Honam Mathematical J. **32** (2010), No. 1 , pp. 73– 76

NOTE ON DEHN FILLINGS CREATING MÖBIUS BANDS AND KLEIN BOTTLES

Seungsang Oh

Abstract. In this paper we investigate the distances between Dehn fillings on a hyperbolic 3-manifold that yield 3-manifolds containing a Möbius band and a Klein bottle.

1. Introduction

Let M be a compact, connected, orientable 3-manifold with a torus boundary T. A *slope* on T is the isotopy class of an essential simple loop. If γ is a slope on T, then as usual we denote by $M(\gamma)$ the 3-manifold obtained by γ-*Dehn filling* on M, that is, attaching a solid torus V_γ to M along T in such a way that γ bounds a disk in the torus.

In the present paper we assume that $M(\gamma_1)$ contains a Möbius band and $M(\gamma_2)$ contains a Klein bottle. Suppose that $M(\gamma_i)$ contains such a surface $\widehat{F}_i$ for $i = 1, 2$. Then we may assume that $\widehat{F}_i$ meets the attached solid torus V_{γ_i} in a finite collection of meridian disks, and is chosen so that the number of disks n_i is minimal among all such surfaces in $M(\gamma_i)$. $\Delta(\gamma_1, \gamma_2)$ denotes their minimal geometric intersection number. This paper gives the proof of the following theorem mostly based on the results of Lee, Oh and Teragaito [2].

Theorem 1. *Suppose that M is hyperbolic. If $M(\gamma_1)$ contains a Möbius band and $M(\gamma_2)$ contains a Klein bottle, then $\Delta(\gamma_1, \gamma_2) \leq 3$ unless $n_1 = 1$ or $n_2 = 1$.*

Let $F_i = \widehat{F}_i \cap M$. By an isotopy we may assume that $F_1 \cap F_2$ has the minimal number of components and they intersect transversely. Then we obtain graphs G_i in $\widehat{F}_i$ as usual. We use the definitions and terminology of [2]. An *x-edge* in G_i is an edge with label x at one endpoint, and *level x-edge* is a positive edge with label x at both endpoints. A vertex x is

Received February 2, 2010. Accepted March 10, 2010.
Key words and phrases: Dehn filling, essential annulus, Klein bottle.
This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MEST) (No. 2009-0074101).

called a *level vertex* if x is the label of a level edge on the other graph. The following lemmas are Lemmas 2.4., 2.5. and 2.6. in that paper.

Lemma 2. *G_1 has at most one level vertices and G_2 has at most two level vertices. Furthermore both G_i cannot have a generalized S-cycle.*

Lemma 3. *G_1 has at most $\frac{n_2}{2}+1$ mutually parallel positive edges and n_2 mutually parallel negative edges. Similarly G_2 has at most $\frac{n_1+1}{2}$ mutually parallel positive edges and n_1 mutually parallel negative edges.*

2. x-face and extremal block

A disk face of the subgraph of G_i consisting of all the vertices and positive x-edges is called an *x-face*. The following theorem is Theorem 3.4. in [2].

Theorem 4. *G_2 cannot contain an x-face where x is not the level vertex of G_1.*

The *reduced graph* $\overline{G}_1$ of G_1 is defined to be the graph obtained from G_1 by amalgamating each family of parallel edges into a single edge. Let G_1^+ denote the subgraph of G_1 consisting of all vertices and positive edges of G_1. For convenience, cap off the boundary of $\widehat{F}_1$ by a disk, which is called a *fake disk*. Then we think of G_1 as a graph in the projective plane. Then each component of G_1^+ has a disk support because any orientation-preserving loop in a projective plane is contractible. A disk support which possibly contains the fake disk is called a *fake disk support*

Lemma 5. *Each non-level vertex of G_1 has at least $(\Delta(\gamma_1,\gamma_2)-1)\, n_2$ positive edge endpoints.*

Proof. Assume that there is a non-level vertex x of G_1 which has more than n_2 negative edges. Then G_2 contains more than n_2 positive x-edges by the parity rule. Thus the subgraph Γ_x of G_2 consisting of all vertices and positive x-edges has more edges than vertices. Then an Euler characteristic calculation shows that Γ_x contains a disk face, which is an x-face. This contradicts Theorem 4. □

From now on, we assume that $\Delta \geq 4$ and $n_1 \geq 2$. Thus G_1 has non-level vertices. Note that each of these vertices has therefore valency at least two in $\overline{G}_1$ by Lemmas 3. Now take an innermost component Λ_0 of G_1^+ with a fake disk support D_0, which means that $D_0 \cap G_1^+ = \Lambda_0$.

Suppose that Λ_0 is a single vertex. Since all positive edges are mutually parallel loops, this vertex has at most n_2+2 positive edge endpoints by Lemma 3. By Lemma 5, it must be a level vertex. So a negative loop is incident there. Thus we can also choose another innermost component of G_1^+ which contains only non-level vertices.

We may therefore assume that Λ_0 has more than one vertex. Then Λ_0 has either no cut vertex or at least two blocks with at most one cut vertex. Thus we can choose an innermost component Λ with a fake disk support D after splitting Λ_0 at all cut vertices, such that Λ has more than one vertex with at most one cut vertex. If it is the case that Λ contains a cut vertex and a distinct level vertex, we can choose another innermost one containing no level vertex.

Such a subgraph Λ of G_1^+ is called an *extremal block* with a fake disk support D. A vertex of Λ is called a *ghost vertex* if it is either a cut vertex or a level vertex. We emphasize that Λ has more than one vertex and at most one ghost vertex y_0. A vertex of Λ is called a *boundary vertex* if there is an arc connecting it to ∂D whose interior is disjoint from Λ, and an *interior vertex* otherwise. Then the preceding argument together with Lemma 5 proves the following theorem, which plays a key role in this paper:

Lemma 6. *Suppose that $\Delta \geq 4$, $n_1 \geq 2$ and $n_2 \geq 2$. G_1 contains an extremal block Λ with a fake disk support D so that each boundary vertex excluding y_0 has at least $3n_2$ consecutive edge endpoints of Λ.*

3. Proof of Theorem 1

Assume that $\Delta(\gamma_1, \gamma_2) \geq 4$, $n_1 \geq 2$ and $n_2 \geq 2$. As mention in [2], $M(\gamma_2)$ is irreducible and boundary irreducible.

Lemma 6 says that G_1^+ contains an extremal block Λ with a fake disk support D so that each boundary vertex, except y_0, has at least $3n_2$ consecutive edge endpoints of Λ. First, assume that $n_2 = 2$. This case is done by the similar argument of [1, Section 6].

Assume that $n_2 \geq 3$. Choose a non-level label x of G_1 by Lemma 2. Let Λ^x be the subgraph of Λ consisting of all vertices and x-edges. Then each boundary vertex of Λ^x, except y_0, has at least three edges attached with label x, and any two among them cannot be parallel by Lemma 3. Note that Λ^x may not be connected. Then, apply the argument in Section 2 to the present situation; choose an extremal block Λ' of Λ^x with a fake disk support D' in D, which we can define in a similar way.

Let v, e and f be the numbers of vertices, edges, and disk faces of Λ', respectively. Also let v_i, v_∂ and v_g be the numbers of interior vertices, boundary vertices and ghost vertices. Hence $v = v_i + v_\partial$ and $v_g = 0$ or 1.

Suppose that Λ' has a bigon which does not contain the fake disk. By Lemma 3, it contains a generalized S-cycle. But this is impossible by Lemma 2. Thus, except at most one, each face of Λ' is a disk with at least 3 sides. Hence we have $3f + v_\partial - 2 \le 2e$. Since Λ' has only disk faces, combined with $v - e + f = \chi(D') = 1$, we get $e \le 3v_i + 2v_\partial - 1$. On the other hand, we have $3(v_\partial - v_g) + 4v_i \le e$, because each boundary vertex of Λ', except y_0, has at least three edges attached with label x, and x is not a label of level edges. These two inequalities give us that $v_i + v_\partial + 1 \le 3v_g$, so $v_g = 1$ and $v_i + v_\partial \le 2$. Λ' may have one interior vertex and one boundary vertex, which is a ghost vertex. Now the boundary vertex which is not the ghost vertex has at least three edges attached with label x connecting both vertices. Even if one face contains the fake disk, some two of these x-edges are parallel through another face, a contradiction.

This completes the theorem.

References

1. G. T. Jin, S. Lee, S. Oh and M. Teragaito, *P^2-reducing and toroidal Dehn fillings*, Math. Proc. Camb. Philos. Soc. **134** (2003), 271–288.
2. S. Lee, S. Oh and M. Teragaito, *Reducing Dehn fillings and small surfaces*, Proc. London Math. Soc. (3) **92** (2006), 203–223.

Seungsang Oh
Department of Mathematics,
Korea University, Anam-dong,
Sungbuk-ku, Seoul 136-701, Korea
E-mail: seungsang@korea.ac.kr