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On a Question of Closed Maps of S. Lin

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Abstract. Let X be a regular T_1-space such that each single point set is a G_δ set. Denote 'hereditarily closure-preserving' by 'HCP'. To consider a question of closed maps of S. Lin in [6], we improve some results of Foged in [1], and prove the following propositions.

Proposition 1. $D = \{x \in X : |\{F \in \mathcal{F} : x \in F\}| \geq \aleph_0\}$ is discrete and closed if $\mathcal{F}$ is a collection of HCP.

Proposition 2. $\mathcal{H} = \{\cap\mathcal{F}' : \mathcal{F}'$ is an fininte subcolletion of $\mathcal{F}_n\}$ is HCP if $\mathcal{F}$ is a collection of HCP.

Proposition 3. Let (X, τ) have a σ-HCP k-network. Then (X, τ) has a σ-HCP k-network $\mathcal{F} = \cup_n \mathcal{F}_n$ such that:

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$,

(ii) $D_n = \{x \in X : |\{F \in \mathcal{F}_n : x \in F\}| \geq \aleph_0\}$ is a discrete closed set and

(iii) each $\mathcal{F}_n$ is closed to finite intersections.

1. Introduction

All spaces are assumed to be regular T_1 and have G_δ-property, and all mappings are assumed to be continuous.

Call closed images of metric spaces *Lašnev spaces* since Lašnev in [5] gave a internal characterization of closed images of metric spaces. Recall that a collection $\mathcal{F}$ of subsets of a space X is a (closed) *k-network* (Michael in [7]) if whenever U is open and contains a compact set K, then $K \subset \cup\mathcal{C} \subset U$ for some finite subcollection $\mathcal{C}$ of $\mathcal{F}$ (and each member of $\mathcal{F}$ is closed).

Denote 'hereditarily closure-preserving' by 'HCP'. Recall that a collection $\mathcal{H}$ of subsets of a space X is HCP if $\mathcal{H}' \subset \mathcal{H}$ and $S(H_\alpha) \subset H_\alpha$ for each $H_\alpha \in \mathcal{H}'$, then $\cup\{Cl(S(H_\alpha)) : H_\alpha \in \mathcal{H}'\} = Cl(\cup\{S(H_\alpha)) : H_\alpha \in \mathcal{H}'\}$. In paper [1], Foged proved the following results.

Let (X, τ) be a Fréchet space with a σ-HCP k-network $\mathcal{F} = \cup_n \mathcal{F}_n$. Then

(i) $D_n = \{x \in X : |\{F \in \mathcal{F}_n : x \in F\}| \geq \aleph_0\}$ is a discrete closed set and

(ii) $\{\cap\mathcal{F}' : \mathcal{F}'$ is an fininte subcolletion of $\mathcal{F}_n\}$ is HCP.

And then, by the above results, a characterization of Lašnev spaces is obtained in [1]: A Hausdorff space X is a Lašnev space iff X is a Fréchet space with a σ-HCP k-network.

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Lin in [6] proved that the closed image of $\aleph$-spaces has a σ-HCP k-network, and asks if the natural analogue of Foged's theorem is valid: Is a space X the closed image of an $\aleph$-space iff X has a σ-HCP k-network?

Gruenhage in [2] gave a survey of progress of general metric spaces and metrization and mentioned the above question.

To consider the above question, we construct closed transfinite sequences and prove the above results without the Fréchet condition by the closed transfinite sequences.

2. A closed transfinite sequence

Let $\mathcal{F} = \{F_\alpha : \alpha \in \aleph\}$ be a collection of HCP with a well order index set $\aleph$. We assume that each $F_\alpha \in \mathcal{F}$ is closed throughout this paper. Pick a $y \in X$. Let $\mathcal{F}_y = \{F_\alpha \in \mathcal{F} : y \in F_\alpha\} = \{F_\alpha \in \mathcal{F} : \alpha \in \aleph_y\}$.

Construction.

We construct a sequence of closed sets by transfinite induction on $\aleph_y$.

To do it pick a $\delta < \aleph_y$. Let $\omega = \omega_0$ be the least limit ordinal. Then $\delta = \alpha\omega + n$. Denote F_δ by F^α_n. Let $\mathcal{F}^\alpha = \{F_{\alpha\omega+n} : n \in \omega\} = \{F^\alpha_n : n \in \omega\}$ and $\mathcal{H}^\alpha_0 = \cup_{\beta<\alpha}\mathcal{F}^\beta$. Let $\mathcal{F}^\alpha_n = \{F^\alpha_j : j < n\}$ and $\mathcal{H}^\alpha_n = \mathcal{H}^\alpha_0 \cup \mathcal{F}^\alpha_n$ for $n \in N$.

Let $O_y = X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$ with $O_y \supset O_1 \supset O_2 \supset ...$ being a sequence of decreasing open sets and $\cap_n O_n = \{y\}$.

A.1. Let $E^0_0 = \emptyset$. Take $\mathcal{F}^0_0 = \{F^0_0\}$. Let $E^0_{1i} = \cup\{F^0_1 \cap F_{\gamma_1} - O_{1+i} : F_{\gamma_1} \in \mathcal{F}^0_0\}$ for $i > 1$. Then $E^0_{1i} \subset F^0_1$, $y \notin E^0_{1i}$ and E^0_{1i} is closed.

A.2. Assume that, for each $j \le n$, there is an E^0_{ji} such that $E^0_{ji} \subset F^0_j$, $y \notin E^0_{ji}$ and E^0_{ji} is closed for $i > 1$. Let

$$\begin{aligned} D^0_{n+1i} =& F^0_{n+1} \cap E^0_{ni+1} = (\cup\{F^0_{n+1} \cap F^0_n \cap F_{\gamma_1} - O_{1+(i+1)} : F_{\gamma_1} \in \mathcal{F}^0_n\}) \cup ... \cup \\ & \cup(\cup\{F^0_{n+1} \cap F^0_n \cap F_{\gamma_1} \cap ... \cap F_{\gamma_n} - O_{n+(i+1)} : F_{\gamma_j} \in \mathcal{F}^0_n \text{ for } 0 < j \le n\}). \end{aligned}$$

Here $\mathcal{F}^* = \{F_{\gamma_j} : F_{\gamma_j} \in \mathcal{F}^0_n \text{ for } 0 < j \le k\}$ expresses that $\mathcal{F}^*$ is a collection of true k **many pairwise different sets** for $k \le n$. It is used always with the same meaning throughout this paper. Then, for $i > 1$

$$\begin{aligned} D^0_{n+1i} =& (\cup\{F^0_{n+1} \cap F_{\gamma_1} \cap F_{\gamma_2} - O_{(1+1)+i} : F_{\gamma_j} \in \mathcal{F}^0_{n+1} \text{ for } 0 < j \le 2\ \}) \cup ... \cup \\ & \cup(\cup\{F^0_{n+1} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_{n+1}} - O_{(n+1)+i} : F_{\gamma_j} \in \mathcal{F}^0_{n+1} \text{ for } 0 < j \le n+1\}). \end{aligned}$$

Let $D^{0*}_{n+1i} = \cup\{F^0_{n+1} \cap F_{\gamma_1} - O_{1+i} : F_{\gamma_1} \in \mathcal{F}^0_{n+1}\}$ and $E^0_{n+1i} = D^{0*}_{n+1i} \cup D^0_{n+1i}$ for $i > 1$. Then $E^0_{n+1i} \subset F^0_{n+1}$, $y \notin E^0_{n+1i}$ and E^0_{n+1i} is closed for $i > 1$. Then, for each $n \in \omega$, there is an E^0_{ni} such that $E^0_{ni} \subset F^0_n$, $y \notin E^0_{ni}$ and E^0_{ni} is closed for $i > 1$ by induction on n.

Let $E^0_i = \cup_{n\in\omega} E^0_{ni}$ for $i > 1$. Then $y \notin E^0_i$ and E^0_i is closed since $\mathcal{F}^0$ is HCP.

B.1. Assume, for each $\beta\omega + n < \alpha\omega$, there is an E^{β}_{ni} such that $E^{\beta}_{ni} \subset F^{\beta}_{n} = F_{\beta\omega+n}$, $y \notin E^{\beta}_{ni}$ and E^{β}_{ni} is closed for $i > 1$. Here,

$$E^{\beta}_{ni} = \cup\{(\cup\{F^{\beta}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i} : F_{\gamma_j} \in \mathcal{H}^{\beta}_{n} \text{ for } 0 < j \leq k\}) : k \in N\}.$$

Let $E^{\alpha}_{i} = (\cup_{\beta<\alpha, n<\omega} E^{\beta}_{ni})$. Then $y \notin E^{\alpha}_{i}$ and E^{α}_{i} is closed since $\mathcal{H}^{\alpha}_{0}$ is HCP. Let $D^{\alpha}_{0i} = \cup\{(\cup\{F^{\alpha}_{0} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i} : F_{\gamma_j} \in \mathcal{H}^{\alpha}_{0} \text{ for } 0 < j \leq k\}) : k \in N\}$ for $i > 1$.

Claim. 2.1. *$F^{\alpha}_{0} \cap E^{\alpha}_{i+1} = D^{\alpha}_{0i}$ for $i > 1$.*

Proof. To see $F^{\alpha}_{0} \cap E^{\alpha}_{i+1} \supset D^{\alpha}_{0i}$, take $I_k = F^{\alpha}_{0} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i} \subset D^{\alpha}_{0i}$ with $F_{\gamma_j} \in \mathcal{H}^{\alpha}_{0}$ for $0 < j \leq k$. Then we may assume $\gamma_1 < \gamma_2 < ... < \gamma_k = \gamma\omega + n < \alpha\omega$ since F_{γ_j} $(0 < j \leq k)$ are pairwise different sets. Then $\gamma < \alpha$ and $F_{\gamma\alpha+0} = F^{\alpha}_{0}$. Then $I_k = F^{\alpha}_{0} \cap F^{\gamma}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_{k-1}} - O_{k+i}$.

Take $I'_k = F^{\gamma}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_{k-1}} - O_{k+i} = F^{\gamma}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_{k-1}} - O_{(k-1)+(i+1)}$ with pairwise different $\{F_{\gamma_j} : 0 < j \leq k-1\} \subset \mathcal{H}^{\gamma}_{n}$ and $F^{\gamma}_{n} \notin \mathcal{H}^{\gamma}_{n}$. Then $I'_k \subset E^{\gamma}_{ni+1}$. Then we have proved $I_k = F^{\alpha}_{0} \cap I'_k \subset F^{\alpha}_{0} \cap E^{\gamma}_{ni+1} \subset E^{\alpha}_{i+1}$.

To see $F^{\alpha}_{0} \cap E^{\alpha}_{i+1} \subset D^{\alpha}_{0i}$, take $I_k = F^{\alpha}_{0} \cap F^{\gamma}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+(i+1)} \subset F^{\alpha}_{0} \cap E^{\gamma}_{ni+1}$. Then $\gamma_1 < \gamma_2 < ... < \gamma_k < \gamma\omega + n < \alpha\omega + 0$. Let $\gamma_{k+1} = \gamma\omega + n$. Then $F_{\gamma_j} \in \mathcal{H}^{\alpha}_{0}$ for $0 < j \leq k+1$ and $F^{\alpha}_{0} \notin \mathcal{H}^{\alpha}_{0}$. Then $I_k = F^{\alpha}_{0} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} \cap F_{\gamma_{k+1}} - O_{(k+1)+i} \subset D^{\alpha}_{0i}$. Then $F^{\alpha}_{0} \cap E^{\alpha}_{i+1} \subset D^{\alpha}_{0i}$. □

Continuoued transfinite induction B. Then $F^{\alpha}_{0} \cap E^{\alpha}_{i+1} = D^{\alpha}_{0i}$ is closed and $y \notin D^{\alpha}_{0i}$ by Claim 2.1. On the other hand, $y \notin F^{\alpha}_{0} \cap F_{\gamma_1} - O_{1+i}$ and $F^{\alpha}_{0} \cap F_{\gamma_1} - O_{1+i}$ is a closed subset of F_{γ_1} for $F_{\gamma_1} \in \mathcal{H}^{\alpha}_{0}$. Then $D^{\alpha *}_{0i} = \cup\{F^{\alpha}_{0} \cap F_{\gamma_1} - O_{1+i} : F_{\gamma_1} \in \mathcal{H}^{\alpha}_{0}\}$ is closed in X for $i > 1$ since $\mathcal{H}^{\alpha}_{0}$ is HCP. Let $E^{\alpha}_{0i} = D^{\alpha *}_{0i} \cup D^{\alpha}_{0i}$ for $i > 1$. Then $E^{\alpha}_{0i} \subset F^{\alpha}_{0}$, $y \notin E^{\alpha}_{0i}$ and E^{α}_{0i} is closed for $i > 1$.

B.2. Assume that, for each $j \leq n$, there is an E^{α}_{ji} such that $E^{\alpha}_{ji} \subset F^{\alpha}_{j} = F_{\alpha\omega+j}$, $y \notin E^{\alpha}_{ji}$ and E^{α}_{ji} is closed for $i > 1$. Let

$$\begin{aligned} & D^{\alpha}_{n+1i} \\ =& F^{\alpha}_{n+1} \cap E^{\alpha}_{ni+1} \\ =& \cup\{\cup\{F^{\alpha}_{n+1} \cap F^{\alpha}_{n} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+(i+1)} : F_{\gamma_j} \in \mathcal{H}^{\alpha}_{n} \text{ for } 0 < j \leq k\} : k \in N\} \\ =& \cup\{\cup\{F^{\alpha}_{n+1} \cap F_{\gamma_1} \cap ... \cap F_{\gamma_{k+1}} - O_{(k+1)+i} : F_{\gamma_j} \in \mathcal{H}^{\alpha}_{n+1} \text{ for } 0<j\leq k+1\} : k \in N\}. \end{aligned}$$

Then D^{α}_{n+1i} is closed and $y \notin D^{\alpha}_{n+1i}$ since E^{α}_{ni+1} is closed with $y \notin E^{\alpha}_{ni+1}$ for $i > 1$. Let $D^{\alpha *}_{n+1i} = \cup\{F^{0}_{n+1} \cap F_{\gamma_1} - O_{1+i} : F_{\gamma_1} \in \mathcal{H}^{0}_{n+1}\}$ for $i > 1$. Then $D^{\alpha *}_{n+1i}$ is closed with $y \notin D^{\alpha *}_{n+1i}$ since $F^{\alpha}_{n+1} \cap F_{\gamma_1} - O_{1+i} \subset F_{\gamma_1}$, $F_{\gamma_1} \in \mathcal{H}^{\alpha}_{n+1}$ and $\mathcal{H}^{\alpha}_{n+1}$ is HCP. Let $E^{\alpha}_{n+1i} = D^{\alpha *}_{n+1i} \cup D^{\alpha}_{n+1i}$ for $i > 1$. Then $E^{\alpha}_{n+1i} \subset F^{\alpha}_{n+1}$, $y \notin E^{\alpha}_{n+1i}$ and E^{0}_{n+1i} is closed for $i > 1$. Then, for each $n \in \omega$, there is an E^{α}_{ni} such that $E^{\alpha}_{ni} \subset F^{\alpha}_{n}$, $y \notin E^{\alpha}_{ni}$ and E^{α}_{ni} is closed for $i > 1$ by induction on n.

Then, by transfinite induction on $\aleph_y$, there is an E^{α}_{ni} such that $E^{\alpha}_{ni} \subset F_{\alpha\omega+n}$, $y \notin E^{\alpha}_{ni}$ and E^{α}_{ni} is closed for each $\gamma = \alpha\omega + n < \aleph_y$. So we have the following proposition.

Proposition 2.2. *Let $y \in X$ and $\{O_n : n \in N\}$ be a sequence of decreasing open sets with $\cap_n O_n = \{y\}$. Let $\mathcal{F}_y = \{F_\alpha \in \mathcal{F} : \alpha \in \aleph_y\}$ be HCP. Then, for each $\gamma = \alpha\omega + n < \aleph_y$, there is an E^α_{ni} such that $E^\alpha_{ni} \subset F_{\alpha\omega+n}$, $y \notin E^\alpha_{ni}$ and E^α_{ni} is closed. Here $E^\alpha_{ni} = \cup\{(\cup\{F^\alpha_n \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i} : F_{\gamma_j} \in \mathcal{H}^\alpha_n \text{ for } 0 < j \leq k\}) : k \in N\}$ for $i > 1$ such that $F_{\gamma_1}, ..., F_{\gamma_k}$ are pairwise different for $k \in N$.*

3. Properties of collections of HCP

Let $\mathcal{F}_y = \{F_\alpha \in \mathcal{F} : y \in F_\alpha\} = \{F_\alpha \in \mathcal{F} : \alpha \in \aleph_y\}$.

Proposition 3.1. *Let $\mathcal{F}$ be HCP. Then $D(\aleph) = \{x \in X : |\mathcal{F}_x| \geq \aleph\}$ is closed for each $\aleph \geq \aleph_0$.*

Proof. Pick a $y \in X - D(\aleph)$. Then $|\mathcal{F}_y| < \aleph$. Let $O_y = X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$. Then, for each $x \in O_y$ and each $F_\alpha \in \mathcal{F}_x$, $x \in O_y \cap F_\alpha$ implies $y \in F_\alpha$. Then $F_\alpha \in \mathcal{F}_y$. This implies $\mathcal{F}_x \subset \mathcal{F}_y$ and $|\mathcal{F}_x| \leq |\mathcal{F}_y| < \aleph$. So $x \notin D(\aleph)$ and $O_y \cap D(\aleph) = \emptyset$. Then $D(\aleph)$ is closed. □

Proposition 3.2. *Let $\mathcal{F}$ be HCP. Then $D = D(\aleph_0)$ is a discrete closed set.*

Proof. Pick a $y \in D$. We prove that there is an open set O with $O \cap D = \{y\}$. To do it assume $\mathcal{F}_y = \mathcal{F}$ and $D \subset O_y = X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$. Let $O_y \supset O_1 \supset O_2 \supset ...$ be a sequence of decreasing open sets with $\cap_n O_n = \{y\}$. We prove it by transfinite induction on $\aleph_y = |\mathcal{F}_y|$.

A. D is a discrete closed set when $\mathcal{F} = \{F_n : n < \aleph_0\}$.

To prove A, note Proposition 2.2. Then, for each $n \in \omega$, there is an E^0_{ni} such that $E^0_{ni} \subset F_n$, $y \notin E^0_{ni}$ and E^0_{ni} is closed for $i > 1$. Let $E^0_i = \cup_{n \in \aleph_0} E^0_{ni}$. Then $y \notin E^0_i$ and E^0_i is closed since $\mathcal{F}$ is HCP.

Suppose $y \in Cl(D - \{y\})$. Pick an $x \in D - \{y\}$. Then $\mathcal{F}_x = \{F_n \in \mathcal{F} : x \in F_n\}$ is infinite. Then $\mathcal{F}_x = \{F_n : n \in N_1\}$ with $i_n < i_{n+1}$ if both i_n and i_{n+1} in N_1. Here N_1 is an infinite subset of N. Let $x \in O_m$ and $x \notin O_{m+1}$. Pick an n with $k = i_n \geq n > m + 2$. Then $O_{(n-1)+i} \subset O_{m+1}$. So $x \notin O_{(n-1)+i}$. Then $x \in I = F_k \cap F_{i_1} \cap ... \cap F_{i_{n-1}} - O_{(n-1)+i} \subset E^0_{ki} \subset E^0_i$. So $D - \{y\} \subset E^0_i$. Then $y \in Cl(D - \{y\}) \subset Cl(E^0_i) = E^0_i$ for $i > 1$. It is a contradiction to $y \notin E^0_i$ for $i > 1$. So $y \notin Cl(D - \{y\})$ for each $y \in D$. Then D is a discrete closed set if $\mathcal{F} = \{F_n : n < \aleph_0\}$.

B. Assume that D is a discrete closed set for each $\aleph < \aleph_y$ when $\mathcal{F} = \{F_\alpha : \alpha < \aleph\}$. Then, by the assumption of induction, it is easy to see the following claims.

Claim 3.3. *If $x \in D - \{y\}$ with $|\mathcal{F}_x| = \aleph < \aleph_y$, then there is an open set O_x such that $x \in O_x$ and $O_x \cap D$ is a discrete closed set.*

Proof. Let $O_x = X - \cup\{F_\alpha \in \mathcal{F} : x \notin F_\alpha\}$. Then $\mathcal{F}_t \subset \mathcal{F}_x$ for each $t \in O_x \cap D$. Then $|\mathcal{F}_t| \leq |\mathcal{F}_x| = \aleph < \aleph_y$. Then, by the assumption of induction, $O_x \cap D$ is a discrete closed set in subspace O_x. □

Claim 3.4. *$X^\alpha = \{x \in D : |\mathcal{F}^\alpha_x| \geq \aleph_0\}$ is a discrete closed subset of X if $\alpha < \aleph_y$*

and $\mathcal{F}^\alpha = \{F_\beta \in \mathcal{F} : \beta < \alpha\}$.

Proof. Suppose that there is an $\alpha < \aleph_y$ and a point t such that t is a cluster of X^α. Note $|\mathcal{F}^\alpha| = |\alpha| = \aleph < \aleph_y$. Then, by the assumption of induction and Claim 3.3, there is an open set O_t such that $t \in O_t$ and $O_t \cap X^\alpha$ is a discrete closed set, a contradiction. □

Continued transfinite induction B. We prove that D is a discrete closed set when $\mathcal{F} = \{F_\alpha : \alpha < \aleph_y\}$.

To do it pick a $y \in D$. Let $O_y = X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$. Then we may assume that $\mathcal{F}_y = \mathcal{F}$, $D \subset O_y$ and $O_y \supset O_1 \supset O_2 \supset ...$ is a decreasing sequence of open sets with $\cap_n O_n = \{y\}$. Then, by Proposition 2.2, for each $\gamma = \alpha\omega + n < \aleph_y$ there is an E^α_{ni} such that $E^\alpha_{ni} \subset F_{\alpha\omega+n}$, $y \notin E^\alpha_{ni}$ and E^α_{ni} is closed.

Let $\mathcal{F}^\alpha = \{F_{\alpha\omega+n} : n \in \omega\} = \{F^\alpha_n : n \in \omega\}$ and $\mathcal{H}^\alpha = \cup_{\beta<\alpha}\mathcal{F}^\beta$ just as Proposition 2.2. Let $\mathcal{F}^\alpha_x = \{F^\alpha_n \in \mathcal{F}^\alpha : x \in F^\alpha_n\}$ and $\mathcal{H}^\alpha_x = \{F_\gamma \in \mathcal{H}^\alpha : x \in F_\gamma\}$. Let $X^\alpha = \{x \in D : |\mathcal{F}^\alpha_x| \geq \aleph_0$ or $|\mathcal{H}^\alpha_x| \geq \aleph_0\}$. Then X^α is a discrete closed set by Claim 3.4. Let $X^\alpha_n = X^\alpha \cap F^\alpha_n$ for each $F^\alpha_n \in \mathcal{F}^\alpha$. Let $G^\alpha_{ni} = E^\alpha_{ni} \cup X^\alpha_n$. Then $G^\alpha_{ni} \subset F^\alpha_n$, $y \notin G^\alpha_{ni}$ and G^α_{ni} is closed for $i > 1$. Let $G_i = \cup_{\alpha<\aleph, n<\omega} G^\alpha_{ni}$. Then $y \notin G_i$ and G_i is closed since $\mathcal{F}$ is HCP.

On the other hand, pick an $x \in D - \{y\}$. Let $x \in O_m$ and $x \notin O_{m+1}$. Let $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\} = \{F_{\beta_0\omega+n_0}, F_{\beta_1\omega+n_1}, ..., F_{\beta_\gamma\omega+n_\gamma}, ...\}$ as a subsequence of the sequence $\mathcal{F}$.

Case 1: there is an $\mathcal{F}^\alpha$ such that $\mathcal{F}^\alpha \cap \mathcal{F}_x$ is infinite. Then $x \in X^\alpha \cap F^\alpha_n = X^\alpha_n \subset G_i$ for each $i > 1$. Especially $x \in G_3$.

Case 2: $\mathcal{F}^\alpha \cap \mathcal{F}_x$ is finite for each $\alpha < \aleph_y$. Consider $\mathcal{F}'_x = \{F_{\beta_j\omega+n_j} : j \in N\}$. Denote $F_{\beta_j\omega+n_j}$ by F_{γ_j} for $j \in N$. Let $k > m + 2$. Then $k - 1 > m + 1$ and $\gamma_1 < \gamma_2 < ... < \gamma_k < ...$ with $F_{\gamma_j} \in \mathcal{H}^{\beta_k}_{n_k}$ for $j < k$ and $F_{\gamma_k} \notin \mathcal{H}^{\beta_k}_{n_k}$ by the definition of $\mathcal{H}^\alpha_n$. Then $x \in F^{\beta_k}_{n_k} \cap (F_{\gamma_1} \cap ... \cap F_{\gamma_{k-1}}) - O_{(k-1)+(i+1)} \subset E^{\beta_k}_{n_k i+1} \subset G_{i+1}$ for each $i > 1$. Especially $x \in G_3$.

Then $D - \{y\} \subset G_3$. This implies $y \notin Cl(D - \{y\})$ since $y \notin G_i$ for each $i > 1$. Then D is a discrete closed set. □

Let $\mathcal{F}_1 \wedge \mathcal{F}_2 = \{F_\alpha \cap F_\beta : F_\alpha \in \mathcal{F}_1$ and $F_\beta \in \mathcal{F}_2\}$. The following Proposition 3.5 was used frequently in other papers.

Proposition 3.5. *$\mathcal{F}_1 \wedge \mathcal{F}_2$ is HCP if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are HCP.*

Let $\mathcal{F}$ be a collection of HCP and $\mathcal{F}'_1 = \mathcal{F}$. Let $\mathcal{F}'_n = \mathcal{F}'_{n-1} \wedge \mathcal{F}$ for $n \geq 2$. Denote $F_{\gamma_1} \cap F_{\gamma_2} \cap ... \cap F_{\gamma_k}$ by $F(\gamma_1, \gamma_2, ..., \gamma_k)$ if $\gamma_1 < \gamma_2 < ... < \gamma_k$ for $k \in N$. Let $\mathcal{F}_n = \{F(\gamma_1, \gamma_2, ..., \gamma_n) \in \mathcal{F}'_n : F(\gamma_1, \gamma_2, ..., \gamma_n) \notin \cup_{i<n}\mathcal{F}'_i\}$ be a collection of pairwise different sets for $n \geq 2$.

Proposition 3.6. *Let $\mathcal{F}$ be a collection of HCP and $\mathcal{H} = \cup_n \mathcal{F}_n$ be the collection of all finite intersections of $\mathcal{F}$. Then $\mathcal{H}$ is HCP.*

Proof. Let $\mathcal{H}_n = \{H(\gamma_1, \gamma_2, ..., \gamma_n) : H(\gamma_1, \gamma_2, ..., \gamma_n) \subset F(\gamma_1, \gamma_2, ..., \gamma_n) \in \mathcal{F}_n\}$ and $H_n = \cup\mathcal{H}_n$. Then each H_n is closed by Proposition 3.5. Let $H = \cup_n H_n$. We prove that H is closed. To do it take a $y \notin H$.

Case 1: $y \in X - D$. D is a closed discrete set by Proposition 3.2. Then there is an open set O with $y \in O$ and $O \subset X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$. Let $\mathcal{F}_y = \{F \in \mathcal{F} : y \in F\}$. Then $\mathcal{F}_y = \{F_i : i \leq n(y)\}$ since $y \in X - D$. Then $F \cap O \neq \emptyset$ implies $F \in \mathcal{F}_y$. So $F \notin \mathcal{F}_y$ and $F \cap F_i \in \mathcal{F}_2$ imply $(F \cap F_i) \cap O = \emptyset$. Then $H(\gamma_1, \gamma_2) \cap O \neq \emptyset$ and $H(\gamma_1, \gamma_2) \subset F_{\gamma_1} \cap F_{\gamma_2}$ imply that both F_{γ_1} and F_{γ_2} are in $\mathcal{F}_y$. Let $\mathcal{H}(i, y) = \{H \in \mathcal{H}_i : y \in H\}$. Then there is a k such that $\mathcal{H}(i, y) = \emptyset$ for $i \geq k$ and $\mathcal{H}(i, y)$ is finite for $i < k$. This implies $y \notin Cl(H)$.

Case 2: $y \in D$. Then there is an open set O such that $y \in O$, $D \cap Cl(O) = \{y\}$ and $Cl(O) \subset X - \cup\{F_\alpha \in \mathcal{F} : y \notin F_\alpha\}$ by Proposition 3.2. Let $O \supset O_1 \supset O_2 \supset ...$ be a decreasing sequence of open sets with $\cap_n O_n = \{y\}$ and $(\cup_{i \leq n} H_i) \cap Cl(O_n) = \emptyset$ since $y \notin H_n$ and each H_n is closed. Let $\mathcal{F} = \{F_\alpha : \alpha < \aleph_y\}$. Then, by Proposition 2.2, for each $\gamma = \alpha\omega + n < \aleph_y$ there is an E^α_{ni} such that $E^\alpha_{ni} \subset F_{\alpha\omega+n}$, $y \notin E^\alpha_{ni}$ and E^α_{ni} is closed. Here, for $i > 1$,

$$E^\alpha_{ni} = \cup\{(\cup\{F^\alpha_n \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i} : F_{\gamma_j} \in \mathcal{H}^\alpha_n \text{ for } 0 < j \leq k\}) : k \in N\}.$$

Then we assume $\gamma_1 < \gamma_2 < ... < \gamma_k < \alpha\omega + n = \gamma_{k+1}$ for each $k \in N$. Take a $H(\gamma_1, \gamma_2, ..., \gamma_{k+1})$ from $\mathcal{H}_{n+1}$. Then we have $H(\gamma_1, \gamma_2, ..., \gamma_{k+1}) \subset H_{k+1}$. This implies $H(\gamma_1, \gamma_2, ..., \gamma_{k+1}) \cap O_{k+1} = \emptyset$. Now we take $I = F^\alpha_n \cap F_{\gamma_1} \cap ... \cap F_{\gamma_k} - O_{k+i}$. Then $H(\gamma_1, \gamma_2, ..., \gamma_{k+1}) \subset I \subset E^\alpha_{ni}$ for $i > 1$. Then $\cup_{n>1} H_n \subset \cup_{\alpha,k} E^\alpha_{ki} = E_i$ and E_i is closed by Proposition 2.2. So $y \notin E_i = Cl(E_i) \supset Cl(\cup_{n>1} H_n)$.

On the other hand, $y \notin H'_1 = \cup\{F_\alpha - O_1 : F_\alpha \in \mathcal{F}\}$ and H'_1 is closed since $\mathcal{F}$ is HCP. Then $y \notin H'_1 \supset H_1$. So $y \notin Cl(H)$. □

Corollary 3.7. *Let (X, τ) have a σ-HCP k-network. Then (X, τ) has a σ-HCP k-network $\mathcal{F} = \cup_n \mathcal{F}_n$ such that:*

1. *$\mathcal{F}_n \subset \mathcal{F}_{n+1}$,*
2. *each $D_n = \{x \in X : |\{F \in \mathcal{F}_n : x \in F\}| \geq \aleph_0\}$ is a discrete closed set and*
3. *each $\mathcal{F}_n$ is closed to finite intersections.*

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