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THE NONEXISTENCE OF CONFORMAL DEFORMATIONS ON SPACE-TIMES

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Abstract. In this paper, when N is a compact Riemannian manifold, we discuss the nonexistence of conformal deformations on space-times $M = (a, \infty) \times_f N$ with prescribed scalar curvature functions.

I. Introduction

In a recent study ([5, 6, 7]), M.C. Leung has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of positive solution to equation

$$\Delta_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}}, \tag{1.1}$$

where Δ_{g_0} is the Laplacian operator for an $n-$dimensional Riemannian manifold (N, g_0) and $d_n = \dfrac{n-2}{4(n-1)}$. Equation (1.1) is derived from the conformal deformation of Riemannian metric(cf. [1, 3, 4, 7]).

Similarly, let (N, g_0) be a compact Riemannian n-dimensional manifold. We consider the $(n+1)$-dimensional Lorentzian warped manifold $M = (a, \infty) \times_f N$ with the metric $g = -dt^2 + f(t)^2 g_0$, where f is a positive function on (a, ∞). Let $u(t, x)$ be a positive smooth function on M and let g have a negative constant scalar curvature equal to $-c$, where $c > 0$. If the conformal metric $\overline{g} = u(t, x)^{\frac{4}{n-1}} g$ has a prescribed function $h(t, x)$ as a scalar curvature, then $u(t, x)$ satisfies equation

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$$\frac{4n}{n-1}\Box_g u(t,x) + cu(t,x) + h(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0, \tag{1.2}$$

where $\Box_g$ is the d'Alembertian for a Lorentzian warped manifold $M = (a,\infty) \times_f N$.

In this paper, we study the nonexistence of a positive solution to equation (1.2).

2. Main Results

In this section, we let (N, g_0) be a compact Riemannian n-dimensional manifold with $n \geq 3$ and without boundary. The following proposition is well known(cf. Theorem 5.4 in [2]).

Proposition 1. Let $M = (a,\infty) \times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2 g_0$. Then the Laplacian $\Box_g$ is given by

$$\Box_g = -\frac{\partial^2}{\partial t^2} - \frac{nf'(t)}{f(t)}\frac{\partial}{\partial t} + \frac{1}{f(t)^2}\Delta_x,$$

where Δ_x is the Laplacian on fiber manifold N.

By Proposition 1, equation (1.2) is changed into the following equation

$$\begin{aligned} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial u(t,x)}{\partial t} - \frac{1}{f(t)^2}\Delta_x u(t,x) \\ - \quad c_n u(t,x) - H(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0, \end{aligned} \tag{2.1}$$

where $c_n = \frac{n-1}{4n}c$ and $H(t,x) = \frac{n-1}{4n}h(t,x)$.

If $u(t,x) = u(t)$ is a positive function with only variable t and if $H(t,x) = H(t)$ is also a function of only variable t, then equation (2.1) becomes

$$u^{''}(t) + \frac{nf^{'}(t)}{f(t)}u^{'}(t) = c_n u(t) + H(t)u(t)^{\frac{n+3}{n-1}}. \tag{2.2}$$

The proof of the following theorem is more extension than that of Theorem 4.9 in [7].

Theorem 2. Let $u(t)$ be a positive solution of equation (2.2) and let $H(t)$ satisfy $H(t) \geq c_1$, where c_1 is a positive constant. Assume that there exist positive constants t_0 and C_0 such that $|\frac{f^{'}(t)}{f(t)}| \leq C_0$ for all $t > t_0$. Then $u(t)$ is bounded from above.

Proof. From equation (2.2) we have

$$\frac{(f^n u^{'})^{'}}{f^n} = c_n u + H(t) u^{\frac{n+3}{n-1}}. \tag{2.3}$$

Let $\chi \in C_0^\infty((a,\infty))$ be a cut-off function. Multiplying both sides of equation (2.3) by $\chi^{n+1}u$ and then using integration by parts we obtain

$$\begin{aligned} -\int_a^\infty (f^n u^{'})(\frac{\chi^{n+1}u}{f^n})^{'} dt &= c_n \int_a^\infty \chi^{n+1}u^2 dt + \int_a^\infty H(t)\chi^{n+1}u^{\frac{2n+2}{n-1}} dt \\ &\geq c_1 \int_a^\infty \chi^{n+1} u^{\frac{2n+2}{n-1}} dt. \end{aligned}$$

From the left side of the above equation, we have

$$-(f^n u^{'})(\frac{\chi^{n+1}u}{f^n})^{'} = -(n+1)\chi^n u \chi^{'} u^{'} - \chi^{n+1}|u^{'}|^2 + n\chi^{n+1} u u^{'} \frac{f^{'}}{f}.$$

Applying the Cauchy inequality, we get

$$\begin{aligned} -(n+1)\chi^n u \chi^{'} u^{'} &= -2(\frac{n+1}{\sqrt{2}}\chi^{\frac{n+1}{2}-1} u \chi^{'})(\frac{1}{\sqrt{2}}\chi^{\frac{n+1}{2}} u^{'}) \\ &\leq \frac{(n+1)^2}{2}\chi^{n-1}u^2|\chi^{'}|^2 + \frac{1}{2}\chi^{n+1}|u^{'}|^2 \end{aligned}$$

and

$$\begin{aligned} n\chi^{n+1} u u^{'} \frac{f^{'}}{f} &= 2(\frac{n}{\sqrt{2}}\chi^{\frac{n+1}{2}} u \frac{f^{'}}{f})(\frac{1}{\sqrt{2}}\chi^{\frac{n+1}{2}} u^{'}) \\ &\leq \frac{n^2}{2}\chi^{n+1}(\frac{f^{'}}{f})^2 u^2 + \frac{1}{2}\chi^{n+1}|u^{'}|^2. \end{aligned}$$

Together with the above equations, we obtain

$$\frac{(n+1)^2}{2}\int_a^\infty (\frac{f'}{f})^2\chi^2u^2dt \ + \ \frac{(n+1)^2}{2}\int_a^\infty \chi^{n-1}u^2|\chi'|^2dt \geq c_1\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt.$$

Applying Young's inequality and using the bound $|\frac{f'}{f}| \leq C_0$, we have

$$c_1\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt \leq C'\int_a^\infty (|\chi'|^{n+1}+\chi^{n+1})dt, \tag{2.4}$$

where C' is a positive constant. Let $\chi \equiv 0$ on $(a,r] \cup [r+3,\infty)$ with $r > t_0$ and $\chi \equiv 1$ on $[r+1,r+2]$, $\chi \geq 0$ on $[a,\infty)$ and $|\chi'| \leq \frac{1}{2}$. From equation (2.4) we have

$$\int_{r+1}^{r+2} u^{\frac{2n+2}{n-1}}dt \leq C''$$

for all $r > t_0$,where C'' is a constant independent on r. Therefore u is bounded from above. □

Theorem 3. Let (M,g) be a Lorentzian manifold with scalar curvature equal to $-c$. Assume that there exist positive constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \leq C_0$ for all $t > t_0$. If $H(t)$ is a scalar curvature satisfying $H(t) \geq c_1$, where c_1 is a positive constant, then equation (2.2) has no positive solution.

Proof. If $u = u(t)$ is a positive solution of equation (2.2), then by Theorem 2 $u(t)$ is bounded from above on (a,∞). Then, by Omori-Yau maximum principle (c.f. [8]), there exists a sequence $\{t_k\}$ such that $lim_{k\to\infty}u(t_k) = sup_{t\in[a,\infty)}u(t)$,$|u'(t_k)| \leq \frac{1}{k}$ and $u''(t_k) \leq \frac{1}{k}$. Since $sup_{t\in[a,\infty)}u(t) = c_2 > 0$, there exist a number $\epsilon > 0$ and K such that

$$(c_nu(t_k) + H(t_k)u(t_k)^{\frac{n+3}{n-1}}) > \epsilon$$

for all $k > K$, which is a contradiction to the fact that

$$u''(t_k) + \frac{nf'(t_k)}{f(t_k)}u'(t_k) \leq \frac{1+nC_0}{k}$$

for all $k > K$. Therefore equation (2.2) has no positive solution. □

The following corollary is derived easily from the previous theorems

Corollary 4. Let $(M, g) = ((a, \infty) \times_f N), g)$ be a Lorentzian manifold with scalar curvature equal to $h(t) \leq 0$. Assume that there exist positive constants t_0 and C_0 such that $|\frac{f^{'}(t)}{f(t)}| \leq C_0$ for all $t > t_0$. If $H(t) = C$, where C is a positive constant, then the following equation

$$u^{''}(t) + \frac{nf^{'}(t)}{f(t)}u^{'}(t) = \frac{4n}{n-1}h(t)u(t) + Cu(t)^{\frac{n+3}{n-1}}$$

also has no positive solution.

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