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A FIXED POINT APPROACH TO GENERALIZED STABILITY OF A MIXED TYPE FUNCTIONAL EQUATION IN RANDOM NORMED SPACES

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Abstract. In this note, by using the fixed point method, we prove the generalized stability for a mixed type functional equation in random normed spaces of which the general solution is either cubic or quadratic.

1. Introduction

The study of stability problems for functional equations originated from a question of S.M. Ulam [23] concerning the stability of group homomorphisms and it was affirmatively answered for Banach spaces by D.H. Hyers [8]. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and Th.M. Rassias for linear mappings [18].

Since then, a great deal of work has been done by a number of authors (for instance, [4, 6, 19]).

Consider the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

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The quadratic function $f(x) = qx^2$ is a solution of this functional equation, and so one usually calls the above functional equation to be *quadratic* [1, 5, 11, 12].

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [22] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [4] and S. Czerwik [5].

The cubic function $f(x) = cx^3$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \qquad (1.1)$$

The functional equation (1.1) is said to be *cubic* and every solution of the equation (1.1) is called a *cubic function*. The stability result of the equation (1.1) was obtained by K.-W. Jun and H.-M. Kim [9].

Here, let us consider the following functional equation:

$$8f(x-3y) + 24f(x+y) + f(8y) = 8[f(x+3y) + 3f(x-y) + 2f(2y)]. \qquad (1.2)$$

It is easy to see that all the real-valued functions $f : \mathbb{R} \to \mathbb{R}$ of mixed type of cubic and quadratic, i.e., either $f(x) = cx^3$ or $f(x) = qx^2$ satisfy the functional equation (1.2). Hence, for the sake of convenience, we promise that the equation (1.2) is called a mixed type functional equation of cubic and quadratic and every solution of the equation (1.2) is said to be a mixed type function of cubic and quadratic.

Almost all proofs in this topic used the Hyers' direct method [8]. In 2003, V. Radu [17] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. This method has recently been used by many authors(see, e.g., [3, 15, 16]).

In this note, we provide the generalized stability problem for the functional equation (1.2) in probabilistic setting by using the fixed point approach as in [15].

2. Preliminaries

For explicitly later use, we first state the following theorem:

Lemma 2.1 ([14]). (*The alternative of fixed point*) *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $J : \Omega \to \Omega$ *with Lipschitz constant* L. *Then, for each given* $x \in \Omega$, *either*

$$d(J^n x, J^{n+1} x) = \infty \text{ for all } \; n \geq 0,$$

or

There exists a natural number n_0 *such that*

- $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
- *The sequence* $(J^n x)$ *is convergent to a fixed point* y^* *of* J;
- y^* *is the unique fixed point of* J *in the set* $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in \Delta$.

A function $H : \mathbb{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing and left-continuous, with $\sup_{t\in\mathbb{R}} H(t) = 1$ and $\inf_{t\in\mathbb{R}} H(t) = 0$. The class of all distribution functions H with $H(0) = 0$ is denoted by D_+. The class D_+ is partially ordered by the usual pointwise ordering of functions, that is, $H \leq G$ iff $H(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for D_+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } \; t \leq 0, \\ 1 & \text{if } \; t > 0. \end{cases}$$

Definition 2.2 ([15]). A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm(briefly, a t-norm) if T satisfies the following conditions:

(i) T is commutative and associative;

(ii) T is continuous;

(iii) $T(a, 1) = a$ for all $a \in [0, 1]$

(iv) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$

Three typical examples of continuous t-norms are $T(a, b) = ab$, $T(a, b) = \max(a + b - 1, 0)$ and $T(a, b) = \min(a, b)$.

Definition 2.3. Let X be a real vector space, F be a mapping from X into D_+ (for any $x \in X$, $F(x)$ is denoted by F_x) and T be a t-norm. The triple (X, F, T) is called a *random normed space* iff the following conditions are satisfied:

(RN1) $F_x = \varepsilon_0$ iff $x = \theta$, the zero vector;

(RN2) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $x \in X$;

(RN3) $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2))$ for all $x, y \in X$ and $t_1, t_2 > 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, F, T_M), where

$$F_u(t) = \begin{cases} \dfrac{t}{t + \|u\|} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases}$$

for all $u \in X$ and T_M is the minimum t-norm. This space is the *induced random normed space.*

Definition 2.4. Let (X, F, T) be a random normed space.

(i) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X(we denote $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} F_{x_n - x}(t) = 1$ for all $t > 0$.

(ii) $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{m,n\to\infty} F_{x_m - x_n}(t) = 1$ for all $t > 0$.

(iii) (X, F, T) is said to be *complete* if every Cauchy sequence in X is convergent.

The usual terminology, notations and conventions of the theory of random normed spaces are due to [7, 20, 21].

3. The Main Result

Let X be a real vector space and (Y, F, T_M) be a complete random normed space, where $T_M(a,b) = \min(a,b)$. Given a function $f : X \to Y$, we set

$$\begin{aligned} Df(x,y) &:= 8f(x-3y) + 24f(x+y) + f(8y) \\ &\quad -8[f(x+3y) + 3f(x-y) + 2f(2y)] \end{aligned}$$

for all $x, y \in X$. Let G be a mapping from $X \times \mathbb{R}$ into $[0,1]$ such that $G(x,\cdot) \in D_+$ for all $x \in X$. Consider the set $E = \{g : X \to Y : g(0) = 0\}$ and the mapping d_G defined on $E \times E$ by

$$d_G(g,h) = \inf\{a \in (0,\infty) : F_{g(x)-h(x)}(at) \geq G(x,t) \text{ for all } x \in X \text{ and } t > 0\},$$

where, as usual, $\inf \emptyset = +\infty$. In [15], it was proved that d_G is a complete generalized metric on E.

Theorem 3.1. *Let X be a real vector space and (Y, F, T_M) be a complete random normed space. Suppose that $\Phi : X \times X \to D_+$ is a symmetric mapping such that for each $k = 3, 4$, there exists $\alpha_k \in (0, 2^k)$ satisfying*

$$\Phi(2^{k-2}x, 2^{k-2}y)(\alpha_k t) \geq \Phi(x,y)(t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$F_{Df(x,y)}(t) \geq \Phi(x,y)(t) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$, then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying the equation (1.2) such that

(3.3)

$$F_{f(x)-(C(x)+Q(x))}(t) \geq T_M\Big(\Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)\Big(\frac{8-\alpha_3}{2}t\Big),\ \Phi\Big(0, \frac{x}{2}\Big)\Big(\frac{16-\alpha_4}{2}t\Big)\Big)$$

$$(3.4)\qquad F_{\frac{f(x)-f(-x)}{2}-C(x)}(t) \geq \Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)((8-\alpha_3)t),$$

and

$$(3.5)\qquad F_{\frac{f(x)+f(-x)}{2}-Q(x)}(t) \geq \Phi\Big(0, \frac{x}{2}\Big)((16-\alpha_4)t)$$

for all $x \in X$ and $t > 0$.

Proof. Let $g : X \to Y$ be the function defined by $g(x) = \frac{1}{2}\,[f(x) - f(-x)]$ for all $x \in X$. Then we have $g(0) = 0$, $g(-x) = -g(x)$ and

$$(3.6)\qquad F_{Dg(x,y)}(t) \geq \Phi(x,y)(t)$$

for all $x, y \in X$ and $t > 0$. In fact, we observe that

$$\begin{aligned} F_{Dg(x,y)}(t) &= F_{\frac{1}{2}[Df(x,y)-Df(-x,-y)]}(t) \\ &= F_{Df(x,y)-Df(-x,-y)}(2t) \\ &= F_{Df(x,y)-Df(-x,-y)}(t+t) \\ &\geq T_M(F_{Df(x,y)}(t), F_{-Df(-x,-y)}(t)) \\ &= T_M(F_{Df(x,y)}(t), F_{Df(-x,-y)}(t)) \end{aligned}$$

for all $x, y \in X$ and $t > 0$. That is,

$$F_{Dg(x,y)}(t) \geq T_M(F_{Df(x,y)}(t), F_{Df(-x,-y)}(t))$$

for all $x, y \in X$ and $t > 0$. Since the identity $F_{Dg(x,y)}(t) = F_{Dg(-x,-y)}(t)$ holds for all $x \in X$ and $t > 0$, it follows from the above inequality and (3.2) that

$$F_{Dg(x,y)}(t) \geq \Phi(x,y)(t)$$

for all $x, y \in X$ and $t > 0$.

Putting $y = x$ in (3.6) yields

$$(3.7)\qquad F_{Dg(x,x)}(t) \geq \Phi(x,x)(t),$$

which, by setting $x = \frac{x}{4}$ in (3.7), gives

$$F_{g(x)-\frac{g(2x)}{8}}(t) \geq \Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)(8t),$$

for all $x \in X$ and $t > 0$. Let

$$\Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)(8t) = U(x,t)$$

for all $x \in X$ and $t > 0$, where U is a mapping from $X \times \mathbb{R}$ into $[0,1]$ such that $U(x,\cdot) \in D_+$ for all $x \in X$. Let $E = \{\rho : X \to Y : \rho(0) = 0\}$. As before, we see that the mapping d_U defined on $E \times E$ by

$$d_U(\rho,\lambda) = \inf\{a \in (0,\infty) : F_{\rho(x)-\lambda(x)}(at) \geq U(x,t) \text{ for all } x \in X \text{ and } t > 0\},$$

is a complete generalized metric on E. Consider the linear mapping $J : E \to E$ defined by $J\rho(x) = \frac{1}{8}\rho(2x)$ for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping of E with the Lipschitz constant $\frac{\alpha_3}{8}$.

Indeed, let ρ, λ in E be given such that $d_U(\rho,\lambda) < \varepsilon$. Then we get

$$F_{\rho(x)-\lambda(x)}(\varepsilon t) \geq U(x,t)$$

for all $x \in X$ and $t > 0$. Hence we see that

$$F_{J\rho(x)-J\lambda(x)}\Big(\frac{\alpha_3}{8}\varepsilon t\Big) = F_{\rho(2x)-\lambda(2x)}(\alpha_3\varepsilon t) \geq U(2x, \alpha_3 t)$$

for all $x \in X$ and $t > 0$. From (3.1) with $k = 3$, it follows that

$$\Phi\Big(\frac{x}{2}, \frac{x}{2}\Big)(\alpha_3 t) \geq \Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)(t)$$

whence we get $U(2x, \alpha_3 t) \geq U(x,t)$ for all $x \in X$ and $t > 0$. Thus we obtain

$$F_{J\rho(x)-J\lambda(x)}\Big(\frac{\alpha_3}{8}\varepsilon t\Big) \geq U(x,t),$$

that is,

$$d_U(\rho,\lambda) < \varepsilon \Rightarrow d_U(J\rho, J\lambda) \leq \frac{\alpha_3}{8}\varepsilon.$$

This means that

$$d_U(J\rho, J\lambda) \leq \frac{\alpha_3}{8} d_U(\rho,\lambda)$$

for all $\rho, \lambda \in E$.

Next, from $F_{g(x)-\frac{g(2x)}{8}}(t) \geq U(x,t)$, it follows that

$$d_U(g, Jg) \leq 1$$

So, using the fixed point alternative, we deduce the *unique* existence of a fixed point C of J, i.e., the existence of a mapping $C : X \to Y$ such that $C(2x) = 8C(x)$ for all $x \in X$. Also,

$$d_U(g, C) \leq \frac{1}{1-L} d_U(g, Jg)$$

implies the inequality

$$d_U(g, C) \leq \frac{1}{1 - \frac{\alpha_3}{8}}$$

from which we obtain

$$F_{g(x)-C(x)}\Big(\frac{8}{8-\alpha_3}t\Big) \geq U(x,t)$$

for all $x \in X$ and $t > 0$ (recall that U is left continuous in second variable). This gives us that

$$F_{g(x)-C(x)}(t) \geq U\Big(x, \frac{8-\alpha_3}{8}t\Big)$$

for all $x \in X$ and $t > 0$, whence we get the estimation (3.4). Since it holds that

$$d_U(u,v) < \delta \Rightarrow F_{u(x)-v(x)}(t) \geq U\Big(x, \frac{t}{\delta}\Big)$$

for all $x \in X$ and $t > 0$, from $\lim_{n\to\infty} d_U(J^n g, C) = 0$, it follows that

$$C(x) = \lim_{n\to\infty} \frac{g(2^n x)}{8^n} \tag{3.8}$$

for all $x \in X$.

We claim that the function C is cubic. Since T_M is continuous, the function $z \mapsto F_z$ is continuous (cf. [20, Chapter 12]). Therefore, for $t > 0$,

$$\begin{aligned} F_{DC(x,y)}(t) &= \lim_{n\to\infty} F_{\frac{Dg(2^n x, 2^n y)}{8^n}}(t) \\ &= \lim_{n\to\infty} F_{Dg(2^n x, 2^n y)}(8^n t) \\ &\geq \lim_{n\to\infty} \Phi(x,y)\Big(\Big(\frac{8}{\alpha_3}\Big)^n t\Big) = 1, \end{aligned}$$

so that we have

$$F_{DC(x,y)}(t) = 1$$

for all $t > 0$ which gives $DC(x, y) = 0$, namely, C satisfies the functional equation (1.2). Since the identity $C(2x) = 8C(x)$ holds for all $x \in X$, the equation (1.2) is reduced to the form

$$C(x + 3y) + 3C(x - y) = C(x - 3y) + 3C(x + y) + 48C(y) \tag{3.9}$$

for all $x, y \in X$. Let us replace x by $-x$ in (3.8). Then it follows from the oddness of g that C is odd, and hence interchanging x and y in (3.9) yields

$$C(3x + y) + C(3x - y) = 3C(x + y) + 3C(x - y) + 48C(x). \tag{3.10}$$

Then it follows from [13] that C is cubic.

Let $h : X \to Y$ be the function defined by $h(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then we have $h(0) = 0$, $h(-x) = h(x)$ and

$$F_{Dh(x,y)}(t) \geq \Phi(x, y)(t) \tag{3.11}$$

for all $x, y \in X$ and $t > 0$. For, we have

$$\begin{aligned} F_{Dh(x,y)}(t) &= F_{\frac{1}{2}[Df(x,y)+Df(-x,-y)]}(t) \\ &= F_{Df(x,y)+Df(-x,-y)}(2t) \\ &= F_{Df(x,y)+Df(-x,-y)}(t + t) \\ &\geq T_M(F_{Df(x,y)}(t), F_{Df(-x,-y)}(t)) \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since the identity $F_{Dh(-x,-y)}(t) = F_{Dh(x,y)}(t)$ is valid for all $x, y \in X$ and $t > 0$, the above inequality and (3.2) yield

$$F_{Dh(x,y)}(t) \geq \Phi(x, y)(t)$$

for all $x, y \in X$ and $t > 0$.

By setting $x = 0$ in (3.12) and then letting $y = x$, we get

$$F_{Dh(0,x)}(t) \geq \Phi(0, x)(t), \tag{3.12}$$

Replacing x by $\frac{x}{2}$ in (3.13), we obtain

$$F_{h(x)-\frac{h(4x)}{16}}(t) \geq \Phi\Big(0, \frac{x}{2}\Big)(16t),$$

for all $x \in X$ and $t > 0$. Let

$$\Phi\Big(0, \frac{x}{2}\Big)(16t) = V(x,t)$$

for all $x \in X$ and $t > 0$, where V is a mapping from $X \times \mathbb{R}$ into $[0,1]$ such that $V(x,\cdot) \in D_+$ for all $x \in X$.

We also consider the linear mapping $S : E \to E$ defined by $S\varrho(x) = \frac{1}{16}\varrho(4x)$ for all $x \in X$. It is immediate to see that S is a strictly contractive self-mapping of E with the Lipschitz constant $\frac{\alpha_4}{16}$. Indeed, let ϱ, η in E be given such that $d_V(\varrho, \eta) < \epsilon$, where d_V is a complete generalized metric on E. Then we get

$$F_{\varrho(x)-\eta(x)}(\epsilon t) \geq V(x,t)$$

for all $x \in X$ and $t > 0$. Hence we see that

$$F_{S\varrho(x)-S\eta(x)}\Big(\frac{\alpha_4}{16}\epsilon t\Big) = F_{\varrho(4x)-\eta(4x)}(\alpha_4 \epsilon t) \geq V(4x, \alpha_4 t)$$

for all $x \in X$ and $t > 0$. From (3.1) with $k = 4$, we deduce that

$$\Phi\Big(0, 2x\Big)(\alpha_4 t) \geq \Phi\Big(0, \frac{x}{2}\Big)(t)$$

which implies that $V(4x, \alpha_4 t) \geq V(x,t)$ for all $x \in X$ and $t > 0$. Therefore we see that

$$F_{S\varrho(x)-S\eta(x)}\Big(\frac{\alpha_4}{16}\epsilon t\Big) \geq V(x,t),$$

that is,

$$d_V(\varrho, \eta) < \epsilon \Rightarrow d_V(S\varrho, S\eta) \leq \frac{\alpha_4}{16}\epsilon.$$

This means that

$$d_V(S\varrho, S\eta) \leq \frac{\alpha_4}{16} d_V(\varrho, \eta)$$

for all $\varrho, \eta \in E$. Next, from $F_{h(x)-\frac{h(4x)}{16}}(t) \geq V(x,t)$, it follows that

$$d_V(h, Sh) \leq 1$$

Again using the fixed point alternative, we arrive at the *unique* existence of a fixed point Q of S, i.e., the existence of a mapping $Q : X \to Y$ such that $Q(4x) = 16Q(x)$ for all $x \in X$. Also,

$$d_V(h,Q) \leq \frac{1}{1-L} d_V(h,Sh)$$

implies the inequality

$$d_V(h,Q) \leq \frac{1}{1-\frac{\alpha_4}{16}}$$

from which we obtain

$$F_{h(x)-Q(x)}\Big(\frac{16}{16-\alpha_4}t\Big) \geq V(x,t)$$

for all $x \in X$ and $t > 0$ (recall that V is left continuous in second variable). This means that

$$F_{h(x)-Q(x)}(t) \geq V\Big(x, \frac{16-\alpha_4}{16}t\Big)$$

for all $x \in X$ and $t > 0$, whence we obtain the inequality (3.5). Since we see that

$$d_V(u,v) < \delta \Rightarrow F_{u(x)-v(x)}(t) \geq V\Big(x, \frac{t}{\delta}\Big)$$

for all $x \in X$ and $t > 0$, it follows from $\lim_{n\to\infty} d_V(S^n h, Q) = 0$ that

$$Q(x) = \lim_{n\to\infty} \frac{h(4^n x)}{16^n} \tag{3.13}$$

for all $x \in X$.

We will show that the function Q is quadratic. Since T_M is continuous, the function $z \mapsto F_z$ is continuous. Thus, for almost all t,

$$\begin{aligned} F_{DQ(x,y)}(t) &= \lim_{n\to\infty} F_{\frac{Dh(4^n x, 4^n y)}{16^n}}(t) \\ &= \lim_{n\to\infty} F_{Dh(4^n x, 4^n y)}(16^n t) \\ &\geq \lim_{n\to\infty} \Phi(x,y)\Big(\Big(\frac{16}{\alpha_4}\Big)^n t\Big) = 1, \end{aligned}$$

so that we have

$$F_{DQ(x,y)}(t) = 1$$

for all $t > 0$ which gives $DQ(x, y) = 0$, that is, Q satisfies the functional equation (1.2). Since the identity $Q(4x) = 16Q(x)$ holds for all $x \in X$, the equation (1.2) is reduced to the form

$$Q(x + 3y) + 3Q(x - y) = Q(x - 3y) + 3Q(x + y) \tag{3.14}$$

for all $x, y \in X$. Then it follows from [10] that Q is quadratic.

Since we have $f(x) = g(x) + h(x)$ for all $x \in X$, we see that

$$\begin{aligned} F_{f(x)-(C(x)+Q(x))}(t) &= F_{[g(x)-C(x)]+[h(x)-Q(x)]}(t) \\ &= F_{[g(x)-C(x)]+[h(x)-Q(x)]}\Big(\frac{t}{2} + \frac{t}{2}\Big) \\ &\geq T_M\Big(F_{g(x)-C(x)}\Big(\frac{t}{2}\Big),\ F_{h(x)-Q(x)}\Big(\frac{t}{2}\Big)\Big) \end{aligned}$$

for all $x \in X$ and $t > 0$. Hence, from (3.4) and (3.5), we obtain the inequality (3.3), i.e.,

$$F_{f(x)-(C(x)+Q(x))}(t) \geq T_M\Big(\Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)\Big(\frac{8 - \alpha_3}{2}t\Big),\ \Phi\Big(0, \frac{x}{2}\Big)\Big(\frac{16 - \alpha_4}{2}t\Big)\Big)$$

for all $x \in X$ and $t > 0$. We complete the proof of the theorem. □

Except for some modifications, we follow the process in the proof of Theorem 3.1 to prove the next complementary case.

Theorem 3.2. *Let X be a real vector space and (Y, F, T_M) be a complete random normed space. Suppose that $\Phi : X \times X \to D_+$ is a symmetric mapping such that for each $k = 3, 4$, there exists $\alpha_k \in (0, 2^k)$ satisfying*

$$\Phi(2^{2-k}x, 2^{2-k}y)(\alpha_k t) \geq \Phi(x, y)(t) \tag{3.15}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$F_{Df(x,y)}(t) \geq \Phi(x, y)(t)$$

for all $x, y \in X$ *and* $t > 0$*, then there exist a unique cubic function* $C : X \to Y$ *and a unique quadratic function* $Q : X \to Y$ *satisfying the equation* (1.2) *such that*

$$F_{f(x)-(C(x)+Q(x))}(t) \geq T_M\Big(\Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)\Big(\frac{8-\alpha_3}{2}t\Big), \Phi\Big(0, \frac{x}{2}\Big)\Big(\frac{16-\alpha_4}{2}t\Big)\Big)$$

$$F_{\frac{f(x)-f(-x)}{2}-C(x)}(t) \geq \Phi\Big(\frac{x}{4}, \frac{x}{4}\Big)((8-\alpha_3)t),$$

and

$$F_{\frac{f(x)+f(-x)}{2}-Q(x)}(t) \geq \Phi\Big(0, \frac{x}{2}\Big)((16-\alpha_4)t)$$

for all $x \in X$ *and* $t > 0$*.*

Proof. Putting $x = \frac{x}{8}$ in (3.7), we have

$$F_{g(x)-8g(\frac{x}{2})}(t) \geq \Phi\Big(\frac{x}{8}, \frac{x}{8}\Big)\Big(\frac{t}{8}\Big),$$

for all $x \in X$ and $t > 0$. Let

$$\Phi\Big(\frac{x}{8}, \frac{x}{8}\Big)\Big(\frac{t}{8}\Big) = U(x, t)$$

for all $x \in X$ and $t > 0$, where U is a mapping from $X \times \mathbb{R}$ into $[0, 1]$ such that $U(x, \cdot) \in D_+$ for all $x \in X$. We define a mapping $J : E \to E$ by $J\rho(x) = 8\rho(\frac{x}{2})$ for all $x \in X$. Then J is a strictly contractive self-mapping of E with the Lipschitz constant $8\alpha_3$.

Replacing x by $\frac{x}{8}$ in (3.13), we obtain

$$F_{h(x)-16h(\frac{x}{4})}(t) \geq \Phi\Big(0, \frac{x}{8}\Big)\Big(\frac{t}{16}\Big),$$

for all $x \in X$ and $t > 0$. Let

$$\Phi\Big(0, \frac{x}{8}\Big)\Big(\frac{t}{16}\Big) = V(x, t)$$

for all $x \in X$ and $t > 0$, where V is a mapping from $X \times \mathbb{R}$ into $[0, 1]$ such that $V(x, \cdot) \in D_+$ for all $x \in X$.

We also consider a mapping $S : E \to E$ defined by $S\varrho(x) = 16\varrho(\frac{x}{4})$ for all $x \in X$. It is immediate to see that S is a strictly contractive self-mapping of E with the Lipschitz constant $16\alpha_4$. The rest is similar to the proof of Theorem 3.1. $\square$

Remark 3.3. Let X and Y be normed spaces and (X, F, T_M) be the induced random normed space. If

$$\Phi(x,y)(t) := \frac{t}{t + \varphi(x,y)}$$

for all $t > 0$, where $\varphi : X \times X \to [0, \infty)$ be a function, then the condition (3.1) holds iff for each $k = 3, 4$, $\varphi(2^{k-2}x, 2^{k-2}y) \le \alpha_k \varphi(x,y)$ for all $x, y \in X$, while the condition (3.28) holds iff for each $k = 3, 4$, $\varphi(2^{2-k}x, 2^{2-k}y) \le \alpha_k \varphi(x,y)$ for all $x, y \in X$. For instance, $\varphi(x,y) = \theta$ $(\theta > 0)$ (resp. $\varphi(x,y) = \|x\|^p + \|y\|^p$ ($p < 2$ or $p > 2$)) verifies the conditions. Since (3.2) reduces to

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in X$, our theorems represent the stability in the sense of Hyers [8] and Aoki [2].

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