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DECOMPOSITION FORMULAS FOR THE GENERALIZID HYPERGEOMETRIC $_4F_3$ FUNCTION

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Abstract. By using the generalized operator method given by Burchnall and Chaundy in 1940, the authors present one-dimensional inverse pairs of symbolic operators. Many operator identities involving these pairs of symbolic operators are first constructed. By means of these operator identities, 11 decomposition formulas for the generalized hypergeometric $_4F_3$ function are then given. Furthermore, the integral representations associated with generalized hypergeometric functions are also presented.

1. Introduction

Over six decades ago, Burchnall and Chaundy [2, 3] and Chaundy [4] presented certain inverse pairs of symbolic operators to investigate some properties of double hypergeometric Appell functions. With the help of inverse pairs of symbolic operators the authors obtained several decomposition formulas and integral representations for Appell ([1, p. 14, (11)-(14)]; see also [5, p. 219, (6)-(9)]) and Humbert [1, p. 126] of two variables. The obtained results were particularly recorded in monograph [5, p. 94, pp. 187-188 and 234-235]. The presented operators decompose the double hypergeometric functions into the product of two one-dimensional functions. Therefore, an investigation of certain properties of double hypergeometric functions can be made in terms of one-dimensional functions. It is interesting to note that this operator method has been forgotten without any particular reason, though this method turned out to be useful in application of other hypergeometric functions. It should be noted that in [15] were given the definitions of

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205 hypergeometric functions, together with their specified regions of convergence. Burchnall and Chaundy [2, 3], also Chaundy [4] presented the following inverse pairs of symbolic operators:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\,(-\delta_2)_k}{(h)_k\,k!}, \tag{1.1}$$

$$\begin{aligned}\Delta_{xy}(h) &:= \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\,(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\,k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\,(h)_{2k}\,(-\delta_1)_k\,(-\delta_2)_k}{(h+k-1)_k\,(h+\delta_1)_k\,(h+\delta_2)_k\,k!},\end{aligned} \tag{1.2}$$

$$\begin{aligned}\nabla_{xy}(h)\,\Delta_{xy}(g) &:= \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}\frac{\Gamma(\delta_1+g)\,\Gamma(\delta_2+g)}{\Gamma(g)\,\Gamma(\delta_1+\delta_2+g)} \\ &= \sum_{k=0}^{\infty} \frac{(g-h)_k\,(g)_{2k}\,(-\delta_1)_k\,(-\delta_2)_k}{(g+k-1)_k\,(g+\delta_1)_k\,(g+\delta_2)_k\,k!} \\ &= \sum_{k=0}^{\infty} \frac{(h-g)_k\,(-\delta_1)_k\,(-\delta_2)_k}{(h)_k\,(1-g-\delta_1-\delta_2)_k\,k!} \quad \left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y}\right).\end{aligned} \tag{1.3}$$

In the definition of operators (1.1)-(1.3) is used the Gauss's well-known summation formula for the hypergeometric function ${}_2F_1(1)$ (see, e.g., [5, p. 112]):

$$\begin{aligned}&{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} : 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \\ &\left(\Re\,(c-a-b) > 0,\ c \notin \mathbb{Z}_0^- := \{0,-1,-2,-3,\ldots\}\right).\end{aligned} \tag{1.4}$$

Further, double hypergeometric operators (1.1)-(1.3) were generalized for multidimensional cases (see, e.g., [1], [5]-[9], [13]-[15]). By means of multidimensional cases the authors obtained decomposition formulas for the Lauricella function [1, p. 114, (1)-(4)], Horn function [5, p. 220, (10)-(11)] and Srivastava function ([13, 14]; also see [15, p. 43, (11)-(13)]). We note that the symbolic operators (1.1)-(1.2) have only two arguments, therefore they are applicable only for double hypergeometric functions. Here we present one-dimensional inverse symbolic operators to investigate generalized hypergeometric functions:

$$H_x(\alpha,\beta) := \frac{\Gamma(\beta)\,\Gamma(\alpha+\delta_x)}{\Gamma(\alpha)\,\Gamma(\beta+\delta_x)} = \sum_{i=0}^{\infty} \frac{(\beta-\alpha)_i\,(-\delta_x)_i}{(\beta)_i\,i!} \tag{1.5}$$

and

$$\bar{H}_x(\alpha,\beta) := \frac{\Gamma(\alpha)\Gamma(\beta+\delta_x)}{\Gamma(\beta)\Gamma(\alpha+\delta_x)} = \sum_{i=0}^{\infty} \frac{(\beta-\alpha)_i(-\delta_x)_i}{(1-\alpha-\delta_x)_i\, i!}, \qquad (1.6)$$

where $\delta_x := x\frac{\partial}{\partial x}$.

We note that the operators (1.5) and (1.6) could be defined for multidimensional cases as well. So, an index x in symbolic operators is appropriate. For clearer illustrations, we apply operators (1.5) and (1.6) to generalized hypergeometric function ${}_4F_3$. The generalized hypergeometric function ${}_4F_3$ ([1, p. 140, (9)]; see also [5, p. 183, (1)]) is defined as follows:

$${}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array} : x\right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m(\alpha_3)_m(\alpha_4)_m}{(\beta_1)_m(\beta_2)_m(\beta_3)_m\, m!} x^m, \qquad (1.7)$$

$\beta_i \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(i=1,2,3)$ and $(a)_n := \Gamma(a+n)/\Gamma(a)$ denotes Pochhammer symbol for all admissible a (real or complex), $n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{N}$ denotes the set of positive integers.

For the specified conditions of convergence of the series ${}_4F_3$ (more generally ${}_pF_q$), for example, refer the readers to [15, p. 20].

In the conditions $\Re(\beta_i) > \Re(\alpha_i) > 0$ $(i=1,2,3)$, function (1.7) has the integral representation of Eulerian type [1, p. 142, (13)]:

$$\begin{aligned} &{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array} : x\right)\\ &= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_3)}\\ &\cdot \int_0^1\cdots\int_0^1 t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\beta_1-\alpha_1-1}(1-t_2)^{\beta_2-\alpha_2-1}\\ &\quad (1-t_3)^{\beta_3-\alpha_3-1}(1-xt_1t_2t_3)^{-\alpha_4}\,dt_1dt_2dt_3. \end{aligned} \qquad (1.8)$$

More detailed information about properties of function (1.7) can be found in [5, 12, 15].

2. A set of operator identities

By applying the pairs of symbolic operators (1.5) and (1.6) we find the following set of operator identities for hypergeometric ${}_4F_3$ function:

$${}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array} : x\right) = H_x(\alpha_1,\beta_1)\,{}_3F_2\left(\begin{array}{c}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{array} : x\right), \qquad (2.1)$$

$$ {}_3F_2\left(\begin{matrix}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{matrix} : x\right)=H_x\left(\beta_1,\alpha_1\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right), \tag{2.2}$$

$$ {}_3F_2\left(\begin{matrix}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{matrix} : x\right)=\bar{H}_x\left(\alpha_1,\beta_1\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right), \tag{2.3}$$

$$ {}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right)=H_x\left(\alpha_1,\beta_1\right)H_x\left(\alpha_2,\beta_2\right){}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right), \tag{2.4}$$

$$ {}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right)=H_x\left(\beta_1,\alpha_1\right)H_x\left(\beta_2,\alpha_2\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right), \tag{2.5}$$

$$ {}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right)=\bar{H}_x\left(\alpha_1,\beta_1\right)\bar{H}_x\left(\alpha_2,\beta_2\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right), \tag{2.6}$$

$$\begin{aligned} &{}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right)\\ &=H_x\left(\alpha_1,\beta_1\right)H_x\left(\alpha_2,\beta_2\right)H_x\left(\alpha_3,\beta_3\right){}_1F_0\left(\begin{matrix}\alpha_4\\ -\end{matrix} : x\right), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &{}_1F_0\left(\begin{matrix}\alpha_4\\ -\end{matrix} : x\right)\\ &=H_x\left(\beta_1,\alpha_1\right)H_x\left(\beta_2,\alpha_2\right)H_x\left(\beta_3,\alpha_3\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &{}_1F_0\left(\begin{matrix}\alpha_4\\ -\end{matrix} : x\right)\\ &=\bar{H}_x\left(\alpha_1,\beta_1\right)\bar{H}_x\left(\alpha_2,\beta_2\right)\bar{H}_x\left(\alpha_3,\beta_3\right){}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right). \end{aligned} \tag{2.9}$$

The proof of operator identities (2.1)-(2.9) is based upon application of Mellin transformation (for instance, [13] and integral representations of Mellin-Barnes [1, p. 142, (14)]) for the generalized hypergeometric functions. The proofs of operator identities will be omitted here in view of simplicity. Yet, it should be *remarked* in passing that operator identities (2.1)-(2.9) can easily be generalized for ${}_pF_q$.

3. Decomposition formulas

In [11, p. 93], it is found that, for any analytical functions $f(\xi)$, the following equalities are true:

$$(\delta_\xi+\alpha)_n\{f(\xi)\}=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\} \tag{3.1}$$

and

$$(-\delta_\xi)_n\{f(\xi)\}=(-1)^n\xi^n\frac{d^n}{d\xi^n}\{f(\xi)\}, \tag{3.2}$$

$$\left(\delta_\xi:=\xi\frac{d}{d\xi};\ \alpha\in\mathbb{C};\ n\in\mathbb{N}_0\right).$$

In view of formulas (3.1) and (3.2), and taking account of the differentiation rules [1, p. 153, (31)], we have

$$\begin{aligned}&\tfrac{d^i}{dx^i}{}_pF_q\left(\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_p\\ \beta_1,\beta_2,...,\beta_p\end{array}:x\right)\\ &=\tfrac{(\alpha_1)_i(\alpha_2)_i\cdots(\alpha_p)_i}{(\beta_1)_i(\beta_2)_i\cdots(\beta_p)_i}{}_pF_q\left(\begin{array}{c}\alpha_1+i,\alpha_2+i,...,\alpha_p+i\\ \beta_1+i,\beta_2+i,...,\beta_p+i\end{array}:x\right)\end{aligned} \tag{3.3}$$

and from operator identities (2.1)-(2.9), we derive

$$\begin{aligned}&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array}:x\right)\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(\beta_1-\alpha_1)_i(\alpha_2)_i(\alpha_3)_i(\alpha_4)_i}{(\beta_1)_i(\beta_2)_i(\beta_3)_i\,i!}x^i\\ &{}_3F_2\left(\begin{array}{c}\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_2+i,\beta_3+i\end{array}:x\right),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&{}_3F_2\left(\begin{array}{c}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{array}:x\right)\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(\alpha_1-\beta_1)_i(\alpha_2)_i(\alpha_3)_i(\alpha_4)_i}{(\beta_1)_i(\beta_2)_i(\beta_3)_i\,i!}x^i\\ &{}_4F_3\left(\begin{array}{c}\alpha_1+i,\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_1+i,\beta_2+i,\beta_3+i\end{array}:x\right),\end{aligned} \tag{3.5}$$

$$\begin{aligned}&{}_3F_2\left(\begin{array}{c}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{array}:x\right)\\ &=\sum_{i=0}^{\infty}\frac{(\beta_1-\alpha_1)_i(\alpha_2)_i(\alpha_3)_i(\alpha_4)_i}{(\beta_1)_i(\beta_2)_i(\beta_3)_i\,i!}x^i{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_1+i,\beta_2+i,\beta_3+i\end{array}:x\right),\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array}:x\right)\\
&=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\alpha_1)_i(\beta_1-\alpha_1)_j(\beta_2-\alpha_2)_i(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\beta_1)_{i+j}(\beta_2)_i(\beta_3)_{i+j}\,i!j!}\\
&\quad\cdot x^{i+j}{}_2F_1\left(\begin{array}{c}\alpha_3+i+j,\alpha_4+i+j\\ \beta_3+i+j\end{array}:x\right),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{array}{c}\alpha_3,\alpha_4\\ \beta_3\end{array}:x\right)\\
&=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(\alpha_2)_{i+j}(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\alpha_2)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\,i!j!}\\
&\quad\cdot x^{i+j}{}_4F_3\left(\begin{array}{c}\alpha_1+i+j,\alpha_2+i+j,\alpha_3+i+j,\alpha_4+i+j\\ \beta_1+i+j,\beta_2+i+j,\beta_3+i+j\end{array}:x\right),
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{array}{c}\alpha_3,\alpha_4\\ \beta_3\end{array}:x\right)\\
&=\sum_{i,j=0}^{\infty}\frac{(\beta_1)_i(\beta_1-\alpha_1)_{i+j}(\beta_2-\alpha_2)_i(\alpha_2)_j(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\beta_1-\alpha_1)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\,i!j!}x^{i+j}\\
&\quad\cdot{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2+j,\alpha_3+i+j,\alpha_4+i+j\\ \beta_1+i+j,\beta_2+i+j,\beta_3+i+j\end{array}:x\right),
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array}:x\right)=(1-x)^{-\alpha_4}\\
&\quad\cdot\sum_{i,j,k=0}^{\infty}\frac{(\beta_1-\alpha_1)_i(\beta_2-\alpha_2)_j(\beta_3-\alpha_3)_k(\alpha_2)_i(\alpha_3)_{i+j}(\alpha_4)_{i+j+k}}{(\beta_1)_i(\beta_2)_{i+j}(\beta_3)_{i+j+k}\,i!j!k!}\\
&\quad\left(\tfrac{x}{x-1}\right)^{i+j+k},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&(1-x)^{-\alpha_4}=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(\alpha_1-\beta_1)_i(\alpha_2-\beta_2)_j(\alpha_3-\beta_3)_k}{(\alpha_1)_i(\alpha_2)_{i+j}}\\
&\cdot\frac{(\beta_2)_i(\beta_3)_{i+j}(\alpha_1)_{i+j+k}(\alpha_2)_{i+j+k}(\alpha_4)_{i+j+k}}{(\beta_1)_{i+j+k}(\beta_2)_{i+j+k}(\beta_3)_{i+j+k}\,i!j!k!}x^{i+j+k}\\
&\cdot{}_4F_3\left(\begin{array}{c}\alpha_1+i+j+k,\alpha_2+i+j+k,\alpha_3+i+j+k,\alpha_4+i+j+k\\ \beta_1+i+j+k,\beta_2+i+j+k,\beta_3+i+j+k\end{array}:x\right),
\end{aligned}
\tag{3.11}
$$

$$(1-x)^{-\alpha_4} = \sum_{i,j,k=0}^{\infty} \frac{(\beta_1-\alpha_1)_i (\beta_2-\alpha_2)_j (\beta_3-\alpha_3)_k (\beta_2)_i (\beta_3)_{i+j}}{(\alpha_2)_i (\beta_1)_{i+j+k}}$$
$$\cdot \frac{(\alpha_1)_{j+k} (\alpha_2)_{i+k} (\alpha_4)_{i+j+k}}{(\beta_2)_{i+j+k} (\beta_3)_{i+j+k}\, i!j!k!}$$
$$\cdot x^{i+j+k} {}_4F_3\left(\begin{array}{c} \alpha_1+j+k, \alpha_2+i+k, \alpha_3+i+j, \alpha_4+i+j+k \\ \beta_1+i+j+k, \beta_2+i+j+k, \beta_3+i+j+k \end{array} : x\right), \tag{3.12}$$

$$\begin{aligned} &{}_4F_3\left(\begin{array}{c} \alpha_1, \alpha_2, \alpha_2, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{array} : x+y-xy\right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i (\alpha_1)_i (\alpha_2)_i (\alpha_3)_i (\alpha_4)_i}{(\beta_1)_i (\beta_2)_i (\beta_3)_i\, i!} x^i y^i \\ &\cdot {}_4F_3\left(\begin{array}{c} \alpha_1+i, \alpha_2+i, , \alpha_3+i, \alpha_4+i \\ \beta_1+i, \beta_2+i, \beta_3+i \end{array} : x+y\right), \end{aligned} \tag{3.13}$$

$$\begin{aligned} &{}_4F_3\left(\begin{array}{c} \alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{array} : x+y\right) \\ &= \sum_{i=0}^{\infty} \frac{(\alpha_1)_i (\alpha_2)_i (\alpha_3)_i (\alpha_4)_i}{(\beta_1)_i (\beta_2)_i (\beta_3)_i\, i!} x^i y^i \\ &{}_4F_3\left(\begin{array}{c} \alpha_1+i, \alpha_2+i, \alpha_3+i, \alpha_4+i \\ \beta_1+i, \beta_2+i, \beta_3+i \end{array} : x+y-xy\right). \end{aligned} \tag{3.14}$$

We prove some of decomposition formulas. For instance, we consider the decomposition formula (3.7). Making use of the principle of superposition of operators, from the operator identity (2.7), we have

$$H_x(\alpha_1,\beta_1)\, H_x(\alpha_2,\beta_2) = \sum_{i,j=0}^{\infty} \frac{(\beta_1-\alpha_1)_i (\beta_2-\alpha_2)_j (\alpha_2)_i (-\delta_x)_{i+j}}{(\beta_1)_i (\beta_2)_{i+j}\, i!j!}. \tag{3.15}$$

Substituting (3.15) into (2.4), we have

$$\begin{aligned} &{}_4F_3\left(\begin{array}{c} \alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{array} : x\right) \\ &= \sum_{i,j=0}^{\infty} \frac{(\beta_1-\alpha_1)_i (\beta_2-\alpha_2)_j (\alpha_2)_i (-\delta_x)_{i+j}}{(\beta_1)_i (\beta_2)_{i+j}\, i!j!} {}_2F_1\left(\begin{array}{c} \alpha_3, \alpha_4 \\ \beta_3 \end{array} : x\right), \end{aligned} \tag{3.16}$$

Therefore, in view of (3.2), and due to differentiation formulas (3.3) we find that

$$\begin{aligned}&(-\delta_x)_{i+j}\,{}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right)\\ &\quad=(-1)^{i+j}\,x^{i+j}\frac{(\alpha_3)_{i+j}\,(\alpha_4)_{i+j}}{(\beta_3)_{i+j}}\,{}_2F_1\left(\begin{matrix}\alpha_3+i+j,\alpha_4+i+j\\ \beta_3+i+j\end{matrix} : x\right).\end{aligned}\tag{3.17}$$

Substituting (3.17) into (3.16) we derive the decomposition formula (3.7). Analogously, the other decomposition formulas for the generalized hypergeometric function ${}_4F_3$ can be proved.

Decomposition formulas (3.13) and (3.14) follow from Taylor's formula [2, p. 257, (17)]:

$$\begin{aligned}&{}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x-h\right)\\ &=\sum_{i=0}^{\infty}\frac{(h)^i}{i!}x^{-i}\,(-\delta_x)_i\,{}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right)\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i\,(\alpha_1)_i\,(\alpha_2)_i\,(\alpha_3)_i\,(\alpha_4)_i}{(\beta_1)_i\,(\beta_2)_i\,(\beta_3)_i\,i!}\,(h)^i\\ &{}_4F_3\left(\begin{matrix}\alpha_1+i,\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_1+i,\beta_2+i,\beta_3+i\end{matrix} : x\right).\end{aligned}\tag{3.18}$$

Furthermore, making use of the following transformations $x \to x+y$, $h \to xy$ and $x \to x+y-xy$, $h \to -xy$, we derive the decompositions (3.13) and (3.14).

4. Some properties of the generalized hypergeometric functions

In this section we define some identities that follow from decomposition formulas (3.4) and (3.14). Combining the internal and external sums in decomposition formula (3.4) we define the relation between hypergeometric ${}_4F_3$ function and Kampé de Fériet function.

$${}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right)=F^{3,1,0}_{2,1,0}\left[\begin{matrix}\alpha_2,\alpha_3,\alpha_4; & -; & \beta_1-\alpha_1;\\ \beta_2,\beta_3; & -; & \beta_1;\end{matrix}\ x,-x\right],\tag{4.1}$$

where ([1, p. 150, (29)], [15, p. 27, (28)])

$$F^{p,q,k}_{l,m,n}\left[\begin{matrix}(a_p); & (b_q); & (c_k);\\ (\alpha_l); & (\beta_m); & (\gamma_n);\end{matrix}\ x,y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s r!s!} x^r y^s. \tag{4.2}$$

In particular case, the identity (4.1) confirms the correctness of general equality (cf., [15, p. 31, (45)])

$$F^{p,1,0}_{q,1,0}\left[\begin{matrix} \alpha_1,...,\alpha_p; & -; & \nu; \\ \beta_1,...,\beta_q; & -; & \mu; \end{matrix}\; x,-x\right] = {}_{p+1}F_{q+1}\left(\begin{matrix} \alpha_1,...,\alpha_p,\mu-\nu \\ \beta_1,...,\beta_q,\mu \end{matrix} \; : \; x\right).$$

If we change the order of summation in decomposition formula (3.4), then we have the following decomposition formula:

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} \alpha_1,\alpha_2,\alpha_3,\alpha_4 \\ \beta_1,\beta_2,\beta_3 \end{matrix} \; : \; x\right) \\ &= \sum_{j=0}^{\infty} \frac{(\alpha_2)_j (\alpha_3)_j (\alpha_4)_j}{(\beta_2)_j (\beta_3)_j j!} x^j {}_4F_3\left(\begin{matrix} \beta_1-\alpha_1,\alpha_2+j,\alpha_3+j,\alpha_4+j \\ \beta_1,\beta_2+j,\beta_3+j \end{matrix} \; : \; -x\right), \end{aligned}$$

i.e., hypergeometric ${}_4F_3$ function can be decomposed via negative argument. Furthermore, if we use formula of analytical continuation for Gauss function in the decomposition formula (3.7) [5, p. 116, (2)]
(4.4)

$$\begin{aligned} F(a,b;c;x) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a} F\left(a,1-c+a;1-b+a;\frac{1}{x}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b} F\left(b,1-c+b;1-a+b;\frac{1}{x}\right), \end{aligned}$$

we have

$$
\begin{aligned}
&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4;\\ \beta_1,\beta_2,\beta_3;\end{array}:x\right)\\
&=\frac{\Gamma(\beta_3)\,\Gamma(\alpha_4-\alpha_3)}{\Gamma(\alpha_4)\,\Gamma(\beta_3-\alpha_3)}(-x)^{-\alpha_3}\\
&\times\sum_{k=0}^{\infty}\frac{(\alpha_3)_k\,(1-\beta_3+\alpha_3)_k}{(1-\alpha_4+\alpha_3)_k\,k!}\left(\frac{1}{x}\right)^k\\
&F_{1,1,0}^{1,2,1}\left[\begin{array}{ccc}\alpha_3+k; & \alpha_1,\beta_2-\alpha_2; & \beta_1-\alpha_1;\\ \beta_1; & \beta_2; & -;\end{array}1,1\right]\\
&+\frac{\Gamma(\beta_3)\,\Gamma(\alpha_3-\alpha_4)}{\Gamma(\alpha_3)\,\Gamma(\beta_3-\alpha_4)}(-x)^{-\alpha_4}\\
&\times\sum_{k=0}^{\infty}\frac{(\alpha_4)_k\,(1-\beta_3+\alpha_4)_k}{(1-\alpha_3+\alpha_4)_k\,k!}\left(\frac{1}{x}\right)^k\\
&F_{1,1,0}^{1,2,1}\left[\begin{array}{ccc}\alpha_4+k; & \alpha_1,\beta_2-\alpha_2; & \beta_1-\alpha_1;\\ \beta_1; & \beta_2; & -;\end{array}1,1\right].
\end{aligned}
\tag{4.5}
$$

With the help of decomposition formula (4.5) we can investigate the behavior of hypergeometric function ${}_4F_3$ at $y=x$ and obtain

$$
\begin{aligned}
&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_2,\alpha_4\\ \beta_1,\beta_2,,\beta_3\end{array}:2x-x^2\right)\\
&=\sum_{i=0}^{\infty}\frac{(-1)^i\,(\alpha_1)_i\,(\alpha_2)_i\,(\alpha_3)_i\,(\alpha_4)_i}{(\beta_1)_i\,(\beta_2)_i\,(\beta_3)_i\,i!}x^{2i}\\
&{}_4F_3\left(\begin{array}{c}\alpha_1+i,\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_1+i,\beta_2+i,\beta_3+i\end{array}:2x\right),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&{}_4F_3\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{array}:2x\right)\\
&=\sum_{i=0}^{\infty}\frac{(\alpha_1)_i\,(\alpha_2)_i\,(\alpha_3)_i\,(\alpha_4)_i}{(\beta_1)_i\,(\beta_2)_i\,(\beta_3)_i\,i!}x^{2i}\\
&{}_4F_3\left(\begin{array}{c}\alpha_1+i,\alpha_2+i,\alpha_3+i,\alpha_4+i\\ \beta_1+i,\beta_2+i,\beta_3+i\end{array}:2x-x^2\right).
\end{aligned}
\tag{4.7}
$$

From the conditions $\Re(\beta_i) > \Re(\alpha_i) > 0$ ($i = 1, 2, 3$), due to (1.4) and decomposition formula (3.4), we have

$$\frac{\Gamma(\beta_3)\Gamma(\beta_3-\alpha_3-\alpha_4)}{\Gamma(\beta_3-\alpha_3)\Gamma(\beta_3-\alpha_4)} = \sum_{i,j=0}^{\infty} \frac{(-1)^i(\beta_1-\alpha_1)_{i+j}(\beta_2-\alpha_2)_i(\beta_1)_i(\alpha_2)_j(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\beta_1-\alpha_1)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\, i!j!} \cdot {}_4F_3\left(\begin{matrix}\alpha_1, \alpha_2+j, \alpha_3+i+j, \alpha_4+i+j \\ \beta_1+i+j, \beta_2+i+j, \beta_3+i+j\end{matrix} : 1\right). \tag{4.8}$$

We note that if we use the known Dougall's formula ([5, p. 191, (4)]; see also [12, p. 57, (2.3.4.7)])

$${}_4F_3\left(\begin{matrix}a, 1+\frac{a}{2}, b, c \\ \frac{a}{2}, 1+a-b, 1+a-c\end{matrix} : -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \tag{4.9}$$

$$\left(\Re\left(b+c-\frac{a}{2}\right) < 1\right),$$

from decomposition formulas (3.4)-(3.14), we can find the values of some summations under certain restrictions.

5. Integrals associated with hypergeometric ${}_4F_3$ function

Making use of integral representation of generalized hypergeometric function we define the integrals associated with ${}_3F_2$ [1, p. 142, (13)], from decomposition formula (3.4), we obtain

$${}_4F_3\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3\end{matrix} : x\right) = \frac{\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_3)} \cdot \int_0^1\int_0^1 t_1^{\alpha_2-1}t_2^{\alpha_3-1}(1-t_1)^{\beta_2-\alpha_2-1}(1-t_2)^{\beta_3-\alpha_3-1} \, {}_2F_1\left(\begin{matrix}\alpha_1, \alpha_4 \\ \beta_1\end{matrix} : xt_1t_2\right) dt_1 dt_2,$$
$$(\Re(\beta_i) > \Re(\alpha_i) > 0,\ i = 1, 2, 3). \tag{5.1}$$

Upon substituting an integral (1.8) into the decomposition formula (3.5) we find that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}\alpha_2,\alpha_3,\alpha_4\\ \beta_2,\beta_3\end{matrix} : x\right)\\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_3)}\\
&\cdot \int_0^1\int_0^1\int_0^1 t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\beta_1-\alpha_1-1}(1-t_2)^{\beta_2-\alpha_2-1}(1-t_3)^{\beta_3-\alpha_3-1}\\
&\qquad \cdot {}_2F_1\left(\begin{matrix}\alpha_4,\beta_1\\ \alpha_1\end{matrix} : xt_1t_2t_3\right)dt_1dt_2dt_3.
\end{aligned} \tag{5.2}
$$

Analogously, we find the following integrals

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4\\ \beta_1,\beta_2,\beta_3\end{matrix} : x\right)\\
&= \frac{\Gamma(\beta_3)}{\Gamma(\alpha_3)\Gamma(\beta_3-\alpha_3)}\int_0^1 t^{\alpha_3-1}(1-t)^{\beta_3-\alpha_3-1}(1-xt)^{-\alpha_4}\\
&\cdot F^{1,2,1}_{1,1,0}\left[\begin{matrix}\alpha_4; & \beta_1-\alpha_1,\alpha_2; & \beta_2-\alpha_2; & \frac{xt}{xt-1},\frac{xt}{xt-1}\\ \beta_2 & \beta_1; & -; & \end{matrix}\right]dt,\\
&\qquad (\Re(\beta_i)>\Re(\alpha_i)>0,\ i=1,\,2,\,3).
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right)\\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_3)}\\
&\cdot \int_0^1\int_0^1\int_0^1 t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\beta_1-\alpha_1-1}(1-t_2)^{\beta_2-\alpha_2-1}(1-t_3)^{\beta_3-\alpha_3-1}\\
&\qquad \cdot (1-xt_1t_2t_3)^{-\alpha_4}\\
&\cdot F^{1,2,1}_{1,1,0}\left[\begin{matrix}\alpha_4; & \alpha_1-\beta_1,\beta_2; & \alpha_2-\beta_2; & \frac{xt_1t_2t_3}{xt_1t_2t_3-1},\frac{xt_1t_2t_3}{xt_1t_2t_3-1}\\ \alpha_2; & \alpha_1; & -; & \end{matrix}\right]dt_1dt_2dt_3,\\
&\qquad (\Re(\beta_i)>\Re(\alpha_i)>0,\ i=1,\,2,\,3).
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\alpha_3,\alpha_4\\ \beta_3\end{matrix} : x\right)\\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_3)}\\
&\cdot\int\limits_0^1\cdots\int\limits_0^1 t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\beta_1+\alpha_1-1}(1-t_2)^{\beta_2-\alpha_2-1}(1-t_3)^{\beta_3-\alpha_3-1}\\
&\cdot(1-xt_1t_3)^{-\alpha_4}\,{}_2F_1\left(\begin{matrix}\alpha_4,\beta_2\\ \alpha_2\end{matrix} : \frac{x(1-t_1)t_2t_3}{1-xt_1t_3}\right)dt_1dt_2dt_3,\\
&\qquad(\Re(\beta_i)>\Re(\alpha_i)>0,\ i=1,2,3).
\end{aligned}
\tag{5.5}
$$

It should be noted that it is impossible to use the formula [15, p. 28, (34)] in the integrals (5.3) and (5.4):

$$
F_{q,1,0}^{p,2,1}\left[\begin{matrix}\alpha_1,...,\alpha_p; & \lambda,\mu; & \nu-\lambda-\mu;\\ \beta_1,...,\beta_q; & \nu; & -;\end{matrix}\ x,x\right]
$$

$$
= {}_{p+2}F_{q+1}\left(\begin{matrix}\alpha_1,...,\alpha_p,\nu-\lambda,\nu-\mu\\ \beta_1,...,\beta_q,\nu\end{matrix} : x\right).
$$

6. Alternative derivations of the decomposition formulas (3.4) to (3.14)

Here, by mainly making use of symbolic operators, we have presented formulas (3.4) to (3.14). Then a question arises: Are there any other methods deriving the decomposition formulas (3.4) to (3.14) ? The answer seems to be affirmative. The decomposition formulas (3.4) to (3.14) can be proved, without using of inverse pairs of symbolic operators, with the help of the method of comparison of coefficients at same degrees of variable x in the both parts of the equality. For instance, consider the decomposition formula (3.8). The right side of decomposition formula

(3.8) we write in the form:

$$\sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(\alpha_2)_{i+j}(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\alpha_2)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\, i!j!} x^{i+j}$$
$$\cdot\, {}_4F_3\left(\begin{matrix}\alpha_1+i+j, \alpha_2+i+j, \alpha_3+i+j, \alpha_4+i+j \\ \beta_1+i+j, \beta_2+i+j, \beta_3+i+j\end{matrix} : x\right)$$
$$= \sum_{m=0}^{\infty} x^m \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(\alpha_2)_{i+j}(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\alpha_2)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\, i!j!}$$
$$\cdot \frac{(\alpha_1+i+j)_{m-i-j}(\alpha_2+i+j)_{m-i-j}(\alpha_3+i+j)_{m-i-j}(\alpha_4+i+j)_{m-i-j}}{(\beta_1+i+j)_{m-i-j}(\beta_2+i+j)_{m-i-j}(\beta_3+i+j)_{m-i-j}(m-i-j)!}. \tag{6.1}$$

Making use of known formulas [5, p. 78, (31)]

$$(\mu+i+j)_{m-i-j} = \frac{(\mu)_m}{(\mu)_{i+j}}, \quad (m-i-j)! = \frac{m!}{(-1)^{i+j}(-m)_{i+j}}$$

in the equality (6.1), we write in the form

$$\sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(\alpha_2)_{i+j}(\alpha_3)_{i+j}(\alpha_4)_{i+j}}{(\alpha_2)_i(\beta_1)_{i+j}(\beta_2)_{i+j}(\beta_3)_{i+j}\, i!j!} x^{i+j}$$
$$\cdot\, {}_4F_3\left(\begin{matrix}\alpha_1+i+j, \alpha_2+i+j, \alpha_3+i+j, \alpha_4+i+j \\ \beta_1+i+j, \beta_2+i+j, \beta_3+i+j\end{matrix} : x\right)$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m(\alpha_3)_m(\alpha_4)_m}{(\beta_1)_m(\beta_2)_m(\beta_3)_m\, m!} x^m$$
$$\sum_{i,j=0}^{\infty} \frac{(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(-m)_{i+j}}{(\alpha_1)_{i+j}(\alpha_2)_i i!j!}. \tag{6.2}$$

By applying the Gauss's summation formula (1.4) twice, we have

$$\sum_{i,j=0}^{\infty} \frac{(\alpha_1-\beta_1)_j(\alpha_2-\beta_2)_i(\beta_1)_i(-m)_{i+j}}{(\alpha_1)_{i+j}(\alpha_2)_i\, i!j!}$$
$$= \sum_{i=0}^{\infty} \frac{(\alpha_2-\beta_2)_i(\beta_1)_i(-m)_i}{(\alpha_1)_i(\alpha_2)_i\, i!} \sum_{j=0}^{\infty} \frac{(\alpha_1-\beta_1)_j(-m+i)_j}{(\alpha_1+i)_j\, j!} = \frac{(\beta_1)_m}{(\alpha_1)_m}\frac{(\beta_2)_m}{(\alpha_2)_m}. \tag{6.3}$$

Substituting (6.3) into (6.2), we finally obtain the decomposition formula (3.8).

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