u_i = l_i \\ & \quad \text{for all } i \neq 1; \\ \vdots & \\ (-1)^{\alpha_k}\binom{u_k-l_k-1}{\alpha_k}, & \text{if } u_k \geq l_k+1 \text{ and } u_i = l_i \\ & \quad \text{for all } i \neq k; \\ (-1)^{\alpha_1+\alpha_2}\binom{u_1-l_1-1}{\alpha_1}\binom{u_2-l_2-1}{\alpha_2}, & \text{if } u_1 \geq l_1+1,\ u_2 \geq l_2+1 \\ & \text{and } u_i = l_i \\ & \quad \text{for all } i \neq 1 \text{ and } 2; \\ \vdots & \\ (-1)^{\alpha_{k-1}+\alpha_k}\binom{u_{k-1}-l_{k-1}-1}{\alpha_{k-1}}\binom{u_k-l_k-1}{\alpha_k}, & \text{if } u_{k-1} \geq l_{k-1}+1, \\ & u_k \geq l_k+1 \text{ and } u_i = l_i \\ & \quad \text{for all } i \neq k-1 \text{ and } k; \\ (-1)^{\alpha_1+\alpha_2+\alpha_3}\binom{u_1-l_1-1}{\alpha_1}\binom{u_2-l_2-1}{\alpha_2}\binom{u_3-l_3-1}{\alpha_3}, & \text{if } u_1 \geq l_1+1,\ u_2 \geq l_2+1, \\ & u_3 \geq l_3+1 \text{ and } u_i = l_i \\ & \quad \text{for all } i \neq 1,\ 2 \text{ and } 3; \\ \vdots & \\ (-1)^{\alpha_1+\cdots+\alpha_k}\prod_{i=1}^{k}\binom{u_i-l_i-1}{\alpha_i}, & \text{if } u_i \geq l_i+1 \\ & \quad \text{for all } i = 1,\ldots,k. \end{cases}$$

Thus we have

$$\sum_{\substack{0\leq v_1\leq\alpha_1 \\ \vdots \\ 0\leq v_k\leq\alpha_k}} (-1)^{v_1+\cdots+v_k}\prod_{i=1}^{k}\binom{l_i+v_i}{v_i} S_{l_1+v_1,\ldots,l_k+v_k} = \quad N_{l_1,\ldots,l_k}$$

$$+ (-1)^{\alpha_1}\sum_{l_1+1\leq u_1} N_{u_1,l_2,\ldots l_k}\binom{u_1}{l_1}\binom{u_1-l_1-1}{\alpha_1}$$

$$\vdots$$

(26)

$$+ (-1)^{\alpha_k} \sum_{l_k+1\le u_k} N_{l_1,\dots,l_{k-1},u_k} \binom{u_k}{l_k} \binom{u_k - l_k - 1}{\alpha_k}$$

$$+ (-1)^{\alpha_1+\alpha_2} \sum_{\substack{l_1+1\le u_1 \\ l_2+1\le u_2}} N_{u_1,u_2,l_3,\dots l_k} \binom{u_1}{l_1} \binom{u_2}{l_2} \binom{u_1 - l_1 - 1}{\alpha_1} \binom{u_2 - l_2 - 1}{\alpha_2}$$

$$\vdots$$

$$+ (-1)^{\alpha_{k-1}+\alpha_k} \sum_{\substack{l_{k-1}+1\le u_{k-1} \\ l_k+1\le u_k}} N_{l_1,\dots,l_{k-2},u_{k-1},u_k} \binom{u_{k-1}}{l_{k-1}} \binom{u_k}{l_k} \binom{u_{k-1} - l_{k-1} - 1}{\alpha_{k-1}} \binom{u_k - l_k - 1}{\alpha_k}$$

$$\vdots$$

$$+ (-1)^{\alpha_1+\alpha_2+\alpha_3} \sum_{\substack{l_1+1\le u_1 \\ l_2+1\le u_2 \\ l_3+1\le u_3}} N_{u_1,u_2,u_3,l_4,\dots l_k} \prod_{i=1}^{3} \binom{u_i}{l_i} \binom{u_i - l_i - 1}{\alpha_i}$$

$$\vdots$$

$$+ (-1)^{\alpha_1+\cdots+\alpha_k} \sum_{\substack{l_1+1\le u_1 \\ \vdots \\ l_k+1\le u_k}} N_{u_1,\dots,u_k} \prod_{i=1}^{k} \binom{u_i}{l_i} \binom{u_i - l_i - 1}{\alpha_i}.$$

If we set all the values of $\{\alpha_i | i = 1, \dots, k\}$ to be even in (26), we have an upper bound for $N_{l_1,\dots,l_k}$. On the other hand, the corresponding lower bound is not easy to obtain.

Now we deal with the main theorem. We state a trivial result which is used for the proof of our main theorem.

Lemma 4. *Suppose α, n and k are positive integers. If $\alpha \ge \frac{n(k-1)-1}{k}$, for $k < n$, we have*

$$\binom{n}{\alpha} \ge (k-1) \binom{n}{\alpha+1}. \tag{27}$$

Theorem 5. *Suppose that $P_1^j, \dots, P_{s_j}^j$ are subsets of U with property j ($j = 1, \dots, k$). For integers l_j with $0 \le l_j \le s_j$ ($j = 1, \dots, k$) define*

$S_{l_1,\ldots,l_k} = \sum \left|\bigcap_{i_1\in I_1} P^1_{i_1} \cap \cdots \cap \bigcap_{i_k\in I_k} P^k_{i_k}\right|$ *where the sum is over all* l_j*-subsets* $I_j \subset [s_j]$ *for* $j = 1,\ldots,k$*. And for* $0 \le l_j \le s_j($ $j = 1,\ldots,k)$*, let* $N_{l_1,\ldots,l_k}$ *be the number of elements in* U *that belong to exactly* l_j *of the sets* $\{P^j_i\}^{s_j}_{i=1}$ *for* $j = 1,\ldots,k$*. Then there are* $\{\bar{\alpha_i}\}$*,* $i = 1,\ldots,k$ *with* $\bar{\alpha_i} > \frac{(s_i-l_i-1)(k-1)-1}{k}$ *such that*

(28)

$$\sum_{\substack{0\le v_1\le \bar{\alpha_1}\\ \vdots \\ 0\le v_k\le \bar{\alpha_k}}} (-1)^{v_1+\cdots+v_k} \prod_{i=1}^{k} \binom{l_i+v_i}{v_i} S_{l_1+v_1,\ldots,l_k+v_k}$$

$$\le \quad N_{l_1,\ldots,l_k} \quad \le \sum_{\substack{0\le v_1\le 2\alpha_1\\ \vdots \\ 0\le v_k\le 2\alpha_k}} (-1)^{v_1+\cdots+v_k} \prod_{i=1}^{k} \binom{l_i+v_i}{v_i} S_{l_1+v_1,\ldots,l_k+v_k}.$$

Proof. Since contribution of the sums on the right side of (17) is positive if all $\alpha_i' s (i = 1, ..., k)$ are even, we have the upper bound. For the lower bound, for convenient, assume that k is even(when k is odd we can do similarly). Let

(29)

$$\Phi^1_{\alpha_1} = \sum_{l_1+1\le u_1} N_{u_1,l_2,\ldots,l_k} \binom{u_1}{l_1} \binom{u_1-l_1-1}{\alpha_1},$$

$$\vdots$$

$$\Phi^1_{\alpha_k} = \sum_{l_k+1\le u_k} N_{l_1\ldots,l_{k-1},u_k} \binom{u_k}{l_k} \binom{u_k-l_k-1}{\alpha_k},$$

$$\Phi^2_{\alpha_1,\alpha_2} = \sum_{\substack{l_1+1\le u_1\\ l_2+1\le u_2}} N_{u_1,u_2,l_3,\ldots,l_k} \binom{u_1}{l_1} \binom{u_2}{l_2} \binom{u_1-l_1-1}{\alpha_1} \binom{u_2-l_2-1}{\alpha_2},$$

$$\vdots$$

$$\Phi^2_{\alpha_{k-1},\alpha_k} = \sum_{\substack{l_{k-1}+1\le u_{k-1}\\ l_k+1\le u_k}} N_{l_1,\ldots,l_{k-2},u_{k-1},u_k}$$

$$\binom{u_{k-1}}{l_{k-1}} \binom{u_k}{l_k} \binom{u_{k-1}-l_{k-1}-1}{\alpha_{k-1}} \binom{u_k-l_k-1}{\alpha_k}$$

$$\Phi^3_{\alpha_1,\alpha_2,\alpha_3} = \sum_{\substack{l_1+1\le u_1\\ l_2+1\le u_2\\ l_3+1\le u_3}} N_{u_1,u_2,u_3,l_4,\dots l_k} \prod_{i=1}^{3} \binom{u_i}{l_i}\binom{u_i-l_i-1}{\alpha_i},$$

$$\vdots$$

$$\Phi^3_{\alpha_{k-2},\alpha_{k-1},\alpha_k} = \sum_{\substack{l_{k-2}+1\le u_{k-2}\\ l_{k-1}+1\le u_{k-1}\\ l_k+1\le u_k}} N_{l_1,\dots,l_{k-3},u_{k-2},u_{k-1},u_k} \prod_{i=k-2}^{k} \binom{u_i}{l_i}\binom{u_i-l_i-1}{\alpha_i},$$

$$\vdots$$

$$\Phi^k_{\alpha_1,\dots,\alpha_k} = \sum_{\substack{l_1+1\le u_1\\ \vdots\\ l_k+1\le u_k}} N_{u_1,\dots,u_k} \prod_{i=1}^{k} \binom{u_i}{l_i}\binom{u_i-l_i-1}{\alpha_i}.$$

Note that there are $\binom{k}{2}$ terms of the form $\Phi^2_{\alpha_{i_1},\alpha_{i_2}}$, $\binom{k}{3}$ terms of the form $\Phi^3_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$, $\dots$, and $\Phi^k_{\alpha_1,\dots,\alpha_k}$ is the only one term in the form of $\Phi^k_{\alpha_1,\dots,\alpha_k}$ since $\binom{k}{k}=1$. Then (26) can be written:

(30)

$$\begin{aligned}
\sum_{\substack{0\le v_1\le\alpha_1\\ \vdots\\ 0\le v_k\le\alpha_k}} (-1)^{v_1+\dots+v_k} \prod_{i=1}^{k}\binom{l_i+v_i}{v_i} S_{l_1+v_1,\dots,l_k+v_k}
&= N_{l_1,\dots,l_k} + (-1)^{\alpha_1}\Phi^1_{\alpha_1} + \dots + (-1)^{\alpha_k}\Phi^1_{\alpha_k}\\
&\quad + (-1)^{\alpha_1+\alpha_2}\Phi^2_{\alpha_1,\alpha_2} + (-1)^{\alpha_1+\alpha_3}\Phi^2_{\alpha_1,\alpha_3} + \dots\\
&\qquad + (-1)^{\alpha_{k-2}+\alpha_k}\Phi^2_{\alpha_{k-2},\alpha_k} + (-1)^{\alpha_{k-1}+\alpha_k}\Phi^2_{\alpha_{k-1},\alpha_k}\\
&\quad + (-1)^{\alpha_1+\alpha_2+\alpha_3}\Phi^3_{\alpha_1,\alpha_2,\alpha_3} + (-1)^{\alpha_1+\alpha_2+\alpha_4}\Phi^3_{\alpha_1,\alpha_2,\alpha_4} + \dots\\
&\qquad + (-1)^{\alpha_{k-2}+\alpha_{k-1}+\alpha_k}\Phi^3_{\alpha_{k-2},\alpha_{k-1},\alpha_k}\\
&\quad + (-1)^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\Phi^4_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} + \dots\\
&\qquad \vdots\\
&\quad + (-1)^{\alpha_1+\dots+\alpha_k}\Phi^k_{\alpha_1,\dots,\alpha_k}.
\end{aligned}$$

It suffices to show that the right side of (30) except for $N_{l_1,\ldots,l_k}$ is less than or equal to zero to obtain the lower bound for $N_{l_1,\ldots,l_k}$.

Suppose there is no such set $\{\alpha_1, \ldots, \alpha_k\}$ that makes the right side of (30) except $N_{l_1,\ldots,l_k}$ less than or equal to zero. That means the right side of (30) except $N_{l_1,\ldots,l_k}$ is always positive for any set $\{\alpha_1, \ldots, \alpha_k\}$. Suppose $\{\alpha_1 + 1, \ldots, \alpha_k + 1\}$ is a set where all $\alpha_i' s (i = 1, ..., k)$ are odd such that $\alpha_i > \frac{(s_i-l_i-1)(k-1)-1}{k}$. Then this set makes the right side of (30) except $N_{l_1,\ldots,l_k}$ positive. We have k more sets which are made by replacing α_i+1 by α_i for each $i = 1, \ldots, k$, from the set $\{\alpha_1+1, \ldots, \alpha_k+1\}$. If we substitute one of this set in (30), the right side of (30) except $N_{l_1,\ldots,l_k}$ still be positive by our assumption. For example, if we substitute $\{\alpha_1+1, \alpha_2+1, \ldots, \alpha_{\omega-1}+1, \alpha_\omega, \alpha_{\omega+1}+1, \ldots, \alpha_k+1\}$ solution set in equation (30), we have the formula which is positive:
(31)

$$
\begin{aligned}
&\Phi^1_{\alpha_1+1} + \Phi^1_{\alpha_2+1} + \cdots + \Phi^1_{\alpha_{\omega-1}+1} \overbrace{-\Phi^1_{\alpha_\omega}} + \Phi^1_{\alpha_{\omega+1}+1} + \cdots + \Phi^1_{\alpha_k+1} \\
&\overbrace{-\Phi^2_{\alpha_\omega,\alpha_1+1} - \Phi^2_{\alpha_\omega,\alpha_2+1} \cdots - \Phi^2_{\alpha_\omega,\alpha_k+1}} + \Phi^2_{\alpha_1+1,\alpha_2+1} + \cdots + \Phi^2_{\alpha_{k-1}+1,\alpha_k+1} \\
&\overbrace{-\Phi^3_{\alpha_\omega,\alpha_1+1,\alpha_2+1} - \Phi^3_{\alpha_\omega,\alpha_1+1,\alpha_3+1} - \cdots - \Phi^3_{\alpha_\omega,\alpha_{k-1}+1,\alpha_k+1}} + \cdots \\
&+ \Phi^3_{\alpha_{k-2}+1,\alpha_{k-1}+1,\alpha_k+1} \\
&\overbrace{-\Phi^4_{\alpha_\omega,\alpha_1+1,\alpha_2+1,\alpha_3+1} - \cdots - \Phi^4_{\alpha_\omega,\alpha_{k-2}+1,\alpha_{k-1}+1,\alpha_k+1}} + \cdots \\
&\quad \vdots \\
&\overbrace{-\Phi^k_{\alpha_1+1,\alpha_2+1,\ldots,\alpha_{\omega-1}+1,\alpha_\omega,\alpha_{\omega+1}+1,\ldots,\alpha_k+1}}
\end{aligned}
$$

Let us call the left side of (31) as $\mathcal{D}_\omega$ where $\omega = 1, 2, \ldots, k$. We need some explanation for the negative/positive terms in (31). Note that $\binom{k-1}{\tau} + \binom{k-1}{\tau-1} = \binom{k}{\tau}$. For all $\tau = 1, \ldots, k$, there are $\binom{k-1}{\tau}$ terms of form $\Phi^\tau_{\alpha_1+1,\ldots,\alpha_\tau+1,}$ which contribute as positive quantities in (31). And there are $\binom{k-1}{\tau-1}$ terms contributing as negative quantities in (31) because $\tau - 1$ terms of the form α_i+1 are chosen from $\{\alpha_1+1, \ldots, \alpha_{\omega-1}+1, \widehat{\alpha_\omega}, \alpha_{\omega+1}+1, \ldots, \alpha_k + 1\}$ along with α_ω which was already chosen in the subscript of $\Phi^\tau_{\omega,\ldots,\alpha_{\tau-2}+1}$. Hence we have k inequalities just like (31). Now consider the sum of $\mathcal{D}_\omega$ for $\omega = 1, 2, \ldots, k$, that is $\sum_{\omega=1}^k \mathcal{D}_\omega$, which is denoted by Δ. By our assumption, we have

$$\Delta > 0.$$

Actually Δ is the formula which is the sum of k formula as (31) which can be obtained when we replace $\alpha_i + 1$ by α_i for each $i = 1, \ldots, k$, from the set $\{\alpha_1 + 1, \ldots, \alpha_k + 1\}$ where all $\alpha_i's (i = 1, ..., k)$ are odd such that $\alpha_i > \frac{(s_i - l_i - 1)(k-1)-1}{k}$.

Now for fixed τ where $1 \leq \tau \leq k$, consider

$$\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_{h-1}}+1,\alpha_{z_h}+1,\alpha_{z_{h+1}}+1,\ldots,\alpha_{z_\tau}+1}, \tag{32}$$

in Δ where $\{z_1, \ldots, z_\tau\}$ is a τ subset of $[k]$. Then overall there are k such terms in Δ, since we have one such term in each $\mathcal{D}_\omega$ for $\omega = 1, 2, \ldots, k$, which have an index set $\{z_1, \ldots, z_\tau\}$. Among them, τ terms are of the form

$$\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_{h-1}}+1,\alpha_{z_h},\alpha_{z_{h+1}}+1,\ldots,\alpha_{z_\tau}+1}$$

for each $h = 1, \ldots, \tau$ (*i.e.* α_{z_h} is odd and others are even) which contribute as negative quantities. And $k - \tau$ of them have the form

$$\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_{h-1}}+1,\alpha_{z_h}+1,\alpha_{z_{h+1}}+1,\ldots,\alpha_{z_\tau}+1}$$

which contribute as positive quantities. Hence for index set $\{z_1, \ldots, z_\tau\}$, consider

$$\begin{aligned} &- \Phi^{\tau}_{\alpha_{z_1},\alpha_{z_2}+1,\ldots,\alpha_{z_\tau}+1} \\ &- \Phi^{\tau}_{\alpha_{z_1}+1,\alpha_{z_2},\alpha_{z_3}+1\ldots,\alpha_{z_\tau}+1} \\ &\qquad \vdots \\ &- \Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_{\tau-1}+1},\alpha_{z_\tau}} \\ &+ (k-\tau)\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_\tau}+1} \end{aligned} \tag{33}$$

which is in Δ. Note that

$$\begin{aligned} (k-\tau)\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_\tau}+1} &\leq (k-1)\Phi^{\tau}_{\alpha_{z_1}+1,\ldots,\alpha_{z_\tau}+1} \\ &\leq \Phi^{\tau}_{\alpha_{z_1},\alpha_{z_2}+1,\ldots,\alpha_{z_\tau}+1}, \end{aligned} \tag{34}$$

since

$$\begin{aligned} (k-1)&\binom{u_{z_1} - l_{z_1} - 1}{\alpha_{z_1} + 1}\binom{u_{z_2} - l_{z_2} - 1}{\alpha_{z_2} + 1}\cdots\binom{u_{z_\tau} - l_{z_\tau} - 1}{\alpha_{z_\tau} + 1} \\ &\leq \binom{u_{z_1} - l_{z_1} - 1}{\alpha_{z_1}}\binom{u_{z_2} - l_{z_2} - 1}{\alpha_{z_2} + 1}\cdots\binom{u_{z_\tau} - l_{z_\tau} - 1}{\alpha_{z_\tau} + 1} \end{aligned} \tag{35}$$

is hold by Lemma 4. Thus (33) cannot be positive for index set $\{z_1, \ldots, z_\tau\}$. Note that Δ is the sum of the values (33) for $\tau = 1, \ldots, k$. This implies

$$\Delta \leq 0$$

which is a contradition to our assumption. This completes the proof.

3. Applications

Here we use the same idea that was used in [1, 3] for representing general cubic graphs with triples edges. The main difference between the computations of our paper and those of [3] is that we consider triple edges along with loops and double edges.

Let $V = \bigcup_{1\le i\le 2n} V_i$ be a partition of V into 3-subsets V_i for $i = 1,\dots,2n$. We define a configuration F with vertex set V where $|V| = 6n$. The edge set of F consists of $3n$ vertex-disjoint edges. Thus F is a 1-factor with vertex set $V = V(G)$ and it is easy to see that the total number of configurations is

$$\frac{(6n)!}{2^{3n}(3n)!}. \tag{36}$$

For any edge uv in F, if both vertices u and v belong to the same set V_i of the partition, the edge is called a 1-*cycle*. Otherwise they are contained in two different sets V_i and V_j. If there are exactly two such edges between V_i and V_j, we call this a 2-*cycle* and if there are three, it is a *triple*.

A graph G is obtained from this configuration by shrinking V_i to a vertex for all i. The graph G will have a vertex set V with $|V| = 2n$, and an edge set $E(G)$ with $|E(G)| = 3n$, where an edge $uv \in E(G)$ may be a loop, a single edge, or a part of double/triple edge where the number of these are denoted by l, s, d, or t, respectively. Note that cubic graphs with $2n$ vertices (*order* $2n$) satisfy the relation $2n = \frac{2s+4d+2l+6t}{3}$.

Now, for $i = 1,\dots,2n$, let P_i^1 be the set of configurations which have a 1-cycle in V_i. Assume the $\binom{2n}{2}$ pairs of sets V_i in the partition are ordered from 1 to $\binom{2n}{2}$. Let P_j^2 be the set of configurations which have a 2-cycle in the j^{th} pair for $j = 1,\dots,\binom{2n}{2}$. Let P_j^3 be the set of configurations which have a triple in the j^{th} pair for $j = 1,\dots,\binom{2n}{2}$. Let $N(l_1,l_2,l_3)$ be number of configurations with exactly l_1 1-cycles, l_2 2-cycles, and l_3 triples, that means the number of configurations which belong to exactly l_1 of $\{P_i^1 | i = 1,\dots,2n\}$, l_2 of $\{P_j^2 | j = 1,\dots,\binom{2n}{2}\}$, and l_3 of $\{P_j^3 | j = 1,\dots,\binom{2n}{2}\}$.

Let S_{l_1,l_2,l_3} be the number of configurations which have at least l_1 1-cycles, l_2 2-cycles and l_3 triple edges. Then S_{l_1,l_2,l_3} can be found by

using its definition in (15);
(37)

$$S_{l_1,l_2,l_3} = \binom{2n}{l_1, 2l_2, 2l_3, 2n-l_1-2l_2-2l_3} 3^{l_1} \frac{(2l_2)!}{2^{l_2}\cdot l_2!} (3^2\cdot 2)^{l_2} \frac{(2l_3)!}{2^{l_3}\cdot l_3!} (6)^{l_3} (1+\mathrm{o}(1)) \frac{(2(3n-l_1-2l_2-3l_3))!}{2^{3n-l_1-2l_2-3l_3}(3n-l_1-2l_2-3l_3)!}.$$

where the term $(1+\mathrm{o}(1))$ allows for a negligible number of new triples from 2-cycles(see Lemma 1 in [3]). Here is a sketch of the justification of this formula. First we choose $l_1+2l_2+2l_3+2n-l_1-2l_2-2l_3$ labels from the $2n$ available. Then it can be seen that the number of ways to form the adjacencies is

$$3^{l_1} \frac{(2l_2)!}{2^{l_2}\cdot l_2!} (3^2\cdot 2)^{l_2} \frac{(2l_3)!}{2^{l_3}\cdot l_3!} (6)^{l_3}$$

for configurations which have l_1 1-cycles, l_2 2-cycles and l_3 triple edges.

$$\frac{(2(3n-l_1-2l_2-3l_3))!}{2^{3n-l_1-2l_2-3l_3}(3n-l_1-2l_2-3l_3)!}$$

is the number of ways to lay down the remaining edges. Then, on substituting (37) into (28) and simplifying it, we have the number of configurations which have exactly l_1 1-cycles, l_2 2-cycles and l_3 triples;

Theorem 6. *For* $l_1, l_2, l_3 = o(\sqrt{n})$,

$$N_{l_1,l_2,l_3} = (1+o(1)) \frac{e^{-2-\frac{1}{18n}}}{l_1!l_2!l_3!(18n)^{l_3}} \frac{(6n)!}{2^{3n}(3n)!}. \tag{38}$$

Proof. It is enough to consider right side of (28), that is upper bound of $N(l_1, l_2, l_3)$. On substituting equation (37) into the right side of (28) and simplifying it with the fact that $\frac{(n)_k}{n^k} = 1+\mathrm{o}(1)$ for $k = \mathrm{o}(n^{1/2})$, we

have, where $k = 3$,
(39)

$$
\begin{aligned}
N(l_1, l_2, l_3) \quad & \leq \sum_{\substack{0 \leq v_1 \leq 2\alpha_1 \\ 0 \leq v_2 \leq 2\alpha_2 \\ 0 \leq v_3 \leq 2\alpha_3}} (-1)^{v_1+v_2+v_3} \prod_{i=1}^{3} \binom{l_i + v_i}{v_i} \\
& \binom{2n}{l_1, 2l_2, 2l_3, 2n - l_1 - 2l_2 - 2l_3} 3^{l_1} \frac{(2l_2)!}{2^{l_2} \cdot l_2!} (3^2 \cdot 2)^{l_2} \frac{(2l_3)!}{2^{l_3} \cdot l_3!} (6)^{l_3} \\
& \frac{(2(3n - l_1 - 2l_2 - 3l_3))!}{2^{3n-l_1-2l_2-3l_3}(3n - l_1 - 2l_2 - 3l_3)!} \\
& = \frac{1}{l_1! l_2! l_3!} \frac{(6n)!}{2^{3n}(3n)!} \sum_{\substack{0 \leq v_1 \leq 2\alpha_1 \\ 0 \leq v_2 \leq 2\alpha_2 \\ 0 \leq v_3 \leq 2\alpha_3}} \frac{(-1)^{v_1+v_2+v_3}}{v_1! v_2! v_3!} \\
& \quad 3^{l_1+v_1+2l_2+2v_2+l_3+v_3} 2^{l_1+v_1+2l_2+2v_2+3(l_3+v_3)} \\
& \quad \frac{(2n)!}{(2n - (l_1 + v_1) - 2(l_2 + v_2) - 2(l_3 + v_3)!} \\
& \quad \frac{(3n)!}{(3n - (l_1 + v_1) - 2(l_2 + v_2) - 3(l_3 + v_3)!} \\
& \quad \frac{(6n - 2(l_1 + v_1) - 4(l_2 + v_2) - 6(l_3 + v_3)!}{(6n)!} \\
& \sim (1 + \mathrm{o}(1)) \frac{1}{l_1!\, l_2!\, l_3!} \frac{1}{(18n)^{l_3}} \frac{(6n)!}{2^{3n} \cdot (3n)!} \\
& \left[\sum_{\substack{0 \leq v_1 \leq 2\alpha_1 \\ 0 \leq v_2 \leq 2\alpha_2 \\ 0 \leq v_3 \leq 2\alpha_3}} \frac{(-1)^{v_1+v_2+v_3}}{v_1!\, v_2!\, v_3!} \frac{1}{(18n)^{v_3}} \right] \\
& \sim (1 + \mathrm{o}(1)) \frac{e^{-2-\frac{1}{18n}}}{l_1! l_2! l_3! (18n)^{l_3}} \frac{(6n)!}{2^{3n}(3n)!}, \\
& \quad (\text{since} \sum (-1)^n / n! = e^{-1}).
\end{aligned}
$$

where both l, $d = \mathrm{o}(\sqrt{n})$ and $\{\bar{\alpha_i}\}$ with $\bar{\alpha_i} > \frac{(s_i - l_i - 1)(k-1)-1}{k} (1 \leq i \leq 3)$. Since we have the same result for the left side of (28), so the proof is completed.

Let $g(2n, l_1, l_2, l_3)$ be the number of labelled cubic general graphs G with exactly l_1 loops, l_2 double edges and l_3 triple edges. Then we have the following relationship between $g(2n, l_1, l_2, l_3)$ and $N(l_1, l_2, l_3) = N_{l_1,l_2,l_3}$ by shrinking each 3-vertex set V_i of a configuration to a single vertex in a graph.

Proposition 7.

$$N(l_1, l_2, l_3) = g(2n, l_1, l_2, l_3) 3^{l_1} \binom{3}{2}^{2l_2} 2^{l_2} 6^{l_3} (3!)^{2n-l_1-2l_2-2l_3}. \tag{40}$$

Then from Theorem 6 and Proposition 7, we have the following corollaries.

Corollary 8. *For $l_1, l_2, l_3 = o(\sqrt{n})$, we have*

$$g(2n, l_1, l_2, l_3) = (1 + o(1)) \frac{e^{-2-\frac{1}{18n}}}{(3!)^{2n}} \frac{(6n)!}{2^{3n}(3n)!} \frac{2^{l_1} 2^{l_2} (\frac{1}{3n})^{l_3}}{l_1! l_2! l_3!}. \tag{41}$$

It is interesting to compare this result to (13). The fact that triple edges are negligible is proved in [3] with tremendous work. It is easy to handle triple edges when we find (41) by using I&E with three types of properties.

We have the following asymptotic number of general cubic graphs on $2n$ vertices.

Corollary 9.

$$g(2n) = (1 + o(1)) \frac{e^{2+\frac{5}{18n}}}{(3!)^{2n}} \cdot \frac{(6n)!}{2^{3n} \cdot (3n)!}. \tag{42}$$

The asymptotic number of general cubic graphs on $2n$ vertices obtained in Corollary 9 looks different to the one in (14) which was found in Wormald [10], Bender and Canfield [1] and Chae [3]. However they are basically the same. Another application of I&E with five types of properties can be found in [4] in which the asymptotic numbers of general 4-regular graphs are computed with a given connectivity. Note that those numbers can not be obtained from the formula in Proposition 3.8 in [10].

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References

[1] E.A. Bender and E.R. Canfield, The asymptotic number of labeled graphs with given degree sequences, *J. Combin. Theory Ser A* **24** (1978) 296–307.

[2] Béla Bollabás, A probabilistic proof of an asymptotic formula for the number of labeled regular graphs, *Europ. J. Combinatorics* **1** (1980), 311–316.

[3] G.-B. Chae, Asymptotic number of General Cubic Graphs with given connectivity, *Journal of the Korean Math. Soc.* **42,** (2005), No. 6, 1187-1203.

[4] G.-B. Chae, Asymptotic number of General 4-Regular Graphs with given connectivity, *Bulletin of the Korean Math. Soc.* **43,** (2006), No. 1, 125-140.

[5] G. Chartrand and L. Lesniak, Graphs and Digraphs, Fourth Edition, Chapman & Hall/CRC, Boca Raton (2005).

[6] Janos Galambos and Italo Simonelli, *Bonferroni-type Inequalities with applications* , (Springer-Verlag, 1996).

[7] R.M. Meyer, Note on a 'multivariate ' form of Bonferroni's inequalities, *The annals of Mathematical Statistics* **40** 2 (1969), 692–693.

[8] B.D. McKay, E.M. Palmer, R.C. Read and R.W. Robinson, The asymptotic number of claw-free cubic graphs, *Discrete Math.* **272** (2003), 107–118.

[9] E.M. Palmer, *Graphical Evolution* , (John Wiley & Sons, 1985).

[10] N.C. Wormald, Some problems in the enumeration of labelled graphs, *Doctoral Thesis,* (University of Newcastle, Callaghan, NSW, Australia, 1978).

Gab-Byoung Chae
Division of Mathematics and Informational Statistics
Wonkwang University, Iksan, Jeollabukdo, Korea
E-mail: rivendell@wonkwang.ac.kr

Minseok Cheong
Division of Mathematics
Sogang University, Seoul, Korea
E-mail: macross@sogang.ac.kr

Sang-Mok Kim
Division of general education-Mathematics
Kwangwoon University, Seoul, Korea
E-mail: smkim@kw.ac.kr