사변형 그래프에 관하여

정내경[*] · 임형규[**] · 박세원[***]

For the quadrangular graphs

Nae-gyeong Jeong[*] · Hyung-kyu Im[**] · Se-won Park[***]

요 약

본 논문은 최소연결 사변형 그래프 G 의 인접행렬을 연구하고 여기서 우리는 그래프 G 에 대한 $|E(G)|$ 상에서 상한을 얻고 그 얻어진 상한에 대한 또 다른 그래프를 얻는다. 더욱이 우리는 임계연결로 덮힌 사변형 그래프 G에 대한 $|E(G)|$의 상한을 얻는다.

ABSTRACT

In this paper, we study the adjacency matrix of a minimal connected quadrangular graph G, and then we obtain an upper bound on $|E(G)|$ for such a graph G, and we obtain the graph for which the upper bound is attained. In addition, we obtain an upper bound on $|E(G)|$ for a critical matching covered quadrangular graph G.

Ⅰ. Introduction

Let G be a graph with vertex set $V(G)=\{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A=[a_{ij}]$ of G is the (0,1)-matrix of order n in which $a_{ij}=1$ if and only if the edge v_iv_j joining v_i and v_j is in $E(G)$. Thus the adjacency matrix of a graph is symmetric.

If G is a bipartite graph with adjacency matrix A, there exist a permutation matrix P and a (0,1)-matrix B such that

$$P^TAP=\begin{bmatrix} O & B \\ B^T & O \end{bmatrix},$$

and we call B a *bi-adjacency* matrix of G.

In this paper, we only consider simple graphs. Thus if G is a nonbipartite graph with n vertices then its adjacency matrix is a symmetric matrix of order n which has zero trace, and

if G is a bipartite graph with partite set $\{v_1, \ldots, v_m\}$ and $\{v_{m+1}, \ldots, v_n\}$ then its adjacency matrix is a symmetric matrix of order n with $m\times(n-m)$ bi-adjacency matrix. And if G is a nonbipartite graph with adjacency matrix A then the number of edges in G equals the half of the number of nonzero entries in A, and if G is a bipartite graph with bi-adjacency matrix B then the number of edges in G equals the number of nonzero entries in B.

Note that if A is neither symmetric matrix nor

---

* 서남대학교 컴퓨터응용수학과(jnk3611@seonam.ac.kr) ** 교신저자, 신경대학교 인터넷정보통신학과(imh@sgu.ac.kr)
*** 신경대학교 교양과(sewpark1079@naver.com)
접수일자 : 2010. 03. 31 심사(수정)일자 : 2010. 04. 30 게재확정일자 : 2010. 06. 14

has zero trace, then we shall consider bipartite graph G with A as a bi-adjacency matrix.

For a graph G, we define the *neighborhood* of a vertex x in G, $N(x)$, to be the set of vertices in G adjacent to x. The *co-degree* of two distinct vertices x and y in G, $c(x,y)$ is the number of vertices in $N(x)\cap N(y)$. A graph G is said to be *quadrangular* provided $c(x,y)\neq 1$ for any two distict vertices x and y in G.

We say that a matrix A is *inseparable* if there do not exist permutation matrices P_1 and P_2 such that

$$P_1AP_2=\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$$

in which A_{11} and A_{22} are nonempty.

For two vectors $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, we define x and y to be *combinatorially orthogonal* provided $|\{i: x_i\cdot y_i\neq 0\}|\neq 1$. Let A be a matrix. If each pair of rows and each pair of columns of A are combinatorially orthogonal then A is called *combinatorially orthogonal*. Since the combinatorial orthogonality of A depends only on the zero-nonzero pattern of a matrix, in discussing the combinatorial orthogonality of A we may assume that A is a (0,1)-matrix.

As noted in [1], combinatorially orthogonal matrices and quadrangular graphs are closely related.

It is easy to show that G is a connected quadrangular nonbipartite (or bipartite) graph if and only if the adjacency (or bi-adjacency) matrix A of G is an inseparable combinatorially orthogonal matrix.

A connected quadrangular graph G is said to be *minimal* provided G containes no proper spanning subgraph that is both connected and quadrangular.

In [1], Gibson and Zhang proved that $|E(G)|\geq 2|V(G)|-4$ for each minimal connected quadrangular bipartite graph G (see also [2]), and $|E(G)|\geq 2|V(G)|-1$ for each minimal connected quadrangular nonbipartite graph G, and characterized those graphs for which the lower bound is attained. And also, in [1], it is attained a minimal connected quadrangular bipartite graph G of order $n\geq 8$ such that $|E(G)|=\lfloor (n+3)^{2/8}\rfloor-3$, and it is given a couple of open questions for the optimal upper bound on $|E(G)|$ for a minimal connected quadrangular bipartite graph G and nonbipartite graph G, respectively.

In this paper, we study the adjacency matrix of a minimal connected quadrangular graph, and then we obtain an upper bound on $|E(G)|$ for a minimal connected quadrangular nonbipartite graph G and bipartite graph G, respectively, and we obtain those graphs for which the upper bound is attained.

In addition, we obtain an upper bound on $|E(G)|$ for a critical connected matching covered quadrangular nonbipartite graph G and bipartite graph G, respectively, These are also solutions of some open questions presented in [1].

Ⅱ. Optimal combinatorially orthogonal matrices and related graphs

A combinatorially orthogonal matrix A is said to be *optimal* provided each matrix obtained from A replacing a nonzero entry in A by 0 is not combinatorially orthogonal. Thus an optimal combinatorially orthogonal matrix is a combinatorially orthogonal matrix in which every nonzero entry is essential. For example, if

$$A_1 = \begin{bmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix}$$

then A_1 is combinatorially orthogonal matrix which is optimal, but A_2 is combinatorially orthogonal matrix which is not optimal.

In this section, we show that optimal combinatorially orthogonal matrices and minimal quadrangular graphs are closely related.

A *cell* is a matrix with exactly one nonzero entry and it equals 1. If the nonzero entry of a cell is in the (i,j) location, we denote the cell by E_{ij}.

Lemma 2.1 If A is an adjacency (or bi-adjacency) matrix of a minimal connected quadrangular graph, then A is an optimal combinatorially orthogonal matrix which is inseparable.

Proof. Let G be a minimal connected quadrangular graph with adjacency (or bi-adjacency) matrix $A=[a_{ij}]$. Then, A is a combinatorially orthogonal matrix which is inseparable.

Now suppose that A is not optimal. Then there exists an $a_{ij} \neq 0$ such that $A' = A - E_{ij} - E_{ji}$ (or $A' = A - E_{ij}$ if A is a bi-adjacency matrix of G) is inseparable and combinatorially orthogonal matrix. Clearly, the graph G' with adjacency (or bi-adjacency) matrix A' is a proper spanning subgraph of G which is connected and quadrangular. Thus the proof is completed.

The converse of Lemma 2.1 is not true. For example,

$$A = \begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&0&0&1&1&1&1 \\ 0&0&0&1&1&1&1 \\ 1&1&1&0&1&0&1 \\ 1&1&1&1&0&1&0 \\ 1&1&1&0&1&0&1 \\ 1&1&1&1&0&1&0 \end{bmatrix} \tag{1}$$

and $A' = A - E_{14} - E_{16} - E_{41} - E_{61}$ are optimal combinatorially orthogonal matrices which are inseparable. Let G and G' be graphs with adjacency matrices A and A' respectively. Then, clearly, G' is a spanning subgraph of G which is connected and quadrangular. It implies that G is not minimal connected quadrangular graph.

By $B \leq A$ we simply mean entrywise $b_{ij} \leq a_{ij}$ for $n \times n$ (0,1)-matrices $A=[a_{ij}]$ and $B=[b_{ij}]$.

The next theorem immediately follows from the above remark and the definition of minimal connected quadrangular graph.

Theorem 2.2 Let A be an optimal combinatorially orthogonal matrix which is inseparable. If A contains no optimal combinatorially orthogonal matrix A' which is inseparable and $A' < A$ and $A' \neq A$, then G is a minimal connected quadrangular graph with the adjacency (or bi-adjacency) matrix A.

Let P_s be the $s \times s$ basic circulant matrix with form:

$$P_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and the $m \times n$ matrix of all 1's is denoted by $J_{m,n}$.

For integers $r \geq 1$ and $s \geq 3$, we define

$$A_{r,s} = \begin{bmatrix} O & J_{r,s} \\ J_{s,r} & P_s + P_s^{-1} \end{bmatrix} \tag{2}$$

Then we can easily check that $A_{r,s}$ is the $(r+s)\times(r+s)$ optimal combinatorially orthogonal matrix which is inseparable.

We consider a graph, say $G_{r,s}$, with $A_{r,s}$ as an adjacency matrix. Then, clearly, $G_{r,s}$ is a connected quadrangular nonbipartite graph with vertex set $\{v_1, v_2, \ldots, v_{r+s}\}$ obtained by adding the s edges $v_{r+1}v_{r+2}$, $v_{r+2}v_{r+3}, \ldots$, $v_{r+(s-1)}v_{r+s}$, and $v_{r+1}v_{r+s}$ to the complete bipartite graph $K_{r,s}$ with partite sets $\{v_1, v_2, \ldots, v_r\}$ and $\{v_{r+1}, v_{r+2}, \ldots, v_{r+s}\}$.

Note that if $r \geq 3$ and $s \geq 4$ is an even integer then $A_{r,s}$ contains an optimal comb-inatorially orthogonal matrix A' which is inseparable and $A' < A_{r,s}$, $A' \neq A_{r,s}$. For example, such a matrix is

$$A' = A_{r,s} - (E_{1,r+2} + E_{1,r+4} + \cdots + E_{1,r+s} +. \quad E_{r+2,1} + E_{r+4,1} + \cdots + E_{r+s,1})$$

We may refer to (1) for $A_{2,4}$. Clearly, A' is the adjacency matrix of a proper sppaning subgraph of $G_{r,s}$ which is both connected and quadrangular. Thus $G_{r,s}$ is not minimal for each even integer $s \geq 4$.

But, if $s \geq 3$ is an odd integer, then it easy to show that $A_{r,s}$ contains no optimal combinatorially orthogonal matrix A' which is inseparable and $A' < A_{r,s}$, $A' \neq A_{r,s}$. Thus $G_{r,s}$ is minimal for each odd integer $s \geq 3$ from Theorem 2.2, and we obtain following theorem from $|E(G_{r,s})| = (r+1)s$ if we take $G = G_{r,s}$.

Theorem 2.3 For each odd integer $s \geq 3$, there exists a minimal connected quadrangular nonbipartite graph G of order $n \geq 4$ such that $|E(G)| = (n-s+1)s$.

Corollary 2.4 There exists a minimal connected quadrangular nonbipartite graph G of order $n \geq 4$ such that

$$|E(G)| = \begin{cases} 2m(2m+1) & \text{if } n = 4m, \\ (2m+1)^2 & \text{if } n = 4m+1, \\ (2m+2)(2m+1) & \text{if } n = 4m+2, \\ 2m+3)(2m+1) & \text{if } n = 4m+3. \end{cases} \quad (3)$$

Proof. Theorem 2.3 implies that if $r + s = n \geq 4$ and $s \geq 3$ is an odd integer then $G = G_{r,s}$ is minimal connected quadrangular nonbipartite graph of order n. In (2), take

$$\begin{cases} r = 2m-1,\ s = 2m+1 & \text{if } n = 4m, \\ r = 2m,\ s = 2m+1 & \text{if } n = 4m+1, \\ r = 2m+1,\ s = 2m+1 & \text{if } n = 4m+2, \\ r = 2m+2,\ s = 2m+1 \text{ or } r = 2m,\ s = 2m+3 & \text{if } n = 4m+3, \end{cases}$$

then (3) follows from $|E(G)| = (r+1)s$.

In the following remark, we obtain more minimal connected quadrangular nonbipartite graphs.

Remark 2.5. For (0,1)-matrices

$$Q_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

define

$$A_1 = \begin{bmatrix} Q_2 & J_{2,2m-2} \\ J_{2m-2,2} & Q_2 \oplus \cdots \oplus Q_2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} Q_3 & J_{3,2m-2} \\ J_{2m-2,3} & Q_2 \oplus \cdots \oplus Q_2 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} Q_2 & J_{2,2m-1} \\ J_{2m-1,2} & Q_2 \oplus \cdots \oplus Q_2 \oplus Q_3 \end{bmatrix}.$$

Then A_1, A_2 and A_3 are optimal combinatorial orthogonal matrices. Furthermore,

it is easy to show that if A_1, A_2 and A_3 are the adjacency matrices of the graphs G_1, G_2 and G_3 respectively, then G_1, G_2 and G_3 are connected minimal quadrangular nonbipartite graphs of order n such that

$$\begin{cases} |E(G_1)| = 5m-4 & \text{if} \quad n=2m, \\ |E(G_2)| = 7m-5 & \text{if} \quad n=2m+1, \\ |E(G_3)| = 5m-1 & \text{if} \quad n=2m+1. \end{cases}$$

Note that if $m=3$ in A_2, that is,

$$A_2 = \begin{bmatrix} 0\,1\,0\,1\,1\,1\,1 \\ 1\,0\,1\,1\,1\,1\,1 \\ 0\,1\,0\,1\,1\,1\,1 \\ 1\,1\,1\,0\,1\,0\,0 \\ 1\,1\,1\,1\,0\,0\,0 \\ 1\,1\,1\,0\,0\,0\,1 \\ 1\,1\,1\,0\,0\,1\,0 \end{bmatrix}$$

then G_2 with the adjacency matrix A_2 is a connected minimal quadrangular nonbipartite graph of order 7, and $|E(G)|=16$. But $|E(G_{4,3})| = |E(G_{2,5})| = 15$.

Thus we have following: for a connected minimal quadrangular nonbipartite graph G of order $n \geq 4$,

$$\max|E(G)| \geq \begin{cases} 2m(2m+1) & \text{if} \quad n=4m, \\ (2m+1)^2 & \text{if} \quad n=4m+1, \\ (2m+2)(2m+1) & \text{if} \quad n=4m+2, \\ (2m+3)(2m+1) & \text{if} \quad n=4m+3\,(m \neq 1), \\ 6 & \text{if} \quad n=7. \end{cases}$$

For integers $r \geq 1$ and $s \geq 2$, we define

$$B_{r,s} = \begin{bmatrix} O & J_{r,s} \\ J_{s,r} & P_s + I_s \end{bmatrix}, \text{ and}$$
$$B_{r,s+1} = \begin{bmatrix} O & J_{r,s} \\ J_{s,r+1} & P_s + I_s \end{bmatrix} \tag{4}$$

where P_s is the basic circulant matrix of order s and I_s is the identity matrix of order s.

Then we can easily check that $B_{r,s}$ and $C_{r,s+1}$ are the $(r+s)\times(r+s)$ and $(r+s)\times(r+s+1)$ optimal combinatorially orthogonal matrices which are inseparable, respectively.

Theorem 2.7 For each integer $r \geq 1$, there exists a minimal connected quadrangular bipartite graph G of order $n \geq 6$ such that

$$|E(G)| = \begin{cases} 2(m-r)(r+1) & \text{if} \quad n=2m, \\ (m-r)(2r+3) & \text{if} \quad n=2m+1. \end{cases} \tag{5}$$

In particular, there exists a minimal connected quadrangular bipartite graph G of order $n \geq 3$ such that

$$|E(G)| = \begin{cases} 2\left(\frac{n}{2} - \lfloor \frac{n}{4} \rfloor\right)\left(\lfloor \frac{n}{4} \rfloor + 1\right) & \text{if} \quad n \text{ is even}, \\ \left(\frac{n-1}{2} - \lfloor \frac{n}{4} \rfloor\right)\left(2\lfloor \frac{n}{4} \rfloor + 3\right) & \text{if} \quad n \text{ is odd}. \end{cases} \tag{6}$$

Proof. Let G be the bipartite graph with the bi-adjacency matrix $B_{r,s}$ if n is an even integer and the bi-adjacency matrix $C_{r,s+1}$ if n is an odd integer in (4), where $r+s=m \geq 3$ for an integer $r \geq 1$. Then we can easily check that both $B_{r,s}$ and $C_{r,s+1}$ contain no optimal combinatorially orthogonal matrices B' and C' which are inseparable and $B' < B_{r,s}$, $B' \neq B_{r,s}$, and $C' < C_{r,s+1}$, $C' \neq C_{r,s+1}$. Thus G is a minimal connected quadrangular bipartite graph of order $n=2m$ or $n=2m+1$ from Theorem 2.2. (5) follows from $|E(G)| = 2s(r+1)$ if $n=2m$ and $|E(G)| = s(2r+3)$ if $n=2m+1$. In particular, if we take $r = \lfloor \frac{n}{4} \rfloor$ then (6) follows from (5).

참고 문헌

[1] P. M. Gibson and G.-H. Zhang, Combina torially orthogonal matrices and related graphs, *Lin. Alg. Applics*, 282 (1998), 83-95.

[2] K. B. Reid and C. Thomassen, Egde sets contained in circuits, *Israel J. Math.* 24 (1976), 305-319.

저자 소개

정내경(Nae-gyeongJeong)

1991년 8월 조선대학교 대학원 석사 졸업
2002년 2월 우석대학교 대학원 이학박사
서남대학교 컴퓨터응용수학과 조교수
※ 주관심분야 : 신호처리

임형규(Hyung-kyu Im)

1982년 2월 전북대학교 전자공학과 졸업 (공학사)
1985년 8월 전북대학교 대학원 전자공학과 졸업(공학석사)
1996년 8월 전북대학교 대학원 전자공학과 졸업(공학박사)
신경대학교 인터넷정보통신학과 조교수
※ 주관심분야 : 신호처리

박세원(Se-won Park)

1990년 2월 성균관대학교 수학과 졸업 (이학사)
1992년 2월 성균관대학교 대학원 수학과 졸업(이학석사)
1996년 8월 성균관대학교 대학원 수학과 졸업(이학박사)
신경대학교 교양과 교수
※ 주관심분야 : 신호처리