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Students' Colloquial and Mathematical Discourses on Infinity and Limit: A Comparison of U.S. and Korean Students

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The study presented in this paper, which serves as a pilot study for a future comprehensive project, was to investigate how students deal with the concepts of infinity and limit. Based on the communicational approach to cognition, according to which mathematics is a kind of discourse, we tried to identify the characteristics of students' discourse on the topics. Four American and four Korean students were interviewed in English on limits and infinity and their discourse was scrutinized with an eye to common characteristics as well as culture, age, and education-related differences.

I. Introduction

The mathematical concepts of *infinity* and *limit* have been interwoven in the history of mathematics. *Infinity* is the conceptual basis for many mathematical topics, such as the number line and infinite decimals. Since the 19^{th} century, the concept of *limit* has been foundational to how calculus and mathematical analysis deal with other notions such as continuity, differentiability, and integration. In addition, as known to teachers and as confirmed by researchers, most students have considerable difficulty with the notions of infinity and limit.

In this study, students' thinking about infinity and limit is investigated based on the *communicational approach to cognition*, according to which mathematics is a kind of discourse. There are several reasons why this kind of study may be important. First, there has been little research on the mathematical concepts of infinity and limit using discourse analysis as a methodology. Discourse analysis holds promise of answering some previously unanswered questions. Second, such investigations may lead to methods for helping students overcome their difficulties in understanding limit and infinity, and may have implications for K-16 curriculum. Third, this approach may have implications for investigating advanced mathematical thinking in other areas. Finally, applying the communicational method to culturally different groups of students may shed light on the impact of culture on how students understand advanced mathematical notions.

II. Theoretical Background

1. Epistemology and History of the Notions of Infinity and Limit

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The histories of the mathematical concepts of *infinity* and *limit* have been interwoven since their beginning. The story of infinity begins with the ancient Greeks. For the Greeks, infinity did not exist in actuality, but rather as a potential construct. Although there was the notion of bounded processes, there was no concept of limit as a concrete bounding entity.

In the Middle Ages, Christianity came to value infinity as a divine property. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find methods for calculating the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). Throughout the 18th century, calculus lacked firm conceptual foundations. At the end of the 18th century, mathematicians became acutely aware of inconsistencies which plagued the theory of infinitesimal magnitudes (Rotman, 1993).

Today's notion of limit emerged gradually in the 19th century as a result of attempts to remedy the uncertainties existing within mathematical analysis at that time. Cauchy and Weierstrass were pioneers of the movement toward a rigorous calculus (Moore, 1990). At this time, limit turned into an arithmetical rather than geometrical concept, as it was before, in the context of infinitesimals. Infinity was now actual rather than potential. In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of the infinite set.

In spite of the mutual interdependence of the concepts of limit and infinity, there has been little research to examine students' understandings and difficulties of both of them simultaneously.

2. Learning the Mathematical Notions of Infinity and Limit

Various aspects of the learning about infinity and limit have been investigated over the last few decades. Anchoring their research in the analysis of the *mathematical structure* of infinity and limit, some researchers have attempted to understand student (or teacher) difficulties with the notions (박임숙, 2000; 홍진곤, 2008; Artigue, 1992; Borasi, 1985; Cornu, 1992; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Jahnke, 2001; Mamona-Downs, 2001; Sierpinska, 1987; Tall, 1992). For instance, Cottrill et al. (1996) report that there are two reasons for student difficulties with limits. One reason is the need to mentally coordinate two processes: $x \to a$, and $f(x) \to L$. The other is the need for a good understanding of quantification related to ε and δ. Borasi (1985) suggests several alternative rules about how to compare infinities based on students' intuitive notions (within this tradition, see also Cornu, 1992; Tall, 1992).

Other research has focused *on misconceptions* and *cognitive obstacles* related to infinity and limit (김기원, 왕수민, 2003; 우정호, 박선화, 2002; 이대현, 2001; Fischbein, Tirosh, & Hess,1979; Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). For instance, Fischbein, Tirosh, & Hess (1979) and Tall (1992) emphasized the role of intuition. One source of difficulty with the notion of infinity is the belief that a part must be smaller than the whole. Other researchers (Cornu, 1992; Davis &

Vinner, 1986) stress the influence of language. Students might have had many life experiences with boundaries, speed limits, minimum wages, etc. that involved the word "limit". These everyday linguistic uses interfere with students' mathematical understandings (Davis & Vinner, 1986). Przenioslo (2004) focuses on the key elements of students' *concept images* of the limits of functions. Still others have focused on *informal models* that act as cognitive obstacles (Fischbein, 2001; Williams, 2001). According to Williams, informal models based on the notion of actual infinity are a primary cognitive obstacle to students' learning.

Finally, some researchers address students' difficulties through the lens of cognitive theory such as reflective abstraction (전명남, 2003), different levels of cognition (박임숙, 김흥기, 2002), and actions, processes, objects, and schemas (APOS; see Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). For instance, Weller et al. speak about the cognitive mechanisms of *interiorization, encapsulation, and thematization* that are used to build and connect actions, processes, objects, and schemas.

3. Conceptual Framework for This Research

The present study takes as an assumption that when students come to the classroom to learn the notions of infinity and limit, they already have a certain amount of knowledge that comes from daily experience. The use of a given concept in everyday language can be crucial for students' future learning. Therefore, for those who teach the subject it is important to find out how students use the notions of infinity and limit in colloquial discourse.

Most of the past research on learning limits and infinity was grounded in a neo-Piagetian, cognitivist framework which does not seem quite appropriate for this type of study as it underestimates not only the inherently social nature of student thinking, but also the role of discourse and communication in learning and in other intellectual activities. Our project is guided by a conceptual framework within which school learning is seen as aiming at a change in ways of communicating. In particular, learning mathematics is seen as tantamount to becoming more skilful in the discourse regarded as mathematical. The word *discourse* signifies any type of communicative activities, whether with others or with oneself, whether verbal or not.

Four distinctive features of mathematical discourses are often considered whenever discourses are being analysed, compared, and watched for changes over time: *words and their use, discursive routines, endorsed narratives, and mediators and their use* (Ben-Yehuda et al., 2004). *Uses of words* are the ways the participants use keywords of colloquial and mathematical discourse regarding *infinity* and *limit. Word uses* are important as a feature of colloquial and mathematical discourses because they can reveal how students understand those words. For instance, how students use their mathematical words can be assessed by the degree of objectification. The degree of objectification will be assessed according to the frequency with which students speak about infinity and limit, as if these words referred to self-sustained, discourse-independent objects. *Discursive routines* are the patterns of repetitive actions in students' discourses.

For example, we can observe such patterns in the mathematical discourses of comparing, counting, and defining. *Endorsed narratives* are propositions that students accepted as true. A narrative endorsed by an interviewee does not have to be a quotation from what the interviewee says. It may be an interpretation of assumptions that tacitly guides him or her. For instance, the expression of "Limit is something…" can be inferred from statements actually made by students. Thus, through discursive activity, students can produce *endorsed narratives*. *Visual mediators* are means that the participants use for their communication activity. While colloquial discourses are communicated with the help of concrete objects or individual images, mathematical discourses are mediated by informal and formal symbols. For example, to identify infinity in mathematical discourse, students can use and refer to such mediators as an infinity symbol (∞), three dots (…), and an "eight sideways" (∞) in multiple ways.

In this article, particular attention has been paid to the participants' uses of the keywords *infinity* and *limit* in colloquial and mathematical discourses and to *endorsed narratives* about limits and infinity, that is, to propositions that the participants accepted as true.

III. Design of Study

1. Research Questions

Our interest in characterizing the mechanisms of students' thinking about infinity and limit led to the following research questions:

1) What are the leading characteristics (in terms of word use and endorsed narratives) of students' colloquial and literate (school) discourse about infinity and limit?
2) Do the students' colloquial and literate discourses on infinity and limit change with age and education?
3) Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit? Can these differences be accounted for in terms of the differences in the colloquial uses of these words in English and in Korean?

2. Methodology

The reason for a comparison between American and Korean students is that the Korean mathematical words for *infinity* and *limit* in a mathematical context rarely appear in colloquial Korean language. Therefore, while American students have experience with the colloquial use of the English words *infinity* and *limit*, Korean students have little experience with colloquial Korean use of the mathematical terms. Each linguistically distinct group included one elementary school student, one middle school student, one high school student, and one university undergraduate (to refer to groups' members, we use symbols such as A_5 for the American 5^{th} grader, K_{10} for the Korean 10^{th} grader, and AU for the American undergraduate). The four American students were English speakers from the United States and the four Korean students were non-native English speakers from South Korea whose first language is Korean. Because the interviews were to be conducted in English, the four Korean students who were selected had been living in the United States and attending US schools for more than 3 years.

The interview questionnaire consisted of 29 questions, organized into eight categories. The first two categories aimed at scrutinizing students' *colloquial* discourses on infinity and limit, whereas the rest were targeted at investigating students' *mathematical* discourses on the topic. Examples of the interview questions are shown in Figure III-1. The interviews lasted 30 to 40 minutes. The conversations were audio- and video-taped and then transcribed in their entirety.

I. Create a sentence with the following word(term):

(a) Infinite, (b) Infinity

II. Say the same thing without using the underlined word.

(b) Eyeglasses are for people with <u>limited</u> eyessight.

III. Which is a greater amount and how do yo know?

(d) A: Odd numbers, B: Integers

IV. $\frac{1}{4}=0.25, \frac{2}{8}=0.25, \frac{3}{12}=0.25, \ldots$

How many such equalities can you write?

V. What do you think will happen later in this table? How do you know?

VI. (a) What is the limit of the following $\frac{1}{x}$ when x goes to infinity?

VII. Read aloud: $\lim_{x \to \infty} \frac{x^2+3x}{2x^2} = \frac{1}{2}$. Explain what it says,

VIII. (a) What is infinity?

[Figure III-1] Representative sample of question from each category.

Data were analyzed so as to identify and describe the four distinctive features of the respondents' discourses: word use, routines, visual mediators, and endorsed narratives. At the next stage, the analysis of the data was guided by the three research questions. In this paper, we present *word uses* and *endorsed narratives* of the respondents' discourses on *infinity* and *limit* in questions I and VIII, and compare the results obtained in the two ethnic groups.

IV. Findings

Students' responses to the two questions relevant to our present subject (questions I and VIII) have been summarized. Students' responses using the words *infinite* and *infinity* are summarized in Table IV-1. Those responses using the words *limited* and *limit* are reported in Table IV-2. Students' responses to question I may count as representative of the colloquial discourse on *infinity* and *limit*, whereas their answers to question VIII seem to be a result of the students' attempt to respond according to the rules of literate (school) discourse. This conjecture seems reasonable because of the fact that in this latter part, the interview was contextually framed as interview on the *mathematical* concepts of limit and infinity. The analysis that follows is guided by the three research questions listed above.

Question 1: What are the leading characteristics of students' colloquial and literate (school) discourse on infinity and limit?

The first research question pertains to the characteristics of students' colloquial and mathematical discourses on *infinity* and *limit* in terms of *word use* and *endorsed narratives*.

1) Infinity

<u>Use of words:</u> In the use of the words *infinite* and *infinity*, *word use* can be characterized according to three properties: their *context, application,* and the degree of their *objectification* (this latter term is explained below).

<Table VI-1> The summary of answers to questions I and VIII (a).

Student		I. Create a sentence with the followoing word: (a) Infinite	(b) Infinity	VIII. (a) What is infinity?
American	A_5 A_7 A_{10} A_U	[1] They have an infinite amount of movies. [2] Outer-space is infinite and forever [3] There is an infinite amount of numbers. [4] There are infinite ways go spell certain things.	[5] The years go to infinity [6] Infinity is a concept··not a number [7] Infinity is the largest number [8] I love you more than infinity	[17] Infinity will go on forever. [18] Infinity is a concept that goes on forever. [19] Infinity keeps going on and increasing. Infinity has no limit. [20] Infinity is never-ending··has no beginning and no end. There is not one thing that is infinite in the world. It's just a concept.
Korean	K_4 K_7 K_{10} K_U	[9] Numbers are infinite [10] Numbers are infinite [11] We don't have infinite amount of natural resource in the planet. [12] Some people like infinite space	[13] One time Infinity is Infinity [14] A line goes to infinity [15] I don't think there is not just a thing in infinity [16] We have to study infinity.	[21] Infinity is like a number that never ends or something that never ends; infinity is not the number. [22] Infinity is like the furthest number keep going··like never-ending. Infinity is like it goes forever like there is no end. [23] It's not a number··it's same like it's not limited··· same that never end. The number system that never ends and keeps coming. [24] It's a not a number because it is a very large amount and cannot be explained to the number.

- *Context.* The first thing to note is that all the students, even those who are too young to have met the notion of infinity in the context of school mathematics, are capable of creating sentences with the words *infinite* and *infinity* - a fact that testifies to these words being a part of everybody's colloquial English discourse. This said, there is a considerable difference between the American and Korean groups in the context in which the words are mentioned. In the American group, *infinite* is used in conjunction with *amount* in [1], [3], [4] and both words are applied mainly in the context of real-life phenomena involving large magnitudes: outer-space [2], ways to spell words [4], number of years [5], love [8], etc. In the Korean group, the context of the sentences is predominantly abstract and mathematical (the sentences mention numbers [9], [10], lines [14], operations on infinity [13], and the infinity as an object of study [16]), whereas the relation to magnitudes and to large amounts is less pronounced. In the mathematical discourse, the prevalent feature of the definitions given by the American students is that they take the object-like character of infinity for granted and characterize this object by saying what it is doing: "go on forever" (A_5, A_7), "keeps going and increasing" (A_{10}), "is never-ending" (A_U). The Korean students begin with an attempt to specify the category to which infinity belongs, and they usually do it with the help of comparative or negative sentences, such as "It is like a number" (K_4, K_7) or "It's not a number" (K_{10}, K_U). Thus, a common property of all the answers in this group is that while stating some *number-like properties* of infinity, they also deny its being a number. Such explicit comparison to number (or to any other entity, for that matter) is absent from the American answers.

- *Application.* There is also a delicate ontological difference between the groups in their application of the word *infinite* to numbers: While all three students who apply the word infinite to numbers (A_{10}, K_4 and K_7) seem to be saying the same thing – that numbers can be "infinite", only A_{10} makes it clear that he means the size *of the set* of all numbers. There is no reference to *the set* of all numbers in the utterances of the Korean students, and these utterances may be interpreted as saying that these are the numbers themselves (as opposed to the set *of numbers*, which is a second order construct) that are unlimited in their size.

- *Objectification.* The use of a noun counts as objectified if this noun is applied as if it referred to a self-sustained, discourse-independent entity. In the colloquial discourse, the Koreans' uses of *infinite* and *infinity* were more abstract and mathematical than those of the Americans; which were more colloquial. Thus, the Americans' uses of *infinite* and *infinity* may be considered less objectified than those of the Koreans because objectification seems to increase with abstraction. In the mathematical discourse of defining *infinity*, the American students characterized *infinity* by saying what it is doing, whereas the Korean students began with an attempt to specify the general category to which it belonged and continue with the presentation of specific features. Thus the definitions given by the American group are *operational*, while the definitions provided by the Korean group are *structural*. The label *operational* refers to statements that describe *infinity* as a process, as opposed to *structural* utterances which present its structure. The more rigorous structure of the descriptions in the Korean group may be evidence of objectification.

Endorsed narratives: The endorsed narratives are the statements - explicit or tacit - that students accept as true. Even if such narratives are not articulated explicitly, they can be inferred from students' actual responses. The following endorsed narratives of *infinity* were found.

- *a process that never ends* – In the majority of statements, *infinity* was used to signify something which goes on and on, as evidenced by such uses as "go on forever" [17], "a concept that goes on forever" [18], "keeps going on" [19], "never-ending" [20, 22], "same that never end" [23], and "something that never ends" [21]. In addition, the American 10^{th} grader used *infinity* as something "increasing" [19] like a *monotonous* process.

- *a number-like thing that never ends* – Only the Korean students (K_4, K_7, and K_{10}) characterized *infinity* in terms of something related to both a number and a never-ending process. This appears in such statements as "like a number that never ends" [21], "like the furthest number keep going" [22], and "the number system that never ends" [23].

- *a concept* – Some students conveyed a view of *infinity* as an abstract concept. This endorsement is evidenced by such statements as "a concept that goes on forever" [18] and "it's just a concept" [20]. Although more evidence is needed, the utterance of [24] can be interpreted as conveying this endorsement because the statement that "it cannot be explained as a number because it's too large" [24] can be understood as a view of infinity, that is an abstract concept.

- *no limit* - Other students included "infinity has no limit" [19] in their definitions. Because limit can be interpreted as an ending, this endorsement can be understood as the endorsed narrative of *infinity* as the opposite of limit ("infinity is the opposite of limit" See [22] in Table VI-2).

2) Limit

<u>Use of words:</u> The use of the words *limit* and *limited* can be characterized according to two characteristics: their area of *application* and the measure of their *objectification*.

- *Application*. Almost all students' responses refer the words *limit* and *limited* to *processes that are bounded* in a certain quantifiable aspect, mostly in time. In the colloquial language the processes are of watching TV ([1]), playing ([5], [10]), eating ([2], [6]), driving ([7], [15]), spending money ([4], [8]), and admitting diners in a restaurant ([16]). In the more school-like discourse (see responses to question VIII - (b) in Table 2), there is no mention of any concrete circumstances, but the observation that the words limit and limited refer to processes is evident from expressions that imply a change in time, such as "limit can go on" ([17]), "numbers can't go past" ([19]), "you ··· never reach the limit" ([20]; cf. [23]), "limit is the furthest something to go" ([22]).

- *Objectification*. Evidence of such use can be seen only in the utterances of 10th graders and university students; in all such cases the entity is presented as a particular number ("the speed limit is 50" ([7]), "ten dollars" ([4]), eighteen (12]), "a certain number" ([23]), "special number" ([24]); the status of the utterance [18] is unwarranted, in

<Table VI-2> The summary of answers to questions I and VIII (b)

Student		I. Create a sentence with the following word: (c) Limited	(d) Limit	VIII. (b) What is limit?
American	A_5 A_7 A_{10} A_U	[1] I am limited on my TV for one hour [2] There are limited amount of cookies [3] I am limited to my ability [4] My spending money each month is limited to ten dollars	[5] I have a limit on video games [6] We must limit the amount of candy we eat [7] The speed limit is fifty [8] There is a spending limit on my credit card	[17] Limit can go on and will stop at some point. [18] The limit is the value of a number. [19] Limit is something that numbers can't go past. They have to stop at a certain point. [20] As not in mathematics, limit is the absolute ending of something like no more. In a mathematical sense, it would always be the ending of···like an answer, but sense··· numbers are infinite. You can get to the answer but never reach the limit because there's infinite numbers... kind of opposite in a way because there isn't··· there is no limit in infinity.
Korean	K_5 K_7 K_{10} K_U	[9] The time is limited [10] I have a limited time for playing outside [11] My ability is limited [12] Let's limited to eighteen	[13] There is a limit [14] Our teacher put the limit to how much time we have to test [15] I can't drive such at the speed limit [16] That restaurant limits people's number	[21] Limit means like the *maximum*. [22] Limit is the farthest something to go··· like where it stops···like boundary you cannot go on··· I think, infinity is the opposite of the limit. [23] Limit is a certain number that you can't reach but you get very close to. [24] Limit is *a special number*···for instance if I say limit four, then there is one, two, three, and four. Not over four.

the present context).

Endorsed narratives: The following responses seem to complete the expression "Limit is…" to a narrative that is endorsed as true by at least some of the students, in some of the contexts (the narratives did not have to be articulated in exactly this form; their endorsement was inferred from statements actually made by the students).

- *an upper boundary* - In the majority of statements *limit* is used to describe as something that limits the process from above; this is true whether one speaks about playing, eating, spending money, driving, or admitting diners. It is true in the more abstract context, as evidenced by expressions such as "limit means like the maximum" ([21]), and "limit is the farthest [for] something to go" ([22]).

- *finite* - It seems that some students use *limited (bounded) as having an end* (that is, stopping at some point in time) and thus as an opposite of *infinite;* see [20] and [22].

- *unreachable* - In the process usage, the "unreachable" meaning occurs when one approaches the limit but does not actually attain it. This narrative appears only in the responses of the university students; see [20] and [23]. (In [20], based on our reading of A_U's responses to other interview questions, we interpreted the words "the answer" as referring to the result one gets while calculating limits according to known mathematical algorithms.)

Question 2: Do the students' colloquial and literate discourse on infinity and limit change with age and education?

Based on the above descriptive findings, there were no significant systematic differences with age in the colloquial and mathematical discourses on *infinity.* However, we can speak about the following age-related characteristics of the students' use of the word *limit* and related endorsed narratives.

Differences in the use of the words: Two such differences seem to appear with age. First, the *area of application* of the discourse on limits evolves hand in hand with shifts in the students' everyday experience: The younger students (grades 4, 5, and 7) speak about limits in the context of children's favorite pastime activities, such as watching TV, playing videogames, or eating sweets. The 10^{th} graders speak about driving and individual abilities, two issues clearly quite significant to young people of this age. Finally, the university students make references to limits and limitations in the context of spending money and visiting restaurants. Second, the discourse limit gradually undergoes *objectification* and the uses of the noun limit as signifying a self-sustained entity become more frequent.

Differences in endorsed narratives: There seems to be no change in the students' endorsement of the colloquial narrative of limit as the *upper bound,* where all the other values produced in the limited process must remain on one side of the limit. One salient education related difference is the endorsement of the narrative on the unreachability of the limit, which we saw only in the mathematical discourse of the older students (thus the conjecture that it was related to education). In older students the discourse on limit becomes problematized, as they start seeing a possible incompatibility between *finiteness* and *unreachability.*

Question 3-a: Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit?

Colloquial discourse about *infinity:* In the usage of the words *infinite* and *infinity*, one important characteristic of the Korean group, as opposed to the American, is that the students' utterances were more abstract and mathematical. Also, the Korean students applied the word *infinite* to numbers themselves, rather than to sets of numbers. One significant characteristic of American students' colloquial discourse regarding *infinity* is their use of real-life contexts involving large magnitude. Another characteristic is the application of the word *infinite* to the *set* of all numbers. All of the American students had an *operational use* of infinite and infinity.

Mathematical discourse about *infinity*: In defining *infinity*, the Korean students' responses reflected more attention to a rigorous and abstract structure than the American students' because their responses specified the general category (i.e., some number-like properties of infinity) with the presentation of specific features (i.e., the use of infinity as a process), that is, a *number-based structural use* of infinity. The definitions of most of the Koreans (K_4, K_7, and K_{10}) included the endorsed narratives of *infinity* as a process and a number-like thing. These endorsements seemed to be grounded in the *structural word use.* While defining *infinity*, all four Americans took the object-like character of infinity for granted and described this object based on an operational use without referring to numbers. In the definition task, the American students' endorsement of infinity as process seems to be based on their *operational word use.*

Unlike in the case of the discourse on infinity, no systematic differences were found in the discourse on limit between native Korean and English speakers.

Question 3-b: Can these differences be accounted for in terms of the differences in the colloquial uses of these words in English and in Korean?

According to the previous descriptive summary of the current study, there were two noticeable differences between the American and Korean groups in the colloquial discourse on *infinite* and *infinity*: the *context* of the sentences and the *application* of the word *infinite* to numbers. The two differences between the American and Korean groups in colloquial discourse on *infinity* seemed to be related to the differences in their mathematical discourse on *infinity* (in terms of *word use* and *endorsed narratives*).

In terms of *context,* as compared with their American counterparts, the Korean students showed a more rigorous and abstract structure in their definitions of *infinity.* This more rigorous and abstract structure in mathematical discourse may be related to the abstract and mathematical contexts in colloquial discourse. In contrast, all four American students took the object-like character of *infinity* for granted and characterized this object based on an *operational* use when they defined *infinity.* An *operational* use in mathematical discourse seemed to be related to an operational use in real-life contexts in colloquial discourse on *infinity.* In the case of *application*, the *number-based structural approach* to infinity, appeared in mathematical discourse of the Korean students, was salient in colloquial discourse (i.e., application of the word *infinite* to numbers themselves). As explained before, it was evident in the Korean students' *endorsed narratives.* In the American group, the characteristics of colloquial discourse on *infinity* (i.e., the *application* of the word *infinite* to the *set* of all numbers and the relation to an *amount*) seemed to be related to the *operational* approach in mathematical discourse on *infinity*. This *operational* approach in the mathematical discourse on *infinity* appeared as in *endorsed narratives*.

V. Discussion and Conclusions

This study is a part of a larger research project that seeks to investigate how students think the concepts of *infinity* and *limit*. As noted earlier, a critical review of previous methodologies in research on infinity and limit revealed the importance of this study's methodology. To discover the mechanisms of student thinking and learning difficulties regarding infinity and limit, various aspects of learning about these concepts have been investigated over the last few decades. They present different interpretations of learning difficulties and suggest different methodological reasons related to the concepts of infinity and limit, such as epistemological factors (e.g., 박임숙, 2000; Cottrill et al., 1996), teaching methods (e.g., 김기원, 왕수민, 2003), and psychological reasons (e.g., 전명남, 2003; Weller et al., 2004). However, our important assumption is that when students come to the classroom to learn mathematical concepts, they already have a certain amount of knowledge of interactions with these concepts that come from daily experience, and thus the use of a given concept in everyday language can affect students' future learning.

In Korea, most mathematical words (e.g., infinite, infinity, element, and set) were borrowed from the Chinese written language, but they were written in native Korean without using Chinese characters. The Korean mathematical word *infinity* is sometimes spoken in colloquial Korean but it is less used than the American word. The Korean mathematical noun 무한 (Mu-han), for *infinity* is first introduced to explain the concept of infinite sets in the 7^{th} grade school curriculum. In a 7^{th} grade textbook, a set is defined as a collection whose objects are clearly clarified (and it is used exclusively in the Korean mathematical discourse). Then an element is described as a constituent member in a set. After the introduction, an infinity-set is defined as a set which includes infinitely many elements with other mathematical words such as finite, set, and element. The Korean mathematical noun 무한대 (Mu-han-dae), for *infinity* and its symbol (∞) is first introduced in the 10^{th} grade mathematics curriculum along with the concepts of infinite sequence and limit (and it is also used exclusively in the Korean mathematical discourse). Based on the concepts of convergent and divergent sequences, the new symbol ($n \to \infty$) is also presented (as 'n gets bigger endlessly').

The two differences noted above (i.e., the *contexts* and the area of *application of* the word *infinite*) may be explained on the basis of the fact that the Korean mathematical words for *infinity* and *infinite* and *set*, with their origins in Chinese characters, do not appear in the Korean colloquial language and the Korean students do not associate them, or even their English counterparts, with anything in particular in the colloquial discourse. One can conjecture that for American students, the colloquial use of the English word *infinite* precedes the mathematical, whereas for the Korean students it may go the other way round.

We have reasons, once again, to speculate that the Korean students' acquaintance with the English word *infinity*, unlike that of the American students, came primarily from the formal mathematical discourse. The more rigorous structure of their descriptions, which, unlike those of the American students, begin with the attempt to specify the general category and continue with the presentation of specific features, may be yet additional evidence that these students' were introduced to the discourse on infinity through mathematical, or as-if mathematical, definitions rather than through casual use.

We observed a considerable difference between the American and Korean groups in both the context in which the words of *infinity* and *infinite* were mentioned and their application to numbers. Our present data, however, indicate almost no difference between the two groups in their discourses on *limit*. We ascribe this finding to the fact that unlike the word *infinity*, the words *limit* and *limited* are quite common in English colloquial discourses. The Korean students, in spite of the fact that English was their second language, might already have been sufficiently familiar with the colloquial uses of the English word *limit* to make it irrelevant that there was no comparable use for the Korean mathematical term *limit* in everyday Korean discourse. Thus, this pilot study implies that in a more extensive future study, native Korean students should be chosen who *have not studied mathematics* in English and should be interviewed about both colloquial and literate discourses on infinity and limit *in Korean.*

One may propose that for the Korean students the colloquial English discourse on *limited* and *limit* has been used more frequently than that on *infinite* and *infinity*. The Korean students, like their English counterparts, use the words *limited* and *limit* in the colloquial English discourse. Although the Korean mathematical words for *limited* and *limit* do rarely appear in the Korean colloquial language, the differences in the colloquial discourse on *limited* and *limit* between the American and Korean groups is insignificant in the context and the way in which the words are mentioned. This may be because of the Korean students' experiences in colloquial English discourse. Because there are different Korean colloquial words for the mathematical words *limited* and *limit*, if Korean students were interviewed in Korea where the word for mathematical limit does not often appear in their own colloquial language, the ways in which they speak of limit could be explored more fully.

Based on our findings, we can conclude that the use of the words *limit* and *limited* coming from students' colloquial discourses conflicts with the one imposed by the formal mathematical definition of limit in at least two respects. First, the narrative on limit as *an upper boundary* implies that all the values produced in the process must be on the same side of the limit. Second, the story of the limit's *unreachability* precludes the possibility that some values produced in the "limiting" process will be equal to the limit itself. Clearly, neither of these interpretations follows from the mathematical definition of limit. In addition, the use of *limited* as *finite* seems in conflict with the narrative about limits as being a property of infinite processes (the possibility that a process with an infinite number of steps may still be finite in time is also difficult to conceive).

Although the sample size of the current study is too small to allow for generalization, what was found in this study may serve as a basis for new hypotheses to be tested in a larger-scale project in the future. On the grounds of these findings, the colloquial discourse does seem to have an impact on the mathematical discourse. This was evidenced by the clear differences between the colloquial and mathematical discourses on infinity of the American students and those of the Korean students. A strong relationship between the colloquial and mathematical discourses on *limited* and *limit* provides additional evidence. Therefore, if students' colloquial discourse is the primary source of their mathematical discourse, then this may have an impact not only on students' later use of the mathematical vocabulary, but also on other aspects of their mathematical discourse, such as routines, use of mediators, and endorsed narratives - a fact that the teachers should keep in mind while planning instruction. Our preliminary findings presented in this report justify future attempts to test this conjecture further.

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학생들의 무한과 극한에 대한 구어적 담화와 수학적 담화: 미국학생과 한국학생의 비교

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포괄적인 연구를 위한 예비 조사로서 본 연구는 학생들이 무한 개념과 극한 개념을 어떻게 다루는지를 조사하였다. 수학은 담화의 하나라는 인식에 관한 상호 의사소통의 접근방식을 바탕으로 이 개념들에 관한 학생들의 담화의 특성들을 확인하려고 하였다. 4명의 미국학생과 4명의 한국학생을 무한과 극한에 대해서 영어로 면담하였고 학생들의 담화를 공통적인 특성들과 문화, 나이, 그리고 교육에 관련된 차이점들에 초점을 맞추어 자세히 조사하였다.

* Key words: Infinity(무한), Limit(극한), Communicational approach to cognition and Discourse.

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