

Lacunary Statically Convergent and Lacunary Strongly Convergent Generalized Difference Sequences of Fuzzy Real Numbers

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ABSTRACT. In this paper we introduce the concept of lacunary statistical and lacunary strongly convergence of generalized difference sequence of fuzzy real numbers. We prove some inclusion relations and also study some of their properties.

1. Introduction

The concept of fuzzy set was introduced by L. A. Zadeh in 1965. Later on many research workers were motivated by the introduced notion of fuzzy sets. It has been applied for the studies in almost all the branches of sciences, where mathematics has been applied. Many workers have studied the applications of fuzzy sets by introducing the notions of fuzzy topology, fuzzy orderings, fuzzy measure, fuzzy linear programming etc.

Workers on sequences spaces have also applied the notion of fuzzy real numbers and have introduced sequences of fuzzy real numbers and have studied their different properties. Tripathy and Sarma [21], Bilgin [2], Altin, Et and Colak [1], Tripathy and Baruah ([13], [14]), Tripathy and Borgohain [15], Tripathy and Dutta ([16], [17]) have studied some classes of sequences of fuzzy real numbers.

The notion of new type of generalized difference for sequence spaces is introduced by Tripathy and Esi [18]. The notion of generalized difference Lacunary sequences is studied by Tripathy and Mahanta [20]. The notion of lacunary statistically convergent sequences of fuzzy numbers is studied by Altin, Et and Colak

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Received May 19, 2010; revised August 25, 2010; accepted September 16, 2010.

2000 Mathematics Subject Classification: 40A05, 40A25, 40A30, 40C05.

Key words and phrases: Lacunary sequence, statical convergence, generalised difference, statically Cauchy sequences, solidness, symmetricity, convergence free.

[1] and Blign [2], which motivated us to introduce and investigate the notion of generalized difference for sequences of lacunary statistical and lacunary strongly convergence of sequences of fuzzy real numbers.

A fuzzy real number X is a fuzzy set on R and is a mapping $X : R \rightarrow I (= [0, 1])$ associating each real number t with its grade membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$, such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*. A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R . The set of all *upper semi-continuous, normal, convex* fuzzy number is denoted by $R(I)$.

The α -level set of a fuzzy real number X , for $0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in R : X(t) \geq \alpha\}$; for $\alpha = 0$ it is the closure of the strong 0 cut (*i.e.* closure of the set $\{t \in R : X(t) > 0\}$).

For each $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq 0. \end{cases}$$

Let $\bar{d} : R(I) \times R(I) \rightarrow R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} \{\max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}\},$$

where $X^\alpha = [X_1^\alpha, X_2^\alpha]$ and $Y^\alpha = [Y_1^\alpha, Y_2^\alpha]$.

It is well known that $(R(I), \bar{d})$ is a complete metric space. The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

2. Definitions and preliminaries

The notion of statistical convergence was studied by Fast [6] and Schoenberg [9] at the initial stage independently. Later on it was studied from sequences space point of view and linked with summability theory by Rath and Tripathy [8], Tripathy ([9], [10]), Tripathy and Sarma [22], Tripathy and Sen ([23], [24]) and many others. The idea of statistical convergence depends on the density of subsets of the set N of natural numbers. The density of a subset E of N is defined by

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k), \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E .

A sequence (x_k) is said to be statistically convergent to L , if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. In this case, we write $\text{stat-lim } x_k = L$.

Let $p = (p_k) \in \ell_\infty$, be a sequence of positive real numbers. Let $(a_k), (b_k)$ be two sequences of complex numbers . Then we have the following well-known inequality

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}),$$

where $K = \max(1, 2^{H-1})$ and $H = \sup_k p_k$.

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, 3, \dots$ where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \rightarrow \infty$, as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$ for $r = 1, 2, 3, \dots$.

Kizmaz [7] defined the difference sequence space for crisp set. The notion was generalized by Colak and Et [3] by introducing the notion of Δ^n -convergence of sequences. Different classes of difference sequence spaces were investigated by Esi, Tripathy and Sarma [4], Et, Altin, Choudhary and Tripathy [5], Tripathy [11], Tripathy, Altin and Et [12], Tripathy and Mahanta [20] and many others. The notion of difference sequences was generalized by Tripathy and Esi [18] in a different way as follows.

Throughout ℓ_∞, c, c_0 denote the classes of *bounded, convergent* and *null* sequence spaces of crisp numbers. Let $m \geq 0$ be an integer. Then $Z(\Delta_m) = \{(x_k) \in \omega : (\Delta_m x_k) \in Z\}$, for $Z = \ell_\infty, c, c_0$, where $\Delta_m x_k = x_k - x_{k+m}$ for all $k \in N$. For $m = 1$, the space $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ are studied by Kizmaz [7].

Different types of generalized difference Cesàro sequence spaces were introduced and investigated by Tripathy, Esi and Tripathy [19]. We introduce lacunary Δ_m^n -statistically convergent and lacunary $\Delta_m^n(p)$ -strongly convergent sequences of fuzzy numbers. The generalized difference Δ_m^n has the following binomial representation:

$$\Delta_m^n X_k = \sum_{\nu=0}^n (-1)^\nu \binom{n}{\nu} X_{k+m\nu}, \text{ for all } k \in N, m, n \geq 0 \text{ are integers.}$$

Definition. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be Δ_m^n -bounded if the set $\{\Delta_m^n X_k : k \in N\}$ of fuzzy numbers is bounded and Δ_m^n -convergent to the fuzzy number X_0 , written as $\lim_{k \rightarrow \infty} \Delta_m^n X_k = X_0$, if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $\bar{d}(\Delta_m^n X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

Let $\ell_\infty^F(\Delta_m^n), c^F(\Delta_m^n)$ denote the set of all Δ_m^n -bounded sequences and all Δ_m^n -convergent sequences of fuzzy numbers respectively.

Definition. Let $\theta = (k_r)_{r=0}^\infty$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary Δ_m^n -statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \bar{d}(\Delta_m^n X_k, X_0) \geq \varepsilon\}| = 0,$$

where the vertical bars denote the cardinality of the set.

We denote this as $X_k \rightarrow X_0(S_\theta^F(\Delta_m^n))$ or $S_\theta^F\text{-}\lim (\Delta_m^n)X_k = X_0$. The set of all lacunary Δ_m^n -statistically convergent sequence is denoted by $S_\theta^F(\Delta_m^n)$. In the special case $\theta = (2^r)$, we denote by $S^F(\Delta_m^n)$.

Definition. Let $\theta = (k_r)_{r=0}^\infty$ be a lacunary sequence, $X = (X_k)$ of fuzzy numbers and $p = (p_k)$ be any sequence of positive real numbers. Then the sequence X is said to be lacunary $\Delta_m^n(p)$ -convergent if there is a fuzzy number X_0 such that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_k, X_0)]^{p_k} = 0.$$

We write $X_k \rightarrow X_0(N_\theta^F(\Delta_m^n, (p)))$ or $N_\theta^F(p)\text{-}\lim (\Delta_m^n)X_k = X_0$. We shall use $N_\theta^F(\Delta_m^n, (p))$ to denote the set of all lacunary strongly $\Delta_m^n(p)$ -convergent sequences of fuzzy numbers.

When $\theta = (2^r)$, we denote $(N_\theta^F(\Delta_m^n, (p)))$ by $|\sigma^F \Delta_m^n(p)|$ and for $p_k = p$, for all $k \in N$, We denote $(N_\theta^F(\Delta_m^n, (p)))$ by $N_\theta^F(\Delta_m^n, p)$.

3. Main results

Theorem 1. *The classes of sequences $\ell_\infty^F(\Delta_m^n, (p))$, $c^F(\Delta_m^n, (p))$, $N_\theta^F(\Delta_m^n, (p))$ and $S_\theta^F(\Delta_m^n, (p))$ are closed under the operations of addition and scalar multiplication.*

Proof. (i) We shall prove the result only for the case $N_\theta^F(\Delta_m^n, (p))$. The other cases can be proved similarly.

Let $X \in N_\theta^F(\Delta_m^n, (p))$ and $c \in R$. Then

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(c\Delta_m^n X_k, cX_0)]^{p_k} &= \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [|c| \bar{d}(\Delta_m^n X_k, X_0)]^{p_k} \\ &\leq \{\max\{1, |c|^H\}\} \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_k, X_0)]^{p_k} \\ &= 0. \end{aligned}$$

Thus $cX \in N_\theta^F(\Delta_m^n, (p))$.

(ii) Let $X, Y \in N_\theta^F(\Delta_m^n, (p))$. Then

$$\begin{aligned} &\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_k \oplus \Delta_m^n Y_k, X_0 \oplus Y_0)]^{p_k} \\ &\leq K \left[\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_k, X_0)]^{p_k} + \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n Y_k, Y_0)]^{p_k} \right] \\ &= 0. \end{aligned}$$

Therefore $X \oplus Y \in N_\theta^F(\Delta_m^n, (p))$ □

Theorem 2. *If $X = (X_k) \in N_\theta^F(\Delta_m^n) \cap |\sigma^F(\Delta_m^n)|$, then $N_\theta^F\text{-}\lim \Delta_m^n X_k = |\sigma^F|\text{-}\lim \Delta_m^n X_k$.*

Proof. Let $|\sigma^F|\text{-}\lim \Delta_m^n X_k = X_0$ and $N_\theta^F\text{-}\lim \Delta_m^n X_k = X'_0$. Also suppose that $X_0 \neq X'_0$.

Then

$$\begin{aligned} \tau_r + \sigma_r &= \frac{1}{h_r} \sum_{i \in I_r} \bar{d}(\Delta_m^n X_i, X_0) + \frac{1}{h_r} \sum_{i \in I_r} \bar{d}(\Delta_m^n X_i, X'_0) \\ &\geq \frac{1}{h_r} \sum_{I_r} \bar{d}(X_0, X'_0) \\ &= \bar{d}(X_0, X'_0). \end{aligned}$$

Since $X \in N_\theta^F(\Delta_m^n)$, we have $\tau_r \rightarrow 0$ as $r \rightarrow \infty$. Thus we have for sufficiently large r , $\tau_r > \frac{1}{2} \bar{d}(X_0, X'_0)$.

Therefore

$$\begin{aligned} \frac{1}{k_r} \sum_{i=1}^{k_r} \bar{d}(\Delta_m^n X_i, X_0) &\geq \frac{1}{k_r} \sum_{i \in I_r} \bar{d}(\Delta_m^n X_i, X_0) \\ &= \frac{h_r}{k_r} \tau_r \\ &= \left(1 - \frac{1}{q_r}\right) \tau_r \\ &\geq \frac{1}{2} \left(1 - \frac{1}{q_r}\right) \bar{d}(X_0, X'_0). \end{aligned}$$

For sufficiently large r , since $X = (X_i) \in |\sigma^F(\Delta_m^n)|$, the left hand side of the above inequality converges to 0, so we have $q_r \rightarrow 1$, i.e. $\sup q_r < \infty$. Thus if, $\sup q_r < \infty$, then $N_\theta^F(\Delta_m^n) \subset |\sigma^F(\Delta_m^n)|$.

Thus we have $N_\theta^F\text{-}\lim \Delta_m^n X_k = X'_0$ imply $|\sigma^F|\text{-}\lim \Delta_m^n X_k = X'_0$.

Which gives

$$\frac{1}{t} \sum_{i=1}^t \bar{d}(\Delta_m^n X_i, X'_0) + \frac{1}{t} \sum_{i=1}^t \bar{d}(\Delta_m^n X_i, X_0) \geq \bar{d}(X_0, X'_0) > 0.$$

Which yields a contradiction as both the summation of left hand side converge to 0. Therefore $X_0 = X'_0$. □

Theorem 3. *The class of sequence $N_\theta^F(\Delta_m^n, (p))$ is not solid in general.*

Proof. The result follows from the following example. □

Example 1. Let $n = 1, m = 2, p_k = 1$, for all k odd and $p_k = 2$, for k even. Consider the sequence (X_k) defined by

$$X_k = \bar{1}, \text{ for all } k \in N.$$

Then

$$X_k \rightarrow \bar{0} (N_\theta(\Delta_2, (p))).$$

Now let

$$\alpha_k = (1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, -, -, -, -, -).$$

Then

$$(\alpha_k X_k) \notin N_\theta^F(\Delta_2, (p)).$$

Hence $N_\theta^F(\Delta_2, (p))$ is not solid.

Example 2. Let $m = 1, n = 1, \theta = (2^r)$ and $p_k = 1$ for all $k \in N$. Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} kt + 1, & \text{for } -\frac{1}{k} \leq t \leq 0, \\ 1 - kt, & \text{for } 0 \leq t \leq \frac{1}{k}, \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\Delta X_k(t) = \begin{cases} \frac{tk^2 + tk + 2k + 1}{2k + 1}, & \text{for } -\frac{1}{k} - \frac{1}{k + 1} \leq t \leq 0, \\ \frac{2k + 1 - tk - tk^2}{2k + 1}, & \text{for } 0 \leq t \leq \frac{1}{k} + \frac{1}{k + 1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly $(X_k) \in N_\theta^F(\Delta, (p))$.

Consider the sequence (β_n) defined by $\beta_k = 1$, for all k odd and $\beta_k = 0$, otherwise.

We have $(\beta_k X_k) = \{X_1, \bar{0}, X_3, \bar{0}, X_5, \bar{0}, -, -, -\} \notin N_\theta^F(\Delta, (p))$. Hence $N_\theta^F(\Delta, (p))$ is not solid. □

Theorem 4. $N_\theta^F(\Delta_m^n, (p))$ is not symmetric in general.

Proof. The result follows from the following example. □

Example 3. Let $m = 2, n = 1$ and $p_k = 3$, for all k odd and $p_k = 4$, for all k even. Consider the sequence (X_k) defined by

$$(X_k) = (\bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{1}, \bar{0}, \dots).$$

Then $(X_k) \rightarrow \bar{0}(N_\theta^F(\Delta_m^n, (p)))$.

Let (Y_k) be a re-arrangement of (X_k) defined by

$$(Y_k) = (\bar{1}, \bar{1}, \bar{0}, \bar{0}, \bar{1}, \bar{1}, \bar{0}, \bar{0}, \bar{1}, \bar{1}, \dots).$$

Then clearly $(Y_k) \notin N_\theta^F(\Delta_m^n, (p))$. Hence $N_\theta^F(\Delta_m^n, (p))$ is not symmetric.

Theorem 5. *The class of sequence $N_\theta^F(\Delta_m^n, (p))$ is not convergence free in general.*

Proof. The result follows from the following example. □

Example 4. Let $m = 1, n = 1, \theta = (2^r)$ and $p_k = 1$ for all $k \in N$.

Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} kt + 1, & \text{for } -\frac{1}{k} \leq t \leq 0, \\ 1 - kt, & \text{for } 0 \leq t \leq \frac{1}{k}, \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\Delta X_k(t) = \begin{cases} \frac{tk^2 + tk + 2k + 1}{2k + 1}, & \text{for } -\frac{1}{k} - \frac{1}{k + 1} \leq t \leq 0, \\ \frac{2k + 1 - tk - tk^2}{2k + 1}, & \text{for } 0 \leq t \leq \frac{1}{k} + \frac{1}{k + 1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly $(X_k) \in N_\theta^F(\Delta)$.

Now consider

$$Y_k(t) = \begin{cases} \frac{t+k}{k}, & \text{for } -k \leq t \leq 0, \\ \frac{k-t}{k}, & \text{for } 0 \leq t \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\Delta Y_k(t) = \begin{cases} \frac{t+k+1}{k+1}, & \text{for } -k - 1 \leq t \leq 0, \\ \frac{k-t+1}{k+1}, & \text{for } 0 \leq t \leq k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

But $(Y_k) \notin N_\theta^F(\Delta)$. Therefore the class of sequences $N_\theta^F(\Delta, (p))$ is not convergence free in general. □

Theorem 6. *Let $\theta = (k_r)_{r=0}^\infty$ be a lacunary sequence and let $X = (X_k)$ be any sequence of fuzzy numbers. Then*

- (i) *For $\lim\text{-inf } q_r > 1$, we have $|\sigma^F(\Delta_m^n, p)| \subseteq N_\theta^F(\Delta_m^n, (p))$.*
- (ii) *For $\lim\text{-sup } q_r < \infty$, we have $N_\theta^F(\Delta_m^n, (p)) \subseteq |\sigma^F(\Delta_m^n, p)|$.*

Proof. (i) Let $X \in |\sigma^F(\Delta_m^n, p)|$ and $\lim\text{-inf } q_r > 1$, then there exist $\delta > 0$ such that

$q_r \geq 1 + \delta$ for all $r \geq 1$. Then for $X \in |\sigma^F(\Delta_m^n, (p))|$, we write

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \\ &= \frac{1}{h_r} \sum_{k=1}^{k_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} - \frac{1}{h_r} \sum_{k=1}^{k_{r-1}} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \\ &= \frac{k_r}{h_r} \left(\frac{1}{k_r} \sum_{k=1}^{k_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \right) - \frac{k_{r-1}}{h_r} \left(\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \right). \end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$.

Then the terms $\frac{1}{k_r} \sum_{k=1}^{k_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k}$ and $\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k}$ converge to 0 as $r \rightarrow \infty$ uniformly in i . Hence $X \in N_\theta^F(\Delta_m^n, (p))$.

(ii) Now suppose that $\lim\text{-sup } q_r < \infty$, then there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $X \in N_\theta^F(\Delta, (p))$ and $\varepsilon > 0$ be given, we can find $n_0 > 0$ and $K > 0$ such that

$$\frac{1}{h_j} \sum_{k \in I_j} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} < \varepsilon, \text{ for every } j \geq n_0 \text{ and all } i$$

and

$$\frac{1}{h_j} \sum_{k \in I_j} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} < K, \text{ for all } j = 1, 2, 3, \dots$$

Now let s be any integer with $k_{r-1} < s < k_r$, where $r > n_0$. Then

$$\begin{aligned} & \frac{1}{s} \sum_{k=1}^s [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \\ & \leq \frac{1}{k_{r-1}} \sum_{k \in I_1} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} + \frac{1}{k_{r-1}} \sum_{k \in I_2} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} + \dots \\ & \quad + \frac{1}{k_{r-1}} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \\ & \leq \frac{k_1}{k_{r-1}} \frac{1}{k_1} \sum_{k \in I_1} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} + \frac{k_2 - k_1}{k_{r-1}} \frac{1}{k_2 - k_1} \sum_{k \in I_2} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} + \dots \\ & \quad + \frac{k_{n_0} - k_{n_0-1}}{k_{r-1}} \frac{1}{k_{n_0} - k_{n_0-1}} \sum_{k \in I_{n_0}} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} + \dots \\ & \quad + \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{k_r - k_{r-1}} \sum_{k \in I_r} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \end{aligned}$$

$$\begin{aligned} &\leq \sup_{j \geq 1} \frac{1}{h_j} \sum_{k \in I_j} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \frac{k_{n_0}}{k_{r-1}} + \sup_{j \geq n_0} \frac{1}{h_j} \sum_{k \in I_j} [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \frac{k_r - k_{n_0}}{k_{r-1}} \\ &\leq K \frac{k_{n_0}}{k_{r-1}} + \varepsilon M. \end{aligned}$$

Since $k_{r-1} \rightarrow \infty$ as $n \rightarrow \infty$, it follows that

$$\frac{1}{n} \sum_{k=1}^n [\bar{d}(\Delta_m^n X_{k+i}, X_0)]^{p_k} \rightarrow 0, \text{ uniformly in } i.$$

Hence $X \in |\sigma^F(\Delta_m^n, (p))|$. □

4. Conclusions

We have introduced the concept of lacunary statistical convergence and lacunary strongly convergence of fuzzy real numbers. We have shown that the spaces are closed under addition and scalar multiplication and are complete metric spaces. We have verified the solidness, symmetricity, convergence free properties of these classes of sequences. It is shown that N_θ^F -lim $\Delta_m^n X_k = |\sigma|$ -lim $\Delta_m^n X_k$. We have provided examples for the results those fail to hold. The result proved in this article generalizes and unifies several existing results.

References

- [1] Y. Altin, M. Et and R. Colak, *Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers*, Computer Math. Appl., **52**(6-7)(2006), 1011-1020.
- [2] T. Bligin, *Lacunary strongly Δ -convergent sequences of fuzzy numbers*, Information Sciences, **160**(1-4)(2004), 201-206.
- [3] R. Colak and M. Et, *On some generalized difference sequence spaces*, Soochow J. Math., **21**(4)(1995), 377-386.
- [4] A. Esi, B. C. Tripathy and B. Sarma, *On some new type generalized difference sequence spaces*, Math. Slovaca, **57**(5)(2007), 475-482.
- [5] M. Et, Y. Altin, B. Choudhary and B. C. Tripathy, *On some classes of sequences defined by sequences of Orlicz functions*, Math. Ineq. Appl., **9**(2)(2006), 335-342.
- [6] H. Fast, *Sur la convergence statistique*, Colloquium Math., **2**(1951), 241-244.
- [7] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., **24**(2)(1981), 168-176.
- [8] D. Rath and B.C. Tripathy, *Matrix maps on sequence spaces associated with sets of integers*, Indian Jour. Pure Appl. Math., **27**(2)(1996), 197-206.
- [9] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**(1959), 361-375.

- [10] B. C. Tripathy, *Matrix transformations between some classes of sequences*, Jour. Math. Anal. Appl., **206**(3)(1997), 448-450.
- [11] B. C. Tripathy, *On generalized difference paranormed statistically convergent sequences*, Indian Jour. Pure Appl. Math., **35**(5)(2004), 655-663.
- [12] B. C. Tripathy, Y. Altin and M. Et, *Generalized difference sequence spaces on seminormed spaces defined by Orlicz functions*, Math. Slovaca, **58**(3)(2008), 315-324.
- [13] B. C. Tripathy and A. Baruah, *New type of difference sequence spaces of fuzzy real numbers*, Mathematical Modelling and Analysis, **14**(3)(2009), 391-397.
- [14] B. C. Tripathy and A. Baruah, *Nörlund and Riesz mean of sequences of fuzzy real numbers*, Appl. Math. Letters, **23**(2010), 651-655.
- [15] B. C. Tripathy and S. Borgogain, *The sequence space $m(M, \Phi, \Delta_m^n, p)^F$* ; Mathematical Modelling and Analysis, **13**(4)(2008), 577-586.
- [16] B. C. Tripathy and A. J. Dutta, *On fuzzy real-valued double sequence spaces ${}_2\ell_F^p$* , Mathematical and Computer Modelling, **46**(9-10)(2007), 1294-1299.
- [17] B. C. Tripathy and A. J. Dutta, *Bounded variation double sequence space of fuzzy real numbers*, Computers Math. Applications, **59**(2)(2010), 1031-1037.
- [18] B. C. Tripathy, A. Esi, *A new type of difference sequence spaces*, International Jour. Sci. Tech., **1**(1)(2006), 11-14.
- [19] B. C. Tripathy, A. Esi and B. K. Tripathy, *on some new type of generalized difference Cesro sequence spaces*, Soochow J. Math., **31**(3)(2005), 333-340.
- [20] B. C. Tripathy and S. Mahanta, *On a class of generalized lacunary difference sequence spaces defined by Orlicz function*, Acta Math. Appl. Sinica, **20**(2)(2004), 231-238.
- [21] B. C. Tripathy and B. Sarma, *Sequence spaces of fuzzy real numbers defined by Orlicz functions*, Math. Slovaca, **58**(5)(2008), 621-628.
- [22] B. C. Tripathy and B. Sarma, *Statistically convergent difference double sequence spaces*, Acta Mathematica Sinica, **24**(5)(2008), 737-742.
- [23] B. C. Tripathy and M. Sen, *On generalized statistically convergent sequences*, Indian Jour. Pure Appl. Math., **32**(11)(2001), 1689-694.
- [24] B. C. Tripathy and M. Sen, *Characterization of some matrix classes involving paranormed sequence spaces*, Tamkang Jour. Math., **37**(2)(2006), 155-162.