

## A Fixed Point Approach to the Stability of a Functional Equation

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ABSTRACT. By using an idea of Cădariu and Radu [4], we prove the generalized Hyers-Ulam stability of the functional equation

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x, z) + 2f(y, w).$$

The quadratic form  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x, y) = ax^2 + by^2$  is a solution of the above functional equation.

### 1. Introduction

In 1940, S. M. Ulam [13] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms : *Let  $G_1$  be a group and let  $G_2$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if a function  $h : G_1 \rightarrow G_2$  satisfies the inequality*

$$d(h(xy), h(x)h(y)) < \delta$$

*for all  $x, y \in G_1$  then there is a homomorphism  $H : G_1 \rightarrow G_2$  with*

$$d(h(x), H(x)) < \varepsilon$$

*for all  $x \in G_1$ .*

The case of approximately additive mappings was solved by D. H. Hyers [6] under the assumption that  $G_1$  and  $G_2$  are Banach spaces. Thereafter, many authors investigated solutions or stability of various functional equations (see [1], [2], [5], [8], [9]–[12]).

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Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty]$  is called a generalized metric on  $X$  if  $d$  satisfies

- (1)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Throughout this paper, let  $X$  and  $Y$  be two real vector spaces and let  $\varphi : X \times X \times X \times X \rightarrow [0, \infty)$  be a function. For a mapping  $f : X \times X \rightarrow Y$ , consider the functional equation:

$$(1.1) \quad f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w).$$

The quadratic form  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x, y) := ax^2 + by^2$  is a solution of the functional equation (1.1).

In 2007, the authors [10] acquired the general solution and proved the stability of the functional equation (1.1) for the case that  $X$  and  $Y$  are real vector spaces.

**Theorem A[10].** *A mapping  $f : X \times X \rightarrow Y$  satisfies (1.1) for all  $x, y, z, w \in X$  if and only if there exist two symmetric bi-additive mappings  $S, T : X \times X \rightarrow Y$  such that*

$$f(x, y) = S(x, x) + T(y, y)$$

for all  $x, y \in X$ .

From now on, let  $Y$  be complete.

**Theorem B[10].** *Assume that  $\varphi$  satisfies the condition*

$$\begin{aligned} & \tilde{\varphi}(x, y, z, w) \\ & := \sum_{j=0}^{\infty} \left( \frac{1}{4^{j+1}} \varphi(2^j x, 2^j y, 2^j z, -2^j w) \right. \\ & \quad + \frac{1}{2 \cdot 4^{j+1}} [\varphi(2^j x, 2^j y, 2^j z, 2^j w) + \varphi(2^j x, 2^j y, -2^j z, -2^j w) \\ & \quad \left. + \varphi(0, 0, 2^j z, -2^j w) + \varphi(0, 0, -2^j z, 2^j w)] \right) < \infty \end{aligned}$$

for all  $x, y, z, w \in X$ . Let  $f : X \times X \rightarrow Y$  be a mapping such that

$$\|f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) - 2f(y, w)\| \leq \varphi(x, y, z, w)$$

for all  $x, y, z, w \in X$ . Then there exists a unique mapping  $F : X \times X \rightarrow Y$  satisfying (1.1) such that

$$(1.2) \quad \|f(x, y) - F(x, y)\| \leq \tilde{\varphi}(x, x, y, y) + \frac{1}{3} \|f(0, 0)\|$$

for all  $x, y \in X$ . The mapping  $F$  is given by  $F(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f(2^j x, 2^j y)$  for all  $x, y \in X$ .

In this paper, we prove the stability of the functional equation (1.1) using the fixed point method.

**2. Stability using the alternative of fixed point**

In this section, we investigate the stability of the functional equation (1.1) using the alternative of fixed point. Before proceeding the proof, we will state the theorem which is the alternative of fixed point.

**Theorem 2.1.** (The alternative of fixed point [7]). Suppose that we are given a complete generalized metric space  $(\Omega, d)$  and a strictly contractive mapping  $T : \Omega \rightarrow \Omega$  with Lipschitz constant  $L$ . Then, for each given  $x \in \Omega$ , either

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or

there exists a positive integer  $n_0$  such that

- $d(T^n x, T^{n+1} x) < \infty$  for all  $n \geq n_0$ ;
- the sequence  $(T^n x)$  is convergent to a fixed point  $y^*$  of  $T$ ;
- $y^*$  is the unique fixed point of  $T$  in the set  $\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\}$ ;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$  for all  $y \in \Delta$ .

From now on, let  $\Omega$  be the set of all the mappings  $g : X \times X \rightarrow Y$  satisfying  $g(0, 0) = 0$ .

**Lemma 2.2.** Let  $\psi : X \times X \rightarrow [0, \infty)$  be a function. Consider the generalized metric  $d$  on  $\Omega$  given by

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h),$$

where  $S_\psi(g, h) := \{K \in [0, \infty] \mid \|g(x, y) - h(x, y)\| \leq K\psi(x, y) \text{ for all } x, y \in X\}$  for all  $g, h \in \Omega$ . Then  $(\Omega, d)$  is complete.

*Proof.* Let  $\{g_n\}$  be a Cauchy sequence in  $(\Omega, d)$ . Then, given  $\varepsilon > 0$ , there exists  $N$  such that  $d(g_n, g_k) < \varepsilon$  if  $n, k \geq N$ . Let  $n, k \geq N$ . Since  $d(g_n, g_k) = \inf S_\psi(g_n, g_k) < \varepsilon$ , there exists  $K \in [0, \varepsilon)$  such that

$$(2.1) \quad \|g_n(x, y) - g_k(x, y)\| \leq K\psi(x, y) \leq \varepsilon\psi(x, y)$$

for all  $x, y \in X$ . So, for each  $x, y \in X$ ,  $\{g_n(x, y)\}$  is a Cauchy sequence in  $Y$ . Since  $Y$  is complete, for each  $x, y \in X$ , there exists  $g(x, y) \in Y$  such that  $g_n(x, y) \rightarrow g(x, y)$  as  $n \rightarrow \infty$ . So  $g(0, 0) = \lim_{n \rightarrow \infty} g_n(0, 0) = 0$ . Thus we have  $g \in \Omega$ . Taking the limit as  $k \rightarrow \infty$  in (2.1), we obtain that

$$\begin{aligned} n \geq N &\implies \|g_n(x, y) - g(x, y)\| \leq \varepsilon\psi(x, y) \text{ for all } x, y \in X \\ &\implies \varepsilon \in S_\psi(g_n, g) \\ &\implies d(g_n, g) = \inf S_\psi(g_n, g) \leq \varepsilon. \end{aligned}$$

Hence  $g_n \rightarrow g \in \Omega$  as  $n \rightarrow \infty$ . □

By using an idea of Cădariu and Radu (see [4]), we will prove the generalized Hyers-Ulam stability of the functional equation related to quadratic forms.

**Theorem 2.3.** *Let  $L \in (0, 1)$  and  $\varphi$  satisfy*

$$(2.2) \quad \varphi(x, y, z, w) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right)$$

for all  $x, y, z, w \in X$ . Suppose that a mapping  $f : X \times X \rightarrow Y$  satisfying  $f(0, 0) = 0$  and the functional inequality

$$(2.3) \quad \|f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) - 2f(y, w)\| \leq \varphi(x, y, z, w)$$

for all  $x, y, z, w \in X$ . Then there exists a unique mapping  $F : X \times X \rightarrow Y$  satisfying (1.1) such that

$$(2.4) \quad \|f(x, y) - F(x, y)\| \leq \frac{L}{1-L}\psi(x, y),$$

where  $\psi : X \times X \rightarrow [0, \infty)$  is a function given by

$$\begin{aligned} \psi(x, y) := & \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, -\frac{y}{2}\right) \\ & + \frac{1}{2}\left[\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{2}, -\frac{y}{2}\right) + \varphi\left(0, 0, \frac{y}{2}, -\frac{y}{2}\right) + \varphi\left(0, 0, -\frac{y}{2}, \frac{y}{2}\right)\right] \end{aligned}$$

for all  $x, y \in X$ .

*Proof.* By the proof of Theorem 4 in [10], we have the inequality

$$\begin{aligned} & \left\|f(x, y) - \frac{1}{4}f(2x, 2y)\right\| \\ & \leq \frac{1}{4}\varphi(x, x, y, -y) + \frac{1}{8}\left[\varphi(x, x, y, y) + \varphi(x, x, -y, -y) + \varphi(0, 0, y, -y) + \varphi(0, 0, -y, y)\right] \end{aligned}$$

for all  $x, y \in X$ . By (2.2), we get

$$(2.5) \quad \left\|f(x, y) - \frac{1}{4}f(2x, 2y)\right\| \leq \frac{1}{4}\psi(2x, 2y) \leq L\psi(x, y)$$

for all  $x, y \in X$ . Consider the generalized metric  $d$  on  $\Omega$  given by

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h)$$

for all  $g, h \in \Omega$ . Then we obtain

$$(2.6) \quad d(f, Tf) \leq L < \infty.$$

By Lemma 2.2, the generalized metric space  $(\Omega, d)$  is complete. Now we define a mapping  $T : \Omega \rightarrow \Omega$  by

$$Tg(x, y) := \frac{1}{4}g(2x, 2y)$$

for all  $g \in \Omega$  and all  $x, y \in X$ . Observe that, for all  $g, h \in \Omega$ ,

$$\begin{aligned} K' \in S_\psi(g, h) \text{ and } K' < K \\ \implies \|g(x, y) - h(x, y)\| &\leq K'\psi(x, y) \leq K\psi(x, y) \text{ for all } x, y \in X \\ \implies K \in S_\psi(g, h). \end{aligned}$$

Let  $g, h \in \Omega$ ,  $K \in [0, \infty]$  and  $d(g, h) < K$ . Then there is a  $K' \in S_\psi(g, h)$  such that  $K' < K$ . By the above observation, we gain  $K \in S_\psi(g, h)$ . So we get  $\|g(x, y) - h(x, y)\| \leq K\psi(x, y)$  for all  $x, y \in X$ . Thus we have

$$\left\| \frac{1}{4}g(2x, 2y) - \frac{1}{4}h(2x, 2y) \right\| \leq \frac{1}{4}K\psi(2x, 2y)$$

for all  $x, y \in X$ . By (2.2), we obtain that

$$\left\| \frac{1}{4}g(2x, 2y) - \frac{1}{4}h(2x, 2y) \right\| \leq LK\psi(x, y)$$

for all  $x, y \in X$ . Hence  $d(Tg, Th) \leq LK$ . Therefore we obtain that

$$d(Tg, Th) \leq Ld(g, h)$$

for all  $g, h \in \Omega$ , that is,  $T$  is a strictly contractive mapping of  $\Omega$  with Lipschitz constant  $L$ . Applying the alternative of fixed point, we see that there exists a fixed point  $F$  of  $T$  in  $\Omega$  such that

$$F(x, y) = \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all  $x, y \in X$ . Replacing  $x, y, z, w$  by  $2^n x, 2^n y, 2^n z, 2^n w$  in (2.3), respectively, and dividing by  $4^n$ , we have

$$\begin{aligned} &\|F(x + y, z - w) + F(x - y, z + w) - 2F(x, z) - 2F(y, w)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(2^n(x + y), 2^n(z - w)) + f(2^n(x - y), 2^n(z + w)) - 2f(2^n x, 2^n z) - 2f(2^n y, 2^n w)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z, 2^n w) \end{aligned}$$

for all  $x, y, z, w \in X$ . By (2.2), the mapping  $F$  satisfies (1.1). By (2.2) and (2.5), we obtain that

$$\begin{aligned} \|T^n f(x, y) - T^{n+1} f(x, y)\| &= \frac{1}{4^n} \left\| f(2^n x, 2^n y) - \frac{1}{4} f(2^{n+1} x, 2^{n+1} y) \right\| \\ &\leq \frac{L}{4^n} \psi(2^n x, 2^n y) \leq \dots \leq \frac{L}{4^n} (4L)^n \psi(x, y) \\ &= L^{n+1} \psi(x, y) \end{aligned}$$

for all  $x, y \in X$  and all  $n \in \mathbb{N}$ , that is,  $d(T^n f, T^{n+1} f) \leq L^{n+1} < \infty$  for all  $n \in \mathbb{N}$ . By the fixed point alternative, there exists a natural number  $n_0$  such that the mapping  $F$  is the unique fixed point of  $T$  in the set  $\Delta = \{g \in \Omega \mid d(T^{n_0} f, g) < \infty\}$ . So we have  $d(T^{n_0} f, F) < \infty$ . Since

$$d(f, T^{n_0} f) \leq d(f, Tf) + d(Tf, T^2 f) + \cdots + d(T^{n_0-1} f, T^{n_0} f) < \infty,$$

we get  $f \in \Delta$ . Thus we have  $d(f, F) \leq d(f, T^{n_0} f) + d(T^{n_0} f, F) < \infty$ . Hence we obtain

$$\|f(x, y) - F(x, y)\| \leq K\psi(x, y)$$

for all  $x, y \in X$  and some  $K \in [0, \infty)$ . Again using the fixed point alternative, we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Tf).$$

By (2.6), we may conclude that

$$d(f, F) \leq \frac{L}{1-L},$$

which implies the inequality (2.4).  $\square$

**Theorem 2.4.**  $L \in (0, 1)$  and  $\varphi$  satisfy

$$(2.7) \quad \varphi(x, y, z, w) \leq \frac{L}{4} \varphi(2x, 2y, 2z, 2w)$$

for all  $x, y, z, w \in X$ . Suppose that a mapping  $f : X \times X \rightarrow Y$  satisfying  $f(0, 0) = 0$  and the functional inequality (2.3). Then there exists a unique mapping  $F : X \times X \rightarrow Y$  satisfying (1.1) such that the inequality

$$(2.8) \quad \|f(x, y) - F(x, y)\| \leq \frac{1}{1-L} \psi(x, y),$$

where  $\psi : X \times X \rightarrow [0, \infty)$  is a function given by

$$\begin{aligned} \psi(x, y) := & \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, -\frac{y}{2}\right) \\ & + \frac{1}{2} \left[ \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{2}, -\frac{y}{2}\right) + \varphi\left(0, 0, \frac{y}{2}, -\frac{y}{2}\right) + \varphi\left(0, 0, -\frac{y}{2}, \frac{y}{2}\right) \right] \end{aligned}$$

for all  $x, y \in X$ .

*Proof.* By the proof of Theorem 4 in [10], we have the inequality

$$\begin{aligned} & \left\| f(x, y) - \frac{1}{4} f(2x, 2y) \right\| \\ & \leq \frac{1}{4} \varphi(x, x, y, -y) \\ & \quad + \frac{1}{8} [\varphi(x, x, y, y) + \varphi(x, x, -y, -y) + \varphi(0, 0, y, -y) + \varphi(0, 0, -y, y)] \end{aligned}$$

for all  $x, y \in X$ . So we get

$$\left\| f(x, y) - 4f\left(\frac{x}{2}, \frac{y}{2}\right) \right\| \leq \psi(x, y)$$

for all  $x, y \in X$ . Consider the generalized metric  $d$  on  $\Omega$  given by

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h)$$

for all  $g, h \in \Omega$ . Then we obtain

$$(2.9) \quad d(f, Tf) \leq 1 < \infty.$$

By Lemma 2.2, the generalized metric space  $(\Omega, d)$  is complete. Now we define a mapping  $T : \Omega \rightarrow \Omega$  by

$$Tg(x, y) := 4g\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all  $g \in \Omega$  and all  $x, y \in X$ . By the same argument of the proof of Theorem 2.3,  $T$  is a strictly contractive mapping of  $\Omega$  with Lipschitz constant  $L$ . Applying the alternative of fixed point, we see that there exists a fixed point  $F$  of  $T$  in  $\Omega$  such that

$$F(x, y) = \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all  $x, y \in X$ . Replacing  $x, y, z, w$  by  $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}$  in (2.3), respectively, and multiplying by  $4^n$ , we have

$$\begin{aligned} & \|F(x + y, z - w) + F(x - y, z + w) - 2F(x, z) - 2F(y, w)\| \\ &= \lim_{n \rightarrow \infty} 4^n \left\| f\left(\frac{x + y}{2^n}, \frac{z - w}{2^n}\right) + f\left(\frac{x - y}{2^n}, \frac{z + w}{2^n}\right) - 2f\left(\frac{x}{2^n}, \frac{z}{2^n}\right) - 2f\left(\frac{y}{2^n}, \frac{w}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) \end{aligned}$$

for all  $x, y, z, w \in X$ . By (2.7), the mapping  $F$  satisfies (1.1). By (2.7), we obtain that

$$\begin{aligned} \|T^n f(x, y) - T^{n+1} f(x, y)\| &= 4^n \left\| f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \right\| \\ &\leq 4^{n-1} L\psi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \leq 4^{n-2} L^2\psi\left(\frac{x}{2^{n-2}}, \frac{y}{2^{n-2}}\right) \\ &\leq \dots \leq L^n \psi(x, y) \end{aligned}$$

for all  $x, y \in X$  and all  $n \in \mathbb{N}$ , that is,  $d(T^n f, T^{n+1} f) \leq L^n < \infty$  for all  $n \in \mathbb{N}$ . By the same reasoning of the proof of Theorem 2.3, we have

$$d(f, F) \leq \frac{1}{1 - L} d(f, Tf).$$

By (2.9), we may conclude that

$$d(f, F) \leq \frac{1}{1 - L},$$

which implies the inequality (2.8).  $\square$

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