KYUNGPOOK Math. J. 50(2010), 545-556

On Approximation of Functions Belonging to $Lip\,(\alpha, r)$ Class and to Weighted $W\,(L_r, \xi\,(t))$ Class by Product Means

Hare Krishna Nigam* and Ajay Sharma
Department of Mathematics, Faculty of Engineering and Technology, Mody Institute of Technology and Science (Deemed University), Laxmangarh-332311, Sikar (Rajasthan), India
e-mail : harekrishnan@yahoo.com *and* ajaymathematicsanand@gmail.com

Abstract. A good amount of work has been done on degree of approximation of functions belonging to $Lip\alpha$, $Lip\,(\alpha, r)\,, Lip\,(\xi\,(t)\,, r)$ and $W\,(L_r, \xi\,(t))$ classes using Cesàro, Nörlund and generalised Nörlund single summability methods by a number of researchers ([1], [10], [8], [6], [7], [2], [3], [4], [9]). But till now, nothing seems to have been done so far to obtain the degree of approximation of functions using $(N, p_n)(C, 1)$ product summability method. Therefore the purpose of present paper is to establish two quite new theorems on degree of approximation of function $f \in Lip\,(\alpha, r)$ class and $f \in W\,(L_r, \xi\,(t))$ class by $(N, p_n)(C, 1)$ product summability means of its Fourier series.

1. Introduction

Let f be 2π-periodic function and Lebesgue integrable. The Fourier series associated with f at a point x is defined by

$$f\,(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n^{th} partial sum $s_n(f; x)$.

L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f\,(x)|^r dx \right)^{\frac{1}{r}}, \; r \geq 1 \tag{1.2}$$

$L_\infty-$ norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup \{\, |f\,(x)\,| : x \in R\} \tag{1.3}$$

---

* Corresponding Author.

Received January 21, 2010; revised September 3,2010; accepted October 28, 2010.
2000 Mathematics Subject Classification: 42B05, 42B08.
Key words and phrases: Degree of approximation, $Lip\,(\alpha, r)$ class, $W\,(L_r, \xi\,(t))$ class of functions, (N, p_n) mean, (C,1) mean, $(N, p_n)(C, 1)$ product means, Fourier series, Lebesgue integral.

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\|\|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{\,|t_n - f(x)| : x \in R\}\,(\text{Zygmund}[12])$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r \tag{1.4}$$

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$|f(x+t) - f(x)| = O(\,|\,t\,|^\alpha)\ for\, 0 < \alpha < 1 \tag{1.5}$$

$f(x) \in Lip(\alpha, r)\ \ for\ 0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O|\,t\,|^\alpha,\ 0 < \alpha \le 1,\ r \ge 1 \tag{1.6}$$

(definition 5.38 of Mc Fadden[5]).

Given a positive increasing function $\xi(t)$ and an integer $r \ge 1$, $f \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{1.7}$$

and that $f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} \left|\{f(x+t) - f(x)\}\sin^\beta x\right|^r dx\right)^{\frac{1}{r}} = O(\xi(t)),\ \beta \ge 0. \tag{1.8}$$

In case $\beta = 0$, we find that $W(L_r, \xi(t))$ class reduces to the $Lip(\xi(t), r)$ class and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the $Lip(\alpha, r)$ class and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the $Lip\alpha$ class.

We observe that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \subseteq W(L_r, \xi(t))\, for\ 0 < \alpha \le 1,\ r \ge 1.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$.

The (C,1) transform is defined as the n^{th} partial sum of (C,1) summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + ... + s_n}{n+1}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \ \ as \ n \to \infty, \tag{1.9}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C,1) method.

Let $\{p_n\}$ be a non-negative, non increasing sequence such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \quad as \quad n \to \infty, \ P_{-1} = p_{-1} = 0.$$

The product of (N, p_n) summability and (C,1) summability defines $(N, p_n)(C, 1)$ summability and we denote it by $N_n^p C_n^1$.

Thus if

$$N_n^p C_n^1 = \frac{1}{P_n}\sum_{k=0}^{n} p_k C_k^1 \to s \quad as \quad n \to \infty, \tag{1.10}$$

where N_n^p denotes the (N, p_n) transform of s_n and C_n^1 denotes the (C,1) transform of s_n, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N, p_n)(C, 1)$ means or summable $(N, p_n)(C, 1)$ to a definite number s.

The (N, p_n) is a regular method of summability.

$$s_n \to s \Rightarrow C_n^1(s_n) = t_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s, \ as \ n \to \infty \quad C_n^1 \text{ method is regular}$$

$$\Rightarrow N_n^p\left(C_n^1(s_n)\right) = N_n^p C_n^1 \to s, \ as \ n \to \infty \quad N_n^p \text{ method is regular}$$

$$\Rightarrow N_n^p C_n^1 \text{ method is regular.}$$

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$M_n(t) = \frac{1}{2\pi P_n}\sum_{k=0}^{n}\left\{p_k\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}\frac{\sin\left(\nu + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

2. Main Theorems

We prove the following theorems.

Theorem 2.1. *Let* (N, p_n) *be a regular Nörlund method defined by a positive, monotonic, non-increasing sequence* $\{p_n\}$. *Let* f *be a* 2π*-periodic function, Lebesgue integrable on* $[0, 2\pi]$ *and is belonging to* $Lip\,(\alpha, r)$ *class,* $r \geq 1$, *then the degree of approximation of* f *by* $N_n^p C_n^1$ *means of its Fourier series* (1.1) *is given by*

$$\left\|N_n^p C_n^1 - f\right\|_r = O\left[\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right] for\, 0 < \alpha \leq 1,$$

where $N_n^p C_n^1$ *is the* $(N, p_n)(C, 1)$ *means of series* (1.1), $\frac{1}{r} + \frac{1}{s} = 1$ *such that* $1 \leq r \leq \infty$.

Theorem 2.2. *Let* (N, p_n) *be a regular Nörlund method defined by a positive, monotonic, non-increasing sequence*$\{p_n\}$. *Let* f *be a* 2π*-periodic function, Lebesgue integrable on* $[0, 2\pi]$ *and is belonging to* $W\,(L_r, \xi\,(t))$ *class,* $r \geq 1$, *then the degree of approximation of* f *by* $N_n^p C_n^1$ *means of its Fourier series* (1.1) *is given by*

$$\left\|N_n^p C_n^1 - f\right\|_r = O\left[(n+1)^{\beta + \frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right] \tag{2.1}$$

provided $\xi\,(t)$ *satisfies the following conditions:*

$$\left\{\frac{\xi\,(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{2.2}$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\phi\,(t)|}{\xi\,(t)}\right)^r \sin^{\beta\, r} t\, dt\right\}^{\frac{1}{r}} = O\left\{\frac{1}{(n+1)}\right\} \tag{2.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\phi\,(t)|}{\xi\,(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\}, \tag{2.4}$$

where δ *is an arbitrary number such that* $s\,(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \leq r \leq \infty$, *conditions* (2.3) *and* (2.4)*hold uniformly in* x.

3. Lemmas

For the proof of our theorem, we require following lemmas.

Lemma 3.1. $|M_n(t)| = O(n+1)\, for\ 0 \le t \le \frac{1}{n+1}$.

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$\begin{aligned}
|M_n(t)| &= \frac{1}{2\pi\, P_n} \left| \sum_{k=0}^{n} \left[p_k \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{\sin(\nu+1)\ t}{\sin\frac{t}{2}} \right] \right| \\
&\le \frac{1}{2\pi\, P_n} \left| \sum_{k=0}^{n} \left[p_k \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \right] \right| \\
&\le \frac{1}{2\pi\, P_n} \left| \sum_{k=0}^{n} \left[p_k\,(k+1) \right] \right| \\
&= O\left[\frac{(n+1)}{P_n} \sum_{k=0}^{n} p_k \right] \\
&= O(n+1).
\end{aligned}$$

□

Lemma 3.2. $|M_n(t)| = O\left(\frac{1}{t}\right) for\ \frac{1}{n+1} \le t \le \pi$.

Proof. For $\frac{1}{n+1} \le t \le \pi$, by applying Jordan's lemma $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $\sin nt \le 1$

$$\begin{aligned}
|M_n(t)| &= \frac{1}{2\pi\, P_n} \left| \sum_{k=0}^{n} \left[p_k \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{\sin\left(\nu+\frac{1}{2}\right)\ t}{\sin\frac{t}{2}} \right] \right| \\
&\le \frac{1}{2\pi\, P_n} \left| \sum_{k=0}^{n} \left[p_k \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} \left(\frac{1}{t/\pi} \right) \right] \right| \\
&= \frac{1}{2t\, P_n} \left| \sum_{k=0}^{n} \left[p_k \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} (1) \right] \right| \\
&= \frac{1}{2t\, P_n} \left| \sum_{k=0}^{n} p_k \right| \\
&= O\left(\frac{1}{t} \right).
\end{aligned}$$

□

Lemma 3.3. (Mc Fadden[5], Lemma 5.40). *If f(x) belongs to* $Lip(\alpha, q)\, on\, [0,\pi]$ *then* $\phi(t)$ *belongs to* $Lip(\alpha, q)\, on\, [0,\pi]$.

Lemma 3.4. *If f(x) belongs to* $Lip(\alpha, r)$ *on* $[0,\pi]$ *then* $\phi(t)$ *belongs to* $Lip(\alpha, r)$ *on* $[0,\pi]$.

Proof. Replacing q by r in above Lemma 3.3, we get Lemma 3.4. □

4. Proof of Theorem 2.1

Following Titchmarsh[11] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (1.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\, \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt.$$

Using (1.9), the (C,1) transform C_n^1 of $s_n(f;x)$ is given by

$$C_n^1 - f(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi} \phi(t) \sum_{k=0}^{n} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt.$$

Now denoting $(N,p_n)(C,1)$ transform of $s_n(f;x)$ by $N_n^p C_n^1$, we write

$$\begin{aligned} N_n^p C_n^1 - f(x) &= \frac{1}{2\pi\, P_n} \sum_{k=0}^{n} \left[p_k \left(\frac{1}{k+1}\right) \int_0^{\pi} \frac{\phi(t)}{\sin\frac{t}{2}} \left\{ \sum_{\nu=0}^{k} \sin\left(\nu+\frac{1}{2}\right) t \right\} dt \right] \\ &= \int_0^{\pi} \phi(t)\ M_n(t)\ dt \\ &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \phi(t)\ M_n(t)\ dt \\ &= I_{1.1} + I_{1.2} (say). \end{aligned} \tag{4.1}$$

We consider,

$$I_{1.1} = \int_0^{\frac{1}{n+1}} |\phi(t)|\ |M_n(t)|\ dt.$$

Using Hölder's inequality and Lemma 3.4,

$$\begin{aligned} |I_{1.1}| &\le \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t\ |\phi(t)|}{t^{\alpha}} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{|M_n(t)|}{t^{1-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \\ &\le \left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{M_n(t)}{t^{1-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \end{aligned}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{(n+1)}{t^{1-\alpha}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by Lemma 3.1}$$

$$= O\left[\int_0^{\frac{1}{n+1}} t^{\alpha\, s - s}\, dt\right]^{\frac{1}{s}}$$

$$= O\left[\left(\frac{1}{n+1}\right)^{\frac{\alpha\, s - s + 1}{s}}\right]$$

$$= O\left[\left(\frac{1}{n+1}\right)^{\alpha - 1 + \frac{1}{s}}\right]$$

$$= O\left[\left(\frac{1}{n+1}\right)^{\alpha - \left(1 - \frac{1}{s}\right)}\right]$$

$$I_{1.1} = O\left[\left(\frac{1}{n}\right)^{\alpha - \frac{1}{r}}\right] \text{ since } \frac{1}{r} + \frac{1}{s} = 1. \tag{4.2}$$

Similarly, as above, we have

$$I_{1.2} \leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\, |\phi(t)|}{t^{\alpha}}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{|M_n(t)|}{t^{-\delta-\alpha}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta} t^{\alpha - \frac{1}{r}}}{t^{\alpha}}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{|M_n(t)|}{t^{-\delta-\alpha}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta} t^{\alpha - \frac{1}{r}}}{t^{\alpha}}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{1}{t^{1-\delta-\alpha}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by Lemma 3.2}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{t^{-\frac{1}{r}-\delta}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} t^{s\alpha + s\delta - s}\, dt\right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} t^{-1-\delta r}\, dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} t^{s\alpha + s\delta - s}\, dt\right]^{\frac{1}{s}}$$

$$= O\left[(n+1)^{\delta}\left\{(n+1)^{-s\alpha-s\delta+s-1}\right\}^{\frac{1}{s}}\right]$$
$$= O\left[(n+1)^{\delta}(n+1)^{-\alpha-\delta+1-\frac{1}{s}}\right]$$
$$= O\left[(n+1)^{-\alpha+\left(1-\frac{1}{s}\right)}\right]$$
$$(4.3)\quad I_{1.2} = O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right].$$

This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

Following the proof of theorem 2.1,

$$N_n^p C_n^1 - f(x) = \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right] \phi(t)\ M_n(t)\ dt$$

$$(5.1)\qquad N_n^p C_n^1 - f(x) = I_{2.1} + I_{2.2} \quad \text{(say)}.$$

We have

$$|\phi(x+t) - \phi(x)| \leq |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|.$$

Hence, by Minkowiski's inequality,

$$\left[\int_0^{2\pi} \left|\{\phi(x+t) - \phi(x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}} \leq \left[\int_0^{2\pi} \left|\{f(u+x+t) - f(u+x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}}$$
$$+ \left[\int_0^{2\pi} \left|\{f(u-x-t) - f(u-x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}} = O\{\xi(t)\}.$$

Then $f \in W(L_r, \xi(t)) \Rightarrow \phi \in W(L_r, \xi(t))$.
We consider

$$|I_{2.1}| \leq \int_0^{\frac{1}{n+1}} |\phi(t)|\ |M_n(t)|\ dt$$

Using Hölder's inequality and the fact that $\phi(t) \in W(L_r, \xi(t))$,

$$|I_{2.1}| \leq \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t\,|\phi(t)|\,\sin^{\beta} t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^{\beta} t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^{\beta} t}\right\}^s dt\right]^{\frac{1}{s}} \text{ by } (2.3).$$

Since $\sin t \geq (2t/\pi)$ and using Lemma 3.1,

$$I_{2.1} = O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{(n+1)\,\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}.$$

Since $\xi(t)$ is a positive increasing function, and using second mean value theorem for integrals,

$$I_{2.1} = O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^{(1+\beta)s}}\right]^{\frac{1}{s}} \text{ for some } \quad 0 < \in < \frac{1}{n+1}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(1+\beta)s+1}}{-(1+\beta)s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\{n+1\}^{1+\beta-\frac{1}{s}}\right]$$

$$I_{2.1} = O\left[\{n+1\}^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right] \text{ since } \frac{1}{r} + \frac{1}{s} = 1. \tag{5.2}$$

Using Hölder's inequality $|\sin t| \leq 1$, $\sin t \geq (2t/\pi)$, conditions (2.2), (2.4), Lemma 3.2 and second mean value theorem for integrals,

$$\begin{aligned}
|I_{2.2}| &\leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\phi(t)|\sin^{\beta}t}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\ |M_n(t)|}{t^{-\delta}\sin^{\beta}t}\right\}^{s} dt\right]^{\frac{1}{s}} \\
&\leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\ |M_n(t)|}{t^{-\delta}\sin^{\beta}t}\right\}^{s} dt\right]^{\frac{1}{s}} \\
&= O\left\{\ (n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\ |M_n(t)|}{t^{-\delta}\sin^{\beta}t}\right\}^{s} dt\right]^{\frac{1}{s}}
\end{aligned}$$

$$I_{2.2} = O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}}.$$

Putting $t = \frac{1}{y}$

$$\begin{aligned}
I_{2.2} &= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}}\right\}^{s} \frac{dy}{y^2}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\eta}^{n+1} \frac{1}{y^{s(\delta-1-\beta)+2}} dy\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq \eta \leq (n+1) \\
&= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{1}^{n+1} \frac{1}{y^{s(\delta-1-\beta)+2}} dy\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq 1 \leq (n+1) \\
&= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[(n+1)^{(\beta+1-\delta)-\frac{1}{s}}\right] \\
&= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\xi\left(\frac{1}{n+1}\right)\right\}
\end{aligned}$$

$$I_{2.2} = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \quad \text{since} \frac{1}{r} + \frac{1}{s} = 1. \tag{5.3}$$

Now combining (5.1), (5.2) and (5.3), we get

$$\left|N_n^p C_n^1 - f(x)\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now using L_r- norm, we get

$$\left\|N_n^p C_n^1 - f\right\|_r = \left\{\int_0^{2\pi} \left|N_n^p C_n^1 - f(x)\right|^r dx\right\}^{\frac{1}{r}}$$

$$= \left\{\int_0^{2\pi} \left\{(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} \left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}.$$

This completes the proof of the Theorem 2.

6. Corollary

Following corollary can be derived from our main theorem.

Corollary 6.1. *If* $\xi(t) = t^\alpha, 0 < \alpha \le 1$, *then the weighted* $W(L_r, \xi(t))$ *class*, $r \ge 1$, *reduces to the class* $Lip(\alpha, r)$ *and the degree of approximation of a* 2π *- periodic function* $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha \le 1$, *is given by*

$$\left|N_n^p C_n^1 - f\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right).$$

Proof: The result follows by setting $\beta = 0$ in (2.1). □

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Translated from German by I Földer. International series of Monograms in Pure and Applied Mathematics, **20** Pergamon Press, New York-Oxford-Paris, 1961.

[2] Prem Chandra, *Trigonometric approximation of functions in L^p norm*, J. Math. Anal. Appl., **275**(1)(2002), 13-26.

[3] H. H. Khan, *On degree of approximation of functions belonging to the class Lip* (α, p), Indian J. Pure Appl. Math., **5**(2)(1974), 132-136.

[4] Lászaló Leindler, *Trigonometric approximation in L^p norm*, J. Math. Anal. Appl., **302**(2005).

[5] Leonard McFadden, *Absolute Nörlund summability*, Duke Math. J., **9**(1942), 168-207.

[6] K. Qureshi, *On the degree of approximation of a periodic function f by almost Nörlund means*, Tamkang J. Math., **12**(1)(1981), 35-38.

[7] K. Qureshi, *On the degree of approximation of a function belonging to the class Lip*α, Indian J. pure Appl. Math., **13**(8)(1982), 898-903.

[8] K. Qureshi; and H. K. Neha, *A class of functions and their degree of approximation*, Ganita, **41**(1)(1990), 37-42.

[9] B. E. Rhaodes, *On degree of approximation of functions belonging to lipschitz class by Hausdorff means of its Fourier series,* Tamkang Journal of Mathematics, **34**(3)(2003), 245-247.

[10] B. N. Sahney; and D. S. Goel, *On the degree of continuous functions*, Ranchi University Math. Jour., **4**(1973), 50-53.

[11] E. C. Titchmarsh, *The Theory of functions*, Oxford Univ. Press,(1939), 402-403.

[12] A. Zygmund, *Trigonometric series, 2nd rev. ed.*, Cambridge Univ. Press, Cambridge, **1**(1959).