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AN EXTENSION OF THE TRIPLE HYPERGEOMETRIC SERIES BY EXTON

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Abstract. The aim of this paper is to extend a number of transformation formulas for the four X_4, X_5, X_7, and X_8 among twenty triple hypergeometric series X_1 to X_{20} introduced earlier by Exton. The results are derived from the generalized Kummer's theorem and Dixon's theorem obtained earlier by Lavoie et al..

1. Introduction

In 1982, Exton [3] introduced a set of twenty triple hypergeometric series X_1 to X_{20} of second order terms of whose series representations contain the factors $(a)_{2m+2n+p}$ and $(a)_{2m+n+p}$. For convenience we write $\sum_{m,\,n,\,p=0}^{\infty}$ for $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}$. Our chosen Exton's series are defined as follows :

$$X_4(a,\,b\,;\,d,\,e,\,f\,;\,x,\,y,\,z) = \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_{n+p}\,x^m\,y^n\,z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!}, \tag{1.1}$$

$$X_5(a,\,b,\,c\,;\,d\,;\,x,\,y,\,z) = \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_{m+n+p}\,m!\,n!\,p!}, \tag{1.2}$$

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$$
\begin{aligned}
&X_7(a,\, b,\, c\,;\, d,\, e\,;\, x,\, y,\, z) \\
&= \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_m\,(e)_{n+p}\,m!\,n!\,p!},
\end{aligned} \tag{1.3}
$$

$$
\begin{aligned}
&X_8(a,\, b,\, c\,;\, d,\, e,\, f\,;\, x,\, y,\, z) \\
&= \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!},
\end{aligned} \tag{1.4}
$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$
(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (\alpha \neq 0,\, -1,\, -2,\, \ldots;\; n = 0,\, 1,\, 2,\, \ldots). \tag{1.5}
$$

The precise three-dimensional region of convergence of (1.1) to (1.4) is given by Srivastava and Karlsson [12, p. 101, Entry 41a]:

$$
4r = (s+t-1)^2, \quad |x| < r, \quad |y| < s, \quad \text{and} \quad |z| < t
$$

where the positive quantities r, s and t are associated radii of convergence. For details about these functions and many other three-variables hypergeometric functions, one refers to Srivastava and Karlsson [12].

In 2007, Kim and Rathie [5] derived twenty five transformation formulas in the form of a single result for X_8 with the help of the generalized Watson's theorem [9]. We obtained 25 results for the series :

$$
X_8(a,\, b,\, c\,;\, d,\, 2b+i,\, 2c+j\,;\, x,\, -x,\, x), \tag{1.6}
$$

for $i, j = 0, \pm1, \pm2$.

In (1.6), if we take $c = b$, then we can again obtain 25 results for the series

$$
X_8(a,\, b,\, b\,;\, d,\, 2b+i,\, 2b+j\,;\, x,\, -x,\, x), \tag{1.7}
$$

for $i, j = 0, \pm1, \pm2$.

The object of this paper is to derive a number of extended transformation formulas for Exton's triple hypergeometric series X_4, X_5, X_7, and X_8 by using a slightly different method from that used in [5] with the help of generalized Kummer's theorem and Dixon's theorem, respectively.

2. Preliminary results

In 1996, Lavoie, Grodin and Rathie [8] have given the generalization of Kummer's theorem on the sum of a ${}_2F_1$ and obtained ten results, in the form of a single result

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\ \ b \\ 1+a-b+i \end{matrix};-1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))} \\
&\times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i))}\right. \\
&\left.+\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{i}{2})}\right\}
\end{aligned}
\tag{2.1}
$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients A_i and B_i are given in [8]. We start with Dixon's classical result [1, p. 13, Eq. (1)] for the well poised ${}_3F_2(1)$:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ \ b,\ \ c \\ 1+a-b,\ 1+a-c \end{matrix};\ 1\right] \\
&=\frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \\
&\qquad(\Re\left(a-2b-2c\right)>-2)\,.
\end{aligned}
\tag{2.2}
$$

A single form of certain extended formulas of (2.2) is

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i+j \end{matrix};1\right) \\
&=\frac{2^{-2c+i+j}\Gamma(a-b+i+1)\Gamma(1+a-c+i+j)\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i))}{\Gamma(b)\Gamma(c)\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)} \\
&\Gamma(c-\frac{1}{2}(i+j+|i+j|)
\end{aligned}
\tag{2.3}
$$

$$\times \left\{ A_{i,j} \frac{\Gamma(\frac{1}{2}a - c + \frac{1}{2} + [\frac{i+j+1}{2}])\Gamma(\frac{1}{2}a - b - c + 1 + i + [\frac{j+1}{2}])}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}a - b + 1 + [\frac{i}{2}])} \right.$$

$$\left. + B_{i,j} \frac{\Gamma(\frac{1}{2}a - c + 1 + [\frac{i+j}{2}])\Gamma(\frac{1}{2}a - b - c + \frac{3}{2} + i + [\frac{j}{2}])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a - b + \frac{1}{2} + [\frac{i+1}{2}])} \right\}$$

$$(\Re\,(a - 2b - 2c) > -2 - 2i - j;\ i = -3, -2, -1, 0, 1, 2;\ j = 0, 1, 2, 3).$$

The coefficients $A_{i,j}$ and $B_{i,j}$ are given in the tables at the end of the paper [7]. More specifically, 39 distinct formulas are given, including Dixon's result which appears when $i = j = 0$.

It is interesting to observe here that if $f_{i,j}$ is the left-hand side of (2.3), the natural symmetry

$$f_{i,j}(a, b, c) = f_{i+j,-j}(a, c, b) \tag{2.4}$$

makes it possible to extend the result to $j = -1, -2, -3$.

We also recall the following well-known identities involving the Pochhammer symbol in (1.5) (see [11, p. 6-8]):

$$(\alpha)_{n-p} = \frac{(-1)^p\,(\alpha)_n}{(1-\alpha-n)_p} \quad \text{and} \quad (n-p)! = \frac{(-1)^p\,n!}{(-n)_p}\ (\alpha = 1); \tag{2.5}$$

(2.6)
$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n \quad \text{and} \quad (\alpha)_{2n+1} = \alpha 2^{2n}\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n\left(\frac{\alpha}{2}+1\right)_n;$$

$$(\alpha + 2m)_{2n} = \frac{2^{2n}\left(\frac{\alpha}{2}\right)_{m+n}\left(\frac{\alpha}{2}+\frac{1}{2}\right)_{m+n}}{\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n}; \tag{2.7}$$

$$(\alpha + 2m)_{2n+1} = \frac{\alpha 2^{2n}\left(\frac{\alpha}{2}+\frac{1}{2}\right)_{m+n}\left(\frac{\alpha}{2}+1\right)_{m+n}}{\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n}; \tag{2.8}$$

$$(\alpha)_m\,(\alpha+m)_n = (\alpha)_{m+n}; \tag{2.9}$$

$$(2n)! = 2^{2n}n!\left(\frac{1}{2}\right)_n,\ \ (2n+1)! = 2^{2n}n!\left(\frac{3}{2}\right)_n. \tag{2.10}$$

3. Main transformation formulas of triple hypergeometric series

The following eleven transformation formulas will be established.

$$\begin{aligned}
&X_4(a,\, b;\, d,\, c,\, c+i;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a}{2})_m\,(\frac{a}{2}+\frac{1}{2})_m(a+2m)_n\,(b)_n\,(-1)^n(4x)^{m+n}}{(d)_m\,(c)_n\,m!\,n!}\\
&\frac{\Gamma(\frac{1}{2})\,\Gamma(c+i)\Gamma(c+n)}{\Gamma(c+n+\frac{1}{2}(i+|i|))} \times \Bigg\{ \frac{A_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}+\frac{1}{2}-[\frac{i+1}{2}])\Gamma(c+\frac{n}{2}+\frac{i}{2})}\\
&+ \frac{B_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}+\frac{1}{2}-[\frac{i}{2}])\Gamma(c+\frac{n}{2}+\frac{i}{2}-\frac{1}{2})} \Bigg\},
\end{aligned} \tag{3.1}$$

provided $\Re\,(2c+2n) > 1-i$, for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The coefficients A_i and B_i can be obtained from the tables of A_i and B_i in [8, p. 298] by replacing a by $-n$ and b by $1-c-n$, respectively.

$$\begin{aligned}
&X_5(a,\, b-i,\, b;\, c+j;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a}{2})_m\,(\frac{a}{2}+\frac{1}{2})_m(a+2m)_n\,(b-i)_n\,(-1)^n(4x)^{m+n}}{(c+j)_m\,(c+m+j)_n\,m!\,n!}\\
&\frac{\Gamma(\frac{1}{2})\,\Gamma(1-b)\Gamma(1-b-n+i)}{\Gamma(1-b+\frac{1}{2}(i+|i|))}\\
&\times \Bigg\{ \frac{A_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}+\frac{1}{2}-[\frac{i+1}{2}])\Gamma(\frac{-n}{2}-b+\frac{i}{2}+1)}\\
&+ \frac{B_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}-[\frac{i}{2}])\Gamma(\frac{-n}{2}-b+\frac{i}{2}+\frac{1}{2})} \Bigg\}
\end{aligned} \tag{3.2}$$

provided $\Re\,(2b-1) < i$, for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The coefficients A_i and B_i can be obtained from the tables of A_i and B_i in [8, p. 298] by replacing a by $-n$.

$$\begin{aligned}
&X_7(a,\, b-i,\, b; c,\, c+j;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a}{2})_m\, (\frac{a}{2}+\frac{1}{2})_m\, (a+2m)_n\, (b-i)_n\, (-1)^n (4x)^{m+n}}{(c)_m\, (c+j)_n\, m!\, n!}\\
&\frac{2^{2n}\Gamma(\frac{1}{2})\,\Gamma(1-b)\Gamma(1-b-n+i)}{\Gamma(1-b+\frac{1}{2}(i+|i|))}\\
&\times \left\{ \frac{A_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}+\frac{1}{2}-[\frac{i+1}{2}])\Gamma(\frac{-n}{2}-b+\frac{i}{2}+1)} \right.\\
&\left. + \frac{B_i}{\Gamma(\frac{-n}{2}+\frac{i}{2}-[\frac{i}{2}])\Gamma(\frac{-n}{2}-b+\frac{i}{2}+\frac{1}{2})} \right\}
\end{aligned} \tag{3.3}$$

provided $\Re\,(2b-1) < i$, for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The coefficients A_i and B_i can be obtained from the tables of A_i and B_i in [8, p. 298] by replacing a by $-n$.

If $\Re\,(2b-2c-n) > 2i+j$ for convergence, then we have

(3.4)

$$\begin{aligned}
&X_8(a,\, b-i,\, b;\, d,\, c,\, c+i+j;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a}{2}+\frac{1}{2})_m\, (\frac{a}{2})_{m+n} (\frac{a}{2}+\frac{1}{2}+n)_m\, (\frac{b-i}{2})_n (\frac{b-i}{2}+\frac{1}{2})_n\, (4x)^m x^{2n}}{(\frac{c}{2})_n\, (\frac{c}{2}+\frac{1}{2})_n (\frac{1}{2})_n (d)_m\;\, m!\, n!}\\
&\cdot \frac{2^{-2(1-c-2n)+i+j}\Gamma(1+i-2n-b)\,\Gamma(2-c+i+j)\Gamma(b-\frac{1}{2}(i+|i|))}{\Gamma(2c+2n+i+j-1)\Gamma(c-b+i+j)\Gamma(b)\Gamma(1-c-2n)}\\
&\times \left\{ D_{i,j} \frac{\Gamma\left(n+c-\frac{1}{2}+[\frac{i+j+1}{2}]\right)\,\Gamma\left(n-b+c+i+[\frac{j+1}{2}]\right)}{\Gamma\left(-n+\frac{1}{2}\right)\Gamma\left(-n-b+1+[\frac{i}{2}]\right)} \right\}\\
&+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\\
&\frac{(\frac{a}{2}+\frac{1}{2})_m\, (\frac{a}{2}+\frac{1}{2}+n)_m\, (b-i)\, (\frac{b-i}{2}+\frac{1}{2})_n (\frac{b-i}{2}+1)_n\, (-1)(4x)^m x^{2n+1}}{c(\frac{c}{2}+\frac{1}{2})_n\, (\frac{c}{2}+1)_n (\frac{3}{2})_n (d)_m\;\, m!\, n!}\\
&\cdot \frac{2^{2(c+2n)+i+j}\Gamma(i-2n-b)\,\Gamma(2-c+i+j)\Gamma(b-\frac{1}{2}(i+|i|))}{\Gamma(2c+2n+i+j)\Gamma(c-b+i+j)\Gamma(b)\Gamma(-c-2n)}\\
&\times \left\{ E_{i,j} \frac{\Gamma\left(n+\frac{1}{2}+c+[\frac{i+j}{2}]\right)\,\Gamma\left(n-b+c+1+i+[\frac{j}{2}]\right)}{\Gamma\left(-n-\frac{1}{2}\right)\Gamma\left(-n-b+[\frac{i+1}{2}]\right)} \right\},
\end{aligned}$$

for $i = 0, \pm 1, \pm 2, \pm 3$, $j = 0, 1, 2, 3$. The coefficients $D_{i,j}$, $E_{i,j}$ are obtained from the coefficients $A_{i,j}$, $B_{i,j}$ in (2.3) by replacing a, b, and c by $-2n$, b and $-c-2n$, respectively. Moreover, in view of the observation (2.4), the results in (3.4) can be extended to $j = -1, -2, -3$.

4. Proofs of (3.1) to (3.4)

Now let us start with the proof of (3.1). Replacing e by c and f by $c+i$, y by $-x$, and z by x, then we have

$$\begin{aligned} X_4 &:= X_4(a,\, b\,;\, d,\, c,\, c+i\,;\, x,\, -x,\, x) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_{n+p}\,(-1)^n\,x^{m+n+p}}{(d)_m\,(c)_n\,(c+i)_p\,m!\,n!\,p!}, \end{aligned} \tag{4.1}$$

which, upon using $(a)_{2m+n+p} = (a)_{2m}\,(a+2m)_{n+p}$, becomes

$$X_4 = \sum_{m=0}^{\infty} \frac{(a)_{2m}\,x^m}{(d)_m\,m!} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a+2m)_{n+p}\,(b)_{n+p}\,(-1)^n\,x^{n+p}}{(c)_n\,(c+i)_p\,n!\,p!}. \tag{4.2}$$

By making use of the well-known formal manipulation for double series [10, p. 56](for other useful formal manipulations of double series, see Choi [2]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,\,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,\,n-k}$$

in (4.2), we obtain

$$X_4 = \sum_{m=0}^{\infty} \frac{(a)_{2m}\,x^m}{(d)_m\,m!} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(a+2m)_n\,(b)_n\,(-1)^{n-p}\,x^n}{(c)_{n-p}\,(c+i)_p\,(n-p)!\,p!}. \tag{4.3}$$

Applying (2.5)-(2.6) to (4.3), we get

$$
\begin{aligned}
X_4 &= \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b)_n\, (-x)^n}{(c)_n\, n!} \\
&\qquad \cdot \sum_{p=0}^{n} \frac{(-n)_p\, (1-c-n)_p}{(c+i)_p\, p!} \\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b)_n\, (-x)^n}{(c)_n\, n!} \\
&\qquad \times {}_2F_1\left[\begin{matrix} -n,\ 1-n-c \\ c+i \end{matrix}; -1\right].
\end{aligned} \tag{4.4}
$$

Applying the generalized Kummer's theorem (2.1) to (4.4), we are lead to the desired formula (3.1). As in the previous proof, we readily find the results (3.2) and (3.3). It remains to derive (3.4). For this, if we denote the left-hand side of (3.4) by X_8, then replacing b by $b-i$, c by b, e and f by c and $c+i+j$ respectively, y by $-x$, and z by x in (1.4), we have

$$
\begin{aligned}
X_8 &:= X_8(a,\, b-i,\, b;\, d,\, c,\, c+i+j\,;\, x,\, -x,\, x) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2m+n+p}\, (b-i)_n\, (b)_p\, (-1)^n\, x^{m+n+p}}{(d)_m\, (c)_n\, (c+i+j)_p\, m!\, n!\, p!}.
\end{aligned} \tag{4.5}
$$

By making use of (2.5)-(2.10) and (4.2), we get

$$
\begin{aligned}
X_8 &= \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b-i)_n\, (-x)^n}{(c)_n\, n!} \\
&\qquad \cdot \sum_{p=0}^{n} \frac{(-n)_p\, (b)_p\, (1-c-n)_p}{(1-b-n+i)_p\, (c+i+j)_p p!} \\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b-i)_n\, (-x)^n}{(c)_n\, n!} \\
&\qquad \times {}_3F_2\left[\begin{matrix} -n,\, b,\, 1-n-c \\ 1-b-n+i,\, c+i+j \end{matrix}; 1\right].
\end{aligned} \tag{4.6}
$$

Applying the generalized Dixon's theorem (3.3) to ${}_3F_2(1)$ in (4.6), we obtain

(4.7)

$$\begin{aligned}
&X_8(a,\, b-i,\, b;\, d,\, c,\, c+i+j;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}\,(a+2m)_n\,(b-i)_n\,(-1)^n x^{m+n}}{(d)_m\,(c)_n\,m!\,(n)!}\\
&\cdot \frac{2^{-2(1-c-n)+i+j}\Gamma(1+i-n-b)\,\Gamma(2-c+i+j)\Gamma(b-\frac{1}{2}(i+|i|))}{\Gamma(2c+n+i+j-1)\Gamma(c-b+i+j)\Gamma(b)\Gamma(1-c-n)}\\
&\times \left\{ {A_{i,j}}' \mathcal{A}(b,\, c\,;\, n) + {B_{i,j}}' \mathcal{B}(b,\, c\,;\, n) \right\},
\end{aligned}$$

where, for convenience,

$$\mathcal{A}(b,\, c\,;\, n) := \frac{\Gamma\left(\frac{n}{2}+c-\frac{1}{2}+[\frac{i+j+1}{2}]\right)\,\Gamma\left(\frac{n}{2}-b+c+i+[\frac{j+1}{2}]\right)}{\Gamma\left(\frac{-n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{-n}{2}-b+1+[\frac{i}{2}]\right)}$$

and

$$\mathcal{B}(b,\, c\,;\, n) := \frac{\Gamma\left(\frac{n}{2}+c+[\frac{i+j}{2}]\right)\,\Gamma\left(\frac{n}{2}-b+c+\frac{1}{2}+i+[\frac{j}{2}]\right)}{\Gamma\left(\frac{-n}{2}\right)\Gamma\left(\frac{-n}{2}-b+\frac{1}{2}+[\frac{i+1}{2}]\right)},$$

the coefficients ${A_{i,j}}'$ and ${B_{i,j}}'$ can be obtained from $A_{i,j}$ and $B_{i,j}$ in (2.3) by replacing a by $-n$ and c by $1-c-n$, respectively.

We see that

$$\mathcal{A}(b,\, c\,;\, n) = 0 \tag{4.8}$$

whenever n is an odd positive integer and because

$$1/\Gamma(-n) = 0 \ (n = 0,\, 1,\, 2, \cdots).$$

It is also seen from the same reasoning just before that

$$\mathcal{B}(b,\, c\,;\, n) = 0 \tag{4.9}$$

whenever n is an even positive integer. Separating into even and odd powers in (4.7), and considering (4.8) and (4.9), we can rewrite X_8 as follows :

(4.10)

$$
\begin{aligned}
&X_8(a,\, b-i,\, b;\, d,\, c,\, c+i+j;\, x,\, -x,\, x)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}\,(a+2m)_{2n}\,(b-i)_{2n}\,x^{m+2n}}{(d)_m\,(c)_2 n\, m!\,(2n)!}\\
&\cdot \frac{2^{-2(1-c-2n)+i+j}\Gamma(1+i-2n-b)\,\Gamma(2-c+i+j)\Gamma(b-\frac{1}{2}(i+|i|))}{\Gamma(2c+2n+i+j-1)\Gamma(c-b+i+j)\Gamma(b)\Gamma(1-c-2n)}\\
&\times D_{i,j}\mathcal{A}(b,\, c\,;\, 2n)\\
&+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}\,(a+2m)_{2n+1}\,(b-i)_{2n+1}\,(-1)\,x^{m+2n+1}}{(d)_m\,(c)_{2n+1}\, m!\,(2n+1)!}\\
&\cdot \frac{2^{2(c+2n)+i+j}\Gamma(i-2n-b)\,\Gamma(2-c+i+j)\Gamma(b-\frac{1}{2}(i+|i|))}{\Gamma(2c+2n+i+j)\Gamma(c-b+i+j)\Gamma(b)\Gamma(-c-2n)}\\
&\times E_{i,j}\mathcal{B}(b,\, c\,;\, 2n+1),
\end{aligned}
$$

where the coefficients $D_{i,j}$ and $E_{i,j}$ can be obtained from the tables of $A'_{i,j}$ and $B'_{i,j}$ given in (4.7) by replacing n by $2n$, and n by $2n+1$, respectively. Applying (2.5)-(2.10) in (4.10), after a little simplification, we arrive at the right-hand side expression in (3.4). This completes the proof of (3.4).

Remark. For $i = j = 0$, (3.4) reduces to the results recorded in [6, p. 258, Eq. (1.5)] obtained by other means.

References

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in math., 1935 Reprinted by Stechert-Hafner, New York, 1964.

[2] J. Choi, *Notes on formal manipulations of double series*, Commun. of Korean Math. Soc. **18** (4) (2003), 781–789.

[3] H. Exton, *Hypergeometric function of three variables*, J. Indian Acad. Maths. **4**(2) (1982), 113–119.

[4] Y. S. Kim, J. Choi and A. K. Rathie, *Remark on two results by Padmanabham for Exton's triple hypergeometric series X_8*, Honam Math. Journal **27** (4) (2005), 603–608.

[5] Y. S. Kim and A. K. Rathie, *On an extension formulas for the triple hypergeometric series X_8 due to Exton*, Bull. Korean Math. Soc. **44** (4) (2007), 743–751.

[6] Y. S. Kim and A. K. Rathie, *Another method for Padmanabham's transformation formula for Exton's triple hypergeometric series X_8*, Commu. of Korean Math. Soc. **24** (4) (2009), 517–521.

[7] J. L. Lavoie, F. grodin and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. **34** (2) (1992), 23–32.
[8] J. L. Lavoie, F. grodin and A. K. Rathie, *Generalizations of Whipple's theorem on the sum of a* ${}_3F_2$, Journal of Computational and Applied Mathematics **72** (1996), 293–300.
[9] J. L. Lavoie, F. grodin, A. K. Rathie and K. Arora *Generalizations of Dixon's theorem on the sum of a* ${}_3F_2$, Mathematics of Computation **62** , No.205 (1994), 267–276.
[10] E. D. Rainville, *Special functions*, Macmillan Company, New York, 1960.
[11] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.
[12] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester); Wiley, New York, Chichester, Brisbane, and Toronto, 1985.
[13] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester); Wiley, New York, Chichester, Brisbane, and Toronto, 1984.

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