Honam Mathematical J. **32** (2010), No. 1, pp. 77–84

ALMOST MINIMAL PRECONTINUOUS FUNCTIONS

Won Keun Min AND **Young Key Kim**

Abstract. In this paper, we introduce the notion of almost minimal precontinuous function and investigate characterizations for such a function.

1. Introduction

In [6], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of m-continuous function [6] as a function defined between a minimal structure and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. In [3], we introduced the notion of m-preopen sets defined on minimal structures and investigated some basic properties. In [4], we introduced and studied the notion of m-precontinuous function which is a generalization of m-continuous function defined between a minimal structure and a topological space. In this paper, we introduce the notion of almost m-precontinuous function defined between a minimal structure and a topological space and investigate characterizations for the function.

2. Preliminaries

Let X be a topological space and $A \subseteq X$. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* [6] on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call

Received January 26, 2010. Accepted March 11, 2010.
2000 Mathematics Subject Classification: 54C08.
Key words and phrases: minimal structures, m-preopen sets, m-continuous, m-precontinuous, almost m-precontinuous, m-precompact.
Corresponding author:wkmin@kangwon.ac.kr.

(X, m_X) a space with a minimal structure m_X on X. Elements in m_X are called m-open sets. Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-closure of A and the m-interior of A are defined as the following [6]:

$$mInt(A) = \cup\{U : U \subseteq A, U \in m_X\};$$
$$mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}.$$

A set A is called an m-preopen set in X if

$$A \subseteq mInt(mCl(A)).$$

A set A is called an *m-preclosed* set if the complement of A is m-preopen.

The *m-pre-closure* and the *m-pre-interior* of A, denoted by $mpCl(A)$ and $mpInt(A)$, respectively, are defined as the following:

$$mpCl(A) = \cap\{F \subseteq X : A \subseteq F, F \text{ is } m\text{-preclosed in } X\}$$
$$mpInt(A) = \cup\{U \subseteq X : U \subseteq A, U \text{ is } m\text{-preopen in } X\}.$$

Theorem 2.1. ([3]) Let (X, m_X) be a space with a minimal structure m_X and $A \subseteq X$. Then

(1) $mpInt(A) \subseteq A \subseteq mpCl(A)$.

(2) If $A \subseteq B$, then $mpInt(A) \subseteq mpInt(B)$ and $mpCl(A) \subseteq mpCl(B)$.

(3) A is m-preopen iff $mpInt(A) = A$.

(4) F is m-preclosed iff $mpCl(F) = F$.

(5) $mpInt(mpInt(A)) = mpInt(A)$ and $mpCl(mpCl(A)) = mpCl(A)$.

(6) $mpCl(X - A) = X - mpInt(A)$ and $mpInt(X - A) = X - mpCl(A)$.

Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with minimal structure m_X and a topological space (Y, τ). Then

(1) f is said to be *m-continuous* [6] if for each x and each open set V containing $f(x)$, there exists an m-open set U containing x such that $f(U) \subseteq V$;

(2) f is said to be *minimal precontinuous* (briefly *m-precontinuous*) [4] if for each x and each open set V containing $f(x)$, there exists an m-preopen set U containing x such that $f(U) \subseteq V$.

3. Almost Minimal Precontinuous Functions

Definition 3.1. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space Y. Then f is said to be *almost minimal precontinuous* (briefly, *almost m-precontinuous*) at x in X if for each open subset V containing $f(x)$, there is an m-preopen set U containing x such that $f(U) \subseteq int(cl(V))$. A function f is said to be *almost minimal precontinuous* if it has the property at each point of X.

m-continuity $\Rightarrow$ m-precontinuity $\Rightarrow$ almost m-precontinuity

In the above diagram, the converse may not be true as seen in the next example.

Example 3.2. In $X = \{a, b, c\}$, consider a minimal structure $m_X = \{\emptyset, \{a\}, \{b\}, X\}$ and a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $f : (X, m_X) \to (X, \tau)$ be a function defined as the following:

$$f(a) = a; f(b) = c; f(c) = b.$$

Then f is almost m-precontinuous but not m-precontinuous.

Theorem 3.3. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following are equivalent:

(1) f is almost m-precontinuous.
(2) $f^{-1}(V) \subseteq mpInt(f^{-1}(int(cl(V))))$ for every open subset V of Y.
(3) $mpCl(f^{-1}(cl(int(F)))) \subseteq f^{-1}(F)$ for every closed set F of Y.
(4) $mpCl(f^{-1}(cl(int(cl(B))))) \subseteq f^{-1}(cl(B))$ for every set B of Y.
(5) $f^{-1}(int(B)) \subseteq mpInt(f^{-1}(int(cl(int(B)))))$ for every set B of Y.
(6) $f^{-1}(V)$ is m-preopen for every regular open subset V of Y.

Proof. (1) $\Rightarrow$ (2) Let V be an open set in Y. For each $x \in f^{-1}(V)$, by hypothesis, there exists an m-preopen set U of X containing x such that $f(U) \subseteq int(cl(V))$. Then since $x \in U \subseteq f^{-1}(int(cl(V)))$, by definition of the interior operator $mpInt$, $x \in mpInt(f^{-1}(int(cl(V))))$. So we have $f^{-1}(V) \subseteq mpInt(f^{-1}(int(cl(V))))$.

(2) $\Rightarrow$ (3) Let F be a closed subset in Y. Then from (2) and Theorem 2.1,

$$\begin{aligned} f^{-1}(Y - F) &\subseteq mpInt(f^{-1}(int(cl(Y - F)))) \\ &= mpInt(f^{-1}(Y - cl(int(F)))) \\ &\subseteq X - mpCl(f^{-1}(cl(int(F)))). \end{aligned}$$

So $mpCl(f^{-1}(cl(int(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) For $B \subseteq Y$, since $cl(B)$ is closed in Y, from (3), it follows $mpCl(f^{-1}(cl(int(cl(B))))) \subseteq f^{-1}(cl(B))$.

(4) $\Rightarrow$ (5) For $B \subseteq Y$, from (4), it follows

$$\begin{aligned} f^{-1}(int(B)) &= X - f^{-1}(cl(Y - B)) \\ &\subseteq X - mpCl(f^{-1}(cl(int(cl(Y - B))))) \\ &= mpInt(f^{-1}(int(cl(int(B))))). \end{aligned}$$

Thus we get the result.

(5) $\Rightarrow$ (6) Obvious.

(6) $\Rightarrow$ (1) For each $x \in X$ and V any open set in Y containing $f(x)$, since $int(cl(V))$ is regular open, by (6), $f^{-1}(int(cl(V)))$ is an m-preopen set. Put $U = f^{-1}(int(cl(V)))$; then the m-preopen set U satisfies $f(x) \in f(U) \subseteq int(cl(V))$. Hence f is almost m-precontinuous. □

Let X be a topological space. A subset S of X is called *semi-open set* [1] (resp. *α-set*, *β-set* [5], *preopen set* [2], *regular open set*) if $S \subseteq cl(int(S))$ (resp. $S \subseteq int(cl(int(S)))$, $S \subseteq cl(int(cl(S)))$, $S \subseteq int(cl(S))$, $S = int(cl(S))$). The complement of a semi-open set (resp. α-set, β-set, preopen set, regular open) is called *semi-closed set* (resp. *α-closed set*, *β-closed set*, *preclosed set*, *regular closed*).

Theorem 3.4. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following are equivalent:
(1) f is almost m-precontinuous.
(2) $f^{-1}(K)$ is m-preclosed for every regular closed set K of Y.
(3) $mpCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every β-set G of Y.
(4) $mpCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every semiopen set G of Y.

Proof. (1) $\Leftrightarrow$ (2) It is obvious from Theorem 3.3.

(2) $\Rightarrow$ (3) Let G be any β-set. Then from $cl(G) = cl(int(cl(G)))$, we know that $cl(G)$ is a regular closed set. So from (2),

$$mpCl(f^{-1}(G)) \subseteq mpCl(f^{-1}(cl(G))) = f^{-1}(cl(G)).$$

Hence $mpCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$.

(3) $\Rightarrow$ (4) It is obvious since every semiopen set is β-open.

(4) $\Rightarrow$ (2) Let V be any regular closed set of Y. Then V also is semiopen. By (4) and $cl(V) = V$,

$$mpCl(f^{-1}(V)) \subseteq f^{-1}(cl(V)) = f^{-1}(V).$$

This implies that $f^{-1}(V)$ is m-preclosed. □

Theorem 3.5. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following are equivalent:

(1) f is almost m-precontinuous.

(2) $mpCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.

(3) $f^{-1}(G) \subseteq mpInt(f^{-1}(int(cl(G))))$ for every preopen set G of Y.

Proof. (1) $\Leftrightarrow$ (2) Let G be any preopen set in Y. Then since G also is β-open, from Theorem 3.4, (2) is obviously obtained.

(1) $\Rightarrow$ (3) Let G be any preopen set of Y; then $int(cl(G))$ is regular open in Y. From Theorem 3.3,

$$f^{-1}(G) \subseteq f^{-1}(int(cl(G))) = mpInt(f^{-1}(int(cl(G)))).$$

Hence $f^{-1}(G) \subseteq mpInt(f^{-1}(int(cl(G))))$.

(3) $\Rightarrow$ (1) Let G be any regular open set in Y. Then G is preopen. By (3) and $G = int(cl(G))$, $f^{-1}(G) \subseteq mpInt(f^{-1}(int(cl(G))) = mpInt(f^{-1}(G))$. It implies that $f^{-1}(G)$ is m-preclosed, and hence by Theorem 3.3, f is almost m-precontinuous. $\square$

We recall that a subset A in a topological space X is said to be δ-*open* [8] if for each $x \in A$ there exists a regular open set G such that $x \in G \subseteq A$. A point $x \in X$ is called a δ-*cluster* point of A if $A \cap int(cl(V)) \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called δ-*closure* of A [8] and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-*closed*. The complement of a δ-closed set is said to be δ-*open*. It is shown in [8] that $cl(A) = cl_\delta(A)$ for every open set A and $cl_\delta(B)$ is closed for every subset B of X. The set $\{x \in X : x \in U \subseteq A$ for some regular open set $U\, of\, X\}$ is called the δ-*interior* of A and is denoted by $int_\delta(A)$.

Theorem 3.6. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following are equivalent:

(1) f is almost m-precontinuous.

(2) $mpCl(f^{-1}(cl(int(cl_\delta(B))))) \subseteq f^{-1}(cl_\delta(B))$ for every set B of Y.

(3) $mpCl(f^{-1}(cl(int(cl(B))))) \subseteq f^{-1}(cl_\delta(B))$ for every set B of Y.

(4) $mpCl(f^{-1}(cl(int(cl(G))))) \subseteq f^{-1}(cl(G))$ for every open set G of Y.

(5) $mpCl(f^{-1}(cl(int(cl(G))))) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let B be any subset in Y. Then since $cl_\delta(B)$ is closed, by Theorem 3.3 (3), the statement (2) is obtained.

(2) $\Rightarrow$ (3) It follows from $cl(B) \subseteq cl_\delta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4) It is obvious since $cl(G) = cl_\delta(G)$ for every open subset G of Y.

(4) $\Rightarrow$ (5) Let G be a preopen subset of Y. Then $cl(G) = cl(int(cl(G)))$. Set $A = int(cl(G))$ then by (4), $mpCl(f^{-1}(cl(int(cl(A))))) \subseteq f^{-1}(cl(A))$. Since $cl(A) = cl(G)$, we have $mpCl(f^{-1}(cl(int(cl(G))))) \subseteq f^{-1}(cl(G))$.

(5) $\Rightarrow$ (1) Let A be a regular closed subset of Y. Then $int(A)$ is preopen. From (5) and $A = cl(int(A))$,

$$\begin{aligned} mpCl(f^{-1}(A)) &= mpCl(f^{-1}(cl(int(A)))) \\ &= mpCl(f^{-1}(cl(int(cl(int(A)))))) \\ &\subseteq f^{-1}(cl(int(A))) \\ &= f^{-1}(A). \end{aligned}$$

It implies $f^{-1}(A)$ is m-preclosed, and so by Theorem 3.4, f is almost m-precontinuous. □

Theorem 3.7. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following are equivalent:

(1) f is almost m-precontinuous.
(2) $f(mpCl(B)) \subseteq cl_\delta(f(B))$ for every set B of X.
(3) $f^{-1}(F)$ is m-preclosed for every δ-closed set F of Y.
(4) $f^{-1}(G)$ is m-preopen for every δ-open set G of Y.
(5) $f^{-1}(int_\delta(B))) \subseteq mpInt(f^{-1}(B))$ for every set B of Y.
(6) $mpCl(f^{-1}(B)) \subseteq f^{-1}(cl_\delta(B))$ for every set B of Y.

Proof. (1) $\Rightarrow$ (2) For $B \subseteq Y$, let $x \in mpCl(B)$ and V any open set of Y containing $f(x)$. Then there exists an m-preopen set U containing x such that $f(U) \subseteq int(cl(V))$. Since $x \in mpCl(B)$, $B \cap U \neq \emptyset$ for the m-preopen set U, and so $\emptyset \neq f(U) \cap f(B) \subseteq int(cl(V)) \cap f(B)$. This implies $f(x) \in cl_\delta(f(B))$. Consequently, $f(mpCl(B)) \subseteq cl_\delta(f(B))$.

(2) $\Rightarrow$ (3) Let F be any cl_δ-closed set of Y. Then from (2), it follows

$$f(mpCl(f^{-1}(F))) \subseteq cl_\delta(f(f^{-1}(F))) \subseteq cl_\delta(F) = F.$$

So this implies $f^{-1}(F)$ is m-preclosed

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (5) For $B \subseteq Y$, since $int_\delta(B)$ is a δ-open set of Y, from (4), $f^{-1}(int_\delta(B)) = mpInt(f^{-1}(int_\delta(B))) \subseteq mpInt(f^{-1}(B))$. Hence $f^{-1}(int_\delta(B)) \subseteq mpInt(f^{-1}(B))$.

(5) $\Rightarrow$ (6) Let B be any subset of Y. From (5), we have $f^{-1}(cl_\delta(B)) = X - f^{-1}(int_\delta(Y - B)) \supseteq X - mpInt(f^{-1}(Y - B)) = mpCl(f^{-1}(B))$. Hence $mpCl(f^{-1}(B)) \subseteq f^{-1}(cl_\delta(B))$.

(6) $\Rightarrow$ (1) For $B \subseteq Y$, since $cl_\delta(B)$ is closed in Y, from (6),

$$\begin{aligned} mpCl(f^{-1}(cl(int(cl_\delta(B))))) &\subseteq f^{-1}(cl_\delta(cl(int(cl_\delta(B))))) \\ &= f^{-1}(cl(int(cl_\delta(B)))) \\ &\subseteq f^{-1}(cl_\delta(B)). \end{aligned}$$

Hence by Theorem 3.6, f is almost m-precontinuous. □

Definition 3.8 ([4])**.** A subset A of a space (X, m_X) with a minimal structure m_X is said to be *m-precompact* relative to A if every collection $\{U_i : i \in J\}$ of m-preopen subsets of X such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subseteq \cup\{U_j : j \in J_0\}$. A subset A of a minimal structure (X, m_X) is said to be *m-precompact* if A is m-precompact as a subspace of X.

A topological space (X, τ) is said to be *nearly compact* [7] if every collection $\{U_i : i \in J\}$ of open subsets of X such that $X \subset \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $X = \cup\{int(cl(U_i)) : i \in J_0\}$.

Theorem 3.9. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). If f is an almost m-precontinuous surjection and if X is m-precompact, then Y is nearly compact.

Proof. Let $\mathcal{C}$ be any open cover of Y. For each $x \in X$, there exists $V \in \mathcal{C}$ such that $f(x) \in V$. Since f is almost m-precontinuous, there exists an m-preopen set U containing x such that $f(U) \subseteq int(cl(V))$. Then the family $\mathcal{U} = \{U : x \in X\}$ is a cover of X by m-preopen sets in X and since X is m-precompact, there is a finite subcover $\{U_j \in \mathcal{U} : j = 1, 2, \cdots, n\}$ such that $X = \cup U_j$. So we have

$$Y = f(\cup U_j) = \cup f(U_j) \subseteq \cup int(cl(V_j)),$$

where $f(U_j) \subseteq int(cl(V_j))$ for $V_j \in \mathcal{C}$.

Hence Y is nearly compact. □

References

1. N. Levine; *Semi-open sets and semi-continuity in topological spaces*, Ams. Math. Monthly, **70**(1963), 36-41.
2. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb; *On precontinuous and weak precontinuous mappings*, Proc. Math. and Phys. Soc. Egypt, **53** (1982), 47-53.
3. W. K. Min and Y. K. Kim; *m-preopen sets and M-precontinuity on spaces with minimal structures*, Advances in Fuzzy Sets and Systems, **4**(3) (2009), 237-245.
4. ———————-; *On minimal precontinuous functions*, Journal of The Chungcheong Mathematical Society, **22**(4) (2009), 667-673.
5. O. Njåstad ; *On some classes of nearly open sets*, pacific journal of mathematics, **15**(3) (1964), 961-970.
6. V. Popa and T. Noiri; *On the definition of some generalized forms of continuity under minimal conditions*, Mem. Fac. Sci. Kochi. Univ. Ser. Math. **22**(2001), 9-19.
7. M. K. Singal and A. Mathur;*On nearly-compact spaces*, Boll. Un. Mat. Ital., **2**(4) (1969), 702-710.
8. N. V. Velicko; *H-closed topological Spaces*, Amer. Math. Soc. Transl., **78**(1968), 103-118.

Department of Mathematics,
Kangwon National University,
Chuncheon, 200-701, Korea
E-mail: wkmin@kangwon.ac.kr

Department of Mathematics,
MyongJi University,
Youngin 449-728, Korea.
E-mail: ykkim@mju.ac.kr