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On Curvature-Adapted and Proper Complex Equifocal Submanifolds

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Abstract. In this paper, we investigate curvature-adapted and proper complex equifocal submanifolds in a symmetric space of non-compact type. The class of these submanifolds contains principal orbits of Hermann type actions as homogeneous examples and is included by that of curvature-adapted and isoparametric submanifolds with flat section. First we introduce the notion of a focal point of non-Euclidean type on the ideal boundary for a submanifold in a Hadamard manifold and give the equivalent condition for a curvature-adapted and complex equifocal submanifold to be proper complex equifocal in terms of this notion. Next we show that the complex Coxeter group associated with a curvature-adapted and proper complex equifocal submanifold is the same type group as one associated with a principal orbit of a Hermann type action and evaluate from above the number of distinct principal curvatures of the submanifold.

1. Introduction

In symmetric spaces, the notion of an equifocal submanifold was introduced in [32]. This notion is defined as a compact submanifold with globally flat and abelian normal bundle such that the focal radius functions for each parallel normal vector field are constant. However, the equifocality is rather weak property in the case where the symmetric spaces are of non-compact type and the submanifold is non-compact. So we [16, 17] have recently introduced the notion of a complex equifocal submanifold in a symmetric space G/K of non-compact type. This notion is defined by imposing the constancy of the complex focal radius functions instead of focal radius functions. Here we note that the complex focal radii are the quantities indicating the positions of the focal points of the extrinsic complexification of the submanifold, where the submanifold needs to be assumed to be complete and of class C^ω (i.e., real analytic). On the other hand, Heintze-Liu-Olmos [11] has recently defined the notion of an isoparametric submanifold with flat section in a general Riemannian manifold as a submanifold such that the normal holonomy

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group is trivial, its sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction and that the image of the normal space at each point by the normal exponential map is flat and totally geodesic. We [17] showed the following fact:

All isoparametric submanifolds with flat section in a symmetric space G/K of non-compact type are complex equifocal and that conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric ones with flat section.

Here the curvature-adaptedness means that, for each normal vector v of the submanifold, the Jacobi operator $R(\cdot, v)v$ preserves the tangent space of the submanifold invariantly and the restriction of $R(\cdot, v)v$ to the tangent space commutes with the shape operator A_v, where R is the curvature tensor of G/K. Note that curvature-adapted hypersurfaces in a complex hyperbolic space (and a complex projective space) mean so-called Hopf hypersurfaces and that curvature-adapted complex equifocal hypersurfaces in the space mean Hopf hypersurfaces with constant principal curvatures, whcih are classified by J. Berndt [1]. Also, he [2] classified curvature-adapted hypersurfaces with constant prinicipal curvatures (i.e., curvature-adapted complex equifocal hypersurfaces) in the quaternionic hyperbolic space. In Appendix 2, we will prove an important fact for a curvature-adapted submanifold. As a subclass of the class of complex equifocal submanifolds, we [18] defined that of the proper complex equifocal submanifolds in G/K as a complex equifocal submanifold whose lifted submanifold to $H^0([0,1], \mathfrak{g})$ ($\mathfrak{g} := \mathrm{Lie}\, G$) through some pseudo-Riemannian submersion of $H^0([0,1], \mathfrak{g})$ onto G/K is proper complex isoparametric in the sense of [16], where we note that $H^0([0,1], \mathfrak{g})$ is a pseudo-Hilbert space. For a proper complex equifocal C^ω-submanifold, we [19] defined the notion of the associated complex Coxeter group as the Coxeter group generated by the complex reflections of order two with respect to complex focal hyperplanes in the normal space of the lift of the complexification of the submanifold to $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ ($\mathfrak{g}^{\mathbf{c}} := \mathrm{Lie}\, G^{\mathbf{c}}$) by some anti-Kaehler submersion of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ onto the anti-Kaehler symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$, where we note that the lifted submanifold is proper anti-Kaehler isoparametric in the sense of [17]. Here we note that the associated complex Coxeter group can be described by only the complex focal structure of the original submanifold without the use of the lifted submanifold. We [19] showed that a proper complex equifocal submanifold is decomposed into the (non-trivial) extrinsic product of such submanifolds if and only if the associated complex Coxeter group is decomposable. Thus it is worth to investigate the complex Coxeter group in detail. According to Theorem 1 of [5], all complete equifocal submanifolds of codimension greater than one on simply connected irreducible symmetric space of compact type are homogeneous. Hence they are principal orbits of hyperpolar actions (see [12]). According to the classification of the hyperpolar actions by A. Kollross ([22]), all hyperpolar actions of cohomogeneity greater than one on the irreducible symmetric space are Hermann ones. On the other hand, O. Goertsches and G. Thorbergsson ([10]) has recently showed that principal orbits of Hermann actions are curvature-adapted. Hence we have the following fact:

All complete equifocal submanifolds of codimension greater than one in simply connected irreducible symmetric spaces of compact type occur as principal orbits of Hermann actions and hence they are curvature-adapted.

Let G/K be a symmetric space of non-compact type and H be a symmetric subgroup of G such that $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ for some involution σ of G, where $\mathrm{Fix}\,\sigma$ is the fixed point group of σ and $(\mathrm{Fix}\,\sigma)_0$ is the identity component of the group. We ([17]) called the action of such a group H on G/K an *action of Hermann type*. In this paper, we call this action Hermann type action for simplicity. We ([17, 18]) showed the following fact:

Principal orbits of a Hermann type action are curvature-adapted and proper complex equifocal.

From these facts, it is conjectured that comparatively many ones among complex equifocal submanifolds of codimension greater than one in irreducible symmetric spaces of non-compact type are curvature-adapted and proper complex equifocal. The following questions are naturally proposed:

Question. *Do all curvature-adapted and proper complex equifocal submanifolds occur as principal orbits of Hermann type actions?*

We defined the notion of a proper complex equifocal submanifold as a complex equifocal submanifold whose lifted submanifold to the above path space is a proper complex isoparametric submanifold. It is important to give an equivalent condition for a complex equifocal submanifold to be proper complex equifocal by using geometric quantities of the original submanifold without the use of those of the lifted submanifold. In this paper, we give such an equivalent condition for a curvature-adapted and complex equifocal submanifold. For its purpose, we first introduce the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ for a submanifold in a Hadamard manifold N in general. By using this notion, we obtain the following equivalent condition.

Theorem A. *Let M be a curvature-adapted and complex equifocal submanifold in a symmetric space $N := G/K$ of non-compact type. Then the following conditions* (i) *and* (ii) *are equivalent:*

(i) *M is proper complex equifocal,*

(ii) *M has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$.*

According to this theorem, we can catch a curvature-adapted and proper complex equifocal submanifold as a curvature-adapted and isoparametric submanifold with flat section which has no focal point of non-Euclidean type on the ideal boundary. In Section 6 of [19], we investigated the complex Coxeter groups associated with principal orbits of Hermann type actions. According to the investigation and Appendix of this paper, it follows that the complex Coxeter group associated with a principal orbit $H(gK)$ of a Hermann type action $H \times G/K \to G/K$ is isomorphic to the affine Weyl group (which is denoted by $W_{\triangle}^{A}$) associated with the root system $\overline{\triangle} := \{\alpha|_{g_*^{-1}T^\perp_{gK}(H(gK))} \,|\, \alpha \in \triangle \text{ s.t. } \alpha|_{g_*^{-1}T^\perp_{gK}(H(gK))} \neq 0\}$, where $\triangle$ is the root system of G/K with respect to a maximal abelian subspace $\mathfrak{a}$ containing $g_*^{-1}T^\perp_{gK}(H(gK))$. See Section 2 about the definition of the affine Weyl group associated with a root system. In order to make sure of whether the above question is solved affirmatively, it is important to investigate whether the complex Coxeter group associated with a curvature-adapted and proper complex equifocal submanifold is isomorphic to the same type group. For the complex Coxeter group associated with this submanifold, we have the following fact.

Theorem B. *Let M be a curvature-adapted and proper complex equifocal C^ω-submanifold in a symmetric space G/K of non-compact type and $\triangle$ be the root system of G/K with respect to a maximal abelian subspace $\mathfrak{a}$ of $T_{eK}(G/K)$ containing $\mathfrak{b} := g_*^{-1}T^\perp_{gK}M$, where gK is an arbitrary point of M. Then $\overline{\triangle} := \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$ is a weakly root system and the complex Coxeter group associated with M is isomorphic to the affine Weyl group associated with $\overline{\triangle}$.*

See Section 2 about the definition of a weakly root system. Thus the complex Coxeter group associated with a curvature-adapted and proper complex equifocal C^ω-submanifold is isomorphic to the same type one as the group associated with a principal orbit of a Hermann type action. Hence the possibility for Question to be solved affirmatively goes up.

Remark 1.1. According to this theorem, in case of $\operatorname{codim} M = 1$, the complex Coxeter group associated with M is isomorphic to $\mathbf{Z}_2 \ltimes \mathbf{Z}$.

By using Theorem 2 of [19] and Theorem B, we obtain the following splitting theorem.

Corollary B.1. *Let M and $\overline{\triangle}$ be as in Theorem B. Then M is decomposed into the extrinsic product of two curvature-adapted and proper complex equifocal submanifolds if and only if $W_{\overline{\triangle}}$ is decomposable, where $W_{\overline{\triangle}}$ is the Coxeter group associated with $\overline{\triangle}$.*

See Section 2 about the definition of the Coxeter group associated with a weakly root system. From this corollary, the following fact follows directly.

Corollary B.2. *Let M be as in Theorem B. If G/K is reducible and* $\operatorname{codim} M = \operatorname{rank} G/K$, *then M is decomposed into the extrinsic product of two curvature-adapted and proper complex equifocal submanifolds.*

For the number of mutually distinct principal curvatures of a curvature-adapted and proper complex equifocal C^ω-submanifold, we have the following fact.

Theorem C. *Let M be a curvature-adapted and proper complex equifocal C^ω-submanifold in a symmetric space G/K of non-compact type and A be the shape tensor of M. Then, for each normal vector v of M at gK, we have*

$$\sharp\operatorname{Spec} A_v \leq \sharp(\overline{\triangle}_+ \setminus \overline{\triangle}^1_+) \times 2 + \sharp\overline{\triangle}^1_+ + \dim \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) - \operatorname{codim} M,$$

where $\mathrm{Spec}\,A_v$ *is the spectrum of* A_v*,* $\overline{\triangle}$ *is as in the statement of Theorem B,* $\overline{\triangle}_+^1 := \{\beta \in \overline{\triangle}_+ \,|\, \text{the multiplicity of } \beta \text{ is equal to } 1\}$*,* $\sharp(\cdot)$ *is the cardinal number of* $(\cdot)$ *and* $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ *is the centralizer of* $\mathfrak{b}$ *in* $\mathfrak{p}$*.*

Remark 1.2. Since $\sharp(\overline{\triangle}_+ \setminus \overline{\triangle}_+^1) \times 2 + \sharp\overline{\triangle}_+^1 \leq \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1$ (where $\triangle_+^1 := \{\alpha \in \triangle_+ \,|\, \text{the multiplicity of } \alpha \text{ is equal to } 1\}$), we have

$$\sharp\mathrm{Spec}\,A_v \leq \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 + \dim \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) - \mathrm{codim}\,M. \tag{1.1}$$

In particular, we have the following fact.

Corollary C.1. *Let* M *be as in Theorem C. Assume that* $\mathrm{codim}\,M = \mathrm{rank}(G/K)$*. Then, for each normal vector* v *of* M*, we have* $\sharp\mathrm{Spec}\,A_v \leq \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1$*, where* $\triangle_+^1 := \{\alpha \in \triangle_+ \,|\, \text{the multiplicity of } \alpha \text{ is equal to } 1\}$*.*

In Table 1, we list up the number $m_{G/K} := \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1$ for irreducible symmetric spaces G/K's of non-compact type. Also, in Appendix 1, we list up the numbers $\max_{v \in T^\perp M} \sharp\mathrm{Spec}\,A_v$ for principal orbits of Hermann type actions H's on irreducible symmetric spaces G/K's of non-compact type satisfying $\mathrm{cohom}\,H = \mathrm{rank}(G/K)$.

Future plan of research. *By using Theorems B and C, we will investigate whether the above question is solved affirmatively in some symmetric spaces of non-compact type.*

For the focal set of a curvature-adapted and proper complex equifocal C^ω-submanifold, we have the following fact.

Theorem D. *Let* M *be as in Theorem B. Then the focal set of* (M, x_0) $(x_0$ *: an arbitrary point of* $M)$ *consists of finitely many totally geodesic hypersurfaces through some point in the section* $\Sigma := \exp^\perp(T_{x_0}^\perp M)$*.*

Let $\{l_i \,|\, i = 1, \cdots, k\}$ be hyperplanes of $T_{x_0}^\perp M$ such that $\bigcup_{i=1}^{k} \exp^\perp(l_i)$ is the focal set of (M, x_0). Denote by $W_{M,\mathbf{R}}$ the group generated by the reflections with respect to l_i's $(i = 1, \cdots, k)$. In this paper, we call this group the *real Coxeter group associated with* M *(at* x_0*)*. Note that this group is independent of the choice of the base point x_0 up to isomorphicness. For this group, we have the following fact.

Theorem E. *Let* M *and* $\overline{\triangle}$ *be as in Theorem B. Then the real Coxeter group associated with* M *is isomorphic to a subgroup of the Coxeter group* $W_{\overline{\triangle}}$*.*

Remark 1.3. We consider the case where M is a principal orbit of a Hermann type action $H \times G/K \to G/K$. Let σ (resp. θ) be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ (resp. $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$), where we may assume $\sigma \circ \theta = \theta \circ \sigma$ without loss of generality. Then the real Coxeter group associated with M is isomorphic to the Weyl group associated with the symmetric space $\mathrm{Fix}(\sigma \circ \theta)/H \cap K$ (see Appendix 1).

Type	G/K	$\sharp\triangle_+$	$\sharp\triangle_+^1$	$m_{G/K}$
(AI)	$SL(n,\mathbf{R})/SO(n)$ $(n\geq 3)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
(AII)	$SU^*(2n)/Sp(n)$ $(n\geq 3)$	$\frac{n(n-1)}{2}$	0	$n(n-1)$
(AIII)	$SU(p,q)/S(U(p)\times U(q))$ $(1\leq p<q)$ $SU(p,p)/S(U(p)\times U(p))$ $(p\geq 2)$	p^2+p p^2	p p	$p(2p+1)$ $p(2p-1)$
(BDI)	$SO_0(p,q)/SO(p)\times SO(q)$ $(2\leq p<q)$ $SO_0(1,q)/SO(1)\times SO(q)$	p^2 1	$\begin{cases} p^2 \\ p(p-1)\end{cases}$ $\begin{cases} 1 \\ 0\end{cases}$	p^2 $(q-p=1)$ $p(p+1)$ $(q-p\geq 2)$ 1 $(q=2)$ 2 $(q\geq 3)$
(BDI′)	$SO_0(p,p)/SO(p)\times SO(p)$	$p(p-1)$	$p(p-1)$	$p(p-1)$
(DIII)	$SO^*(2n)/U(n)$ $(n\geq 4)$	$\begin{cases} \frac{n^2-1}{4} \\ \frac{n^2}{4}\end{cases}$	$\frac{n-1}{2}$ $\frac{n}{2}$	$\frac{n(n-1)}{2}$ $(n:\text{odd})$ $\frac{n(n-1)}{2}$ $(n:\text{even})$
(CI)	$Sp(n,\mathbf{R})/U(n)$ $(n\geq 2)$	n^2	n^2	n^2
(CII)	$Sp(p,q)/Sp(p)\times Sp(q)$ $(p<q)$ $Sp(p,p)/Sp(p)\times Sp(p)$ $(p\geq 2)$	$p(p+1)$ p^2	0 0	$2p(p+1)$ $2p^2$
(EI)	$E_6^6/Sp(4)$	36	36	36
(EII)	$E_6^2/SU(6)\cdot SU(2)$	24	12	36
(EIII)	$E_6^{-14}/Spin(10)\cdot U(1)$	6	2	10
(EIV)	E_6^{-26}/F_4	3	0	6
(EV)	$E_7^7/(SU(8)/\{\pm 1\})$	63	63	63
(EVI)	$E_7^{-5}/SO'(12)\cdot SU(2)$	24	12	36
(EVII)	$E_7^{-25}/E_6\cdot U(1)$	9	3	15
(EVIII)	$E_8^8/SO'(16)$	120	120	120
(EIX)	$E_8^{-24}/E_7\cdot Sp(1)$	24	12	36
(FI)	$F_4^4/Sp(3)\cdot Sp(1)$	24	24	24
(FII)	$F_4^{-20}/Spin(9)$	2	0	4
(G)	$G_2^2/SO(4)$	6	6	6
(II-A)	$SL(n,\mathbf{C})/SU(n)$ $(n\geq 3)$	$\frac{n(n-1)}{2}$	0	$n(n-1)$
(II-BD)	$SO(n,\mathbf{C})/SO(n)$ $(n\geq 4)$	$\begin{cases} \frac{(n-1)^2}{4} \\ \frac{n(n-2)}{4}\end{cases}$	0 0	$\frac{(n-1)^2}{2}$ $(n:\text{odd})$ $\frac{n(n-2)}{2}$ $(n:\text{even})$
(II-C)	$Sp(n,\mathbf{C})/Sp(n)$	n^2	0	$2n^2$
(II-E_6)	$E_6^{\mathbf{c}}/E_6$	36	0	72
(II-E_7)	$E_7^{\mathbf{c}}/E_7$	63	0	126
(II-E_8)	$E_8^{\mathbf{c}}/E_8$	120	0	240
(II-F_4)	$F_4^{\mathbf{c}}/F_4$	24	0	48
(II-G_2)	$G_2^{\mathbf{c}}/G_2$	6	0	12

Table 1.

2. Basic notions and facts

In this section, we recall basic notions introduced in [16-19]. We first recall the notion of a complex equifocal submanifold. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T^\perp_x M$ and $X \in T_x M$ $(x = gK)$. Denote by γ_v the geodesic in N with $\dot\gamma_v(0) = v$. The strongly M-Jacobi field Y along γ_v with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D^{co}_{sv} - sD^{si}_{sv} \circ A_v))(X),$$

where $Y'(0) = \widetilde\nabla_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$ and D^{co}_{sv} (resp. D^{si}_{sv}) is given by

$$D^{co}_{sv} = g_* \circ \cos(\sqrt{-1}\mathrm{ad}(sg_*^{-1}v)) \circ g_*^{-1}$$
$$\left(\text{resp. } D^{si}_{sv} = g_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}\right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M along γ_v are obtained as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v} - s_0 D^{si}_{s_0v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v} - z_0 D^{si}_{z_0v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D^{co}_{z_0v} - z_0 D^{si}_{z_0v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $A^{\mathbf{c}}_v$ is the complexification of A_v and $D^{co}_{z_0v}$ (resp. $D^{si}_{z_0v}$) is a $\mathbf{C}$-linear transformation of $(T_xN)^{\mathbf{c}}$ defined by

$$D^{co}_{z_0v} = g^{\mathbf{c}}_* \circ \cos(\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v)) \circ (g^{\mathbf{c}}_*)^{-1}$$
$$\left(\text{resp. } D^{si}_{sv} = g^{\mathbf{c}}_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v))}{\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v)} \circ (g^{\mathbf{c}}_*)^{-1}\right),$$

where $g^{\mathbf{c}}_*$ (resp. $\mathrm{ad}^{\mathbf{c}}$) is the complexification of g_* (resp. ad). Here we note that, in the case where M is of class C^ω, complex focal radii along γ_v indicate the positions of focal points of the extrinsic complexification $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}})$ of M along the complexified geodesic $\gamma^{\mathbf{c}}_{\iota_* v}$, where $G^{\mathbf{c}}/K^{\mathbf{c}}$ is the anti-Kaehler symmetric space associated with G/K and ι is the natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$. See Section 4 of [17] about the definitions of $G^{\mathbf{c}}/K^{\mathbf{c}}$, $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}})$ and $\gamma^{\mathbf{c}}_{\iota_* v}$. Also, for a complex focal radius z_0 of M along γ_v, we call $z_0 v$ $(\in (T^\perp_x M)^{\mathbf{c}})$ a *complex focal normal vector of M at x*. Furthermore, assume that M has globally flat normal bundle, that is, the normal holonomy group of M is trivial. Let $\tilde v$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde v_x}$ is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots\}$ be the set of all complex focal radii along $\gamma_{\tilde v_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} < 0$". Let r_i $(i = 1, 2, \cdots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \cdots)$ *complex focal radius functions for*

$\tilde{v}$. We call $r_i\tilde{v}$ a *complex focal normal vector field for* $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*.

Next we shall recall the notion of a proper complex equifocal submanifold. For its purpose, we first recall the notion of a proper complex isoparametric submanifold in a pseudo-Hilbert space. Let M be a pseudo-Riemannian Hilbert submanifold in a pseudo-Hilbert space $(V, \langle\ ,\ \rangle)$ immersed by f. See Section 2 of [16] about the definitions of a pseudo-Hilbert space and a pseudo-Riemannian Hilbert submanifold. Denote by A the shape tensor of M and by $T^\perp M$ the normal bundle of M. Note that, for $v \in T^\perp M$, A_v is not necessarily diagonalizable with respect to an orthonormal base. We call M a *Fredholm pseudo-Riemannian Hilbert submanifold* (or simply *Fredholm submanifold*) if the following conditions hold:

(F-i) M is of finite codimension,

(F-ii) There exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ such that $(V, \langle\ ,\ \rangle_{V_\pm})$ is a Hilbert space and that, for each $v \in T^\perp M$, A_v is a compact operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$.

Since A_v is a compact operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$, the operator $\mathrm{id} - A_v$ is a Fredholm operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$ and hence the normal exponential map $\exp^\perp : T^\perp M \to V$ of M is a Fredholm map with respect to the metric of $T^\perp M$ naturally defined from $f^*\langle\ ,\ \rangle_{V_\pm}$ and $\langle\ ,\ \rangle_{V_\pm}$, where id is the identity transformation of TM. The spectrum of the complexification $A_v^{\mathbf{c}}$ of A_v is described as $\{0\} \cup \{\lambda_i \,|\, i = 1, 2, \cdots\}$, where "$|\lambda_i| > |\lambda_{i+1}|$" or "$|\lambda_i| = |\lambda_{i+1}|$ & $\mathrm{Re}\,\lambda_i > \mathrm{Re}\,\lambda_{i+1}$" or "$|\lambda_i| = |\lambda_{i+1}|$ & $\mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_{i+1}$ & $\mathrm{Im}\,\lambda_i = -\mathrm{Im}\,\lambda_{i+1} > 0$". We call λ_i the *i-th complex principal curvature of direction* v. Assume that M has globally flat normal bundle. Fix a parallel normal vector field $\tilde{v}$ on M. Assume that the number (which may be ∞) of distinct complex principal curvatures of $\tilde{v}_x$ is independent of the choice of $x \in M$. Then we can define functions $\tilde{\lambda}_i$ $(i = 1, 2, \cdots)$ on M by assigning the i-th complex principal curvature of direction $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\lambda}_i$ the *i-th complex principal curvature function of direction* $\tilde{v}$. If M is a Fredholm submanifold with globally flat normal bundle satisfying the following condition (CI), then we call M a *complex isoparametric submanifold*:

(CI) for each parallel normal vector field $\tilde{v}$, the number of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$ and each complex principal curvature function of direction $\tilde{v}$ is constant on M and has constant multiplicity.

Furthermore, if, for each $v \in T^\perp M$, there exists a pseudo-orthonormal base of $(T_xM)^{\mathbf{c}}$ (x : the base point of v) consisting of the eigenvectors of the complexified shape operator $A_v^{\mathbf{c}}$, then we call M a *proper complex isoparametric submanifold*. Then, for each $x \in M$, there exists a pseudo-orthonormal base of $(T_xM)^{\mathbf{c}}$ consisting of the common-eigenvectors of the complexified shape operators $A_v^{\mathbf{c}}$'s ($v \in T_x^\perp M$)

because $A_v^{\mathbf{c}}$'s are commutative. Let $\{E_i \mid i \in I\}$ $(I \subset \mathbf{N})$ be the family of subbundles of $(TM)^{\mathbf{c}}$ such that, for each $x \in M$, $\{E_i(x) \mid i \in I\}$ is the set of all common-eigenspaces of $A_v^{\mathbf{c}}$'s $(v \in T_x^\perp M)$. Note that $(T_xM)^{\mathbf{c}} = \overline{\bigoplus_{i\in I} E_i(x)}$ holds. There exist smooth sections λ_i $(i \in I)$ of $((T^\perp M)^{\mathbf{c}})^*$ such that $A_v^{\mathbf{c}} = \lambda_i(v)\mathrm{id}$ on $E_i(\pi(v))$ for each $v \in T^\perp M$, where π is the bundle projection of $(T^\perp M)^{\mathbf{c}}$. We call λ_i $(i \in I)$ *complex principal curvatures of* M and call subbundles E_i $(i \in I)$ of $(T^\perp M)^{\mathbf{c}}$ *complex curvature distributions of* M. Note that $\lambda_i(v)$ is one of the complex principal curvatures of direction v. Set $l_i := \lambda_i^{-1}(1)\,(\subset (T_x^\perp M)^{\mathbf{c}})$ and R_i be the complex reflection of order two with respect to l_i, where $i \in I$. Denote by W_M the group generated by R_i's $(i \in I)$ which is independent of the choice of $x \in M$ up to isomorphicness. We call l_i's *complex focal hyperplanes of* (M, x). Let $N = G/K$ be a symmetric space of non-compact type and π be the natural projection of G onto G/K. Let $(\mathfrak{g}, \sigma)$ be the orthogonal symmetric Lie algebra of G/K, $\mathfrak{f} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, which is identified with the tangent space $T_{eK}N$. Let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form of $\mathfrak{g}$ inducing the Riemannian metric of N. Note that $\langle\ ,\ \rangle|_{\mathfrak{f}\times\mathfrak{f}}$ (resp. $\langle\ ,\ \rangle|_{\mathfrak{p}\times\mathfrak{p}}$) is negative (resp. positive) definite. Denote by the same symbol $\langle\ ,\ \rangle$ the bi-invariant pseudo-Riemannian metric of G induced from $\langle\ ,\ \rangle$ and the Riemannian metric of N. Set $\mathfrak{g}_+ := \mathfrak{p}$, $\mathfrak{g}_- := \mathfrak{f}$ and $\langle\ ,\ \rangle_{\mathfrak{g}_\pm} := -\pi_{\mathfrak{g}_-}^*\langle\ ,\ \rangle + \pi_{\mathfrak{g}_+}^*\langle\ ,\ \rangle$, where $\pi_{\mathfrak{g}_-}$ (resp. $\pi_{\mathfrak{g}_+}$) is the projection of $\mathfrak{g}$ onto $\mathfrak{g}_-$ (resp. $\mathfrak{g}_+$). Let $H^0([0,1], \mathfrak{g})$ be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}$ (with respect to $\langle\ ,\ \rangle_{\mathfrak{g}_\pm}$). Let $H^0([0,1], \mathfrak{g}_-)$ (resp. $H^0([0,1], \mathfrak{g}_+)$) be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}_-$ (resp. $u : [0,1] \to \mathfrak{g}_+$) with respect to $-\langle\ ,\ \rangle|_{\mathfrak{g}_-\times\mathfrak{g}_-}$ (resp. $\langle\ ,\ \rangle|_{\mathfrak{g}_+\times\mathfrak{g}_+}$). It is clear that $H^0([0,1], \mathfrak{g}) = H^0([0,1], \mathfrak{g}_-) \oplus H^0([0,1], \mathfrak{g}_+)$. Define a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle_0$ of $H^0([0,1], \mathfrak{g})$ by $\langle u, v\rangle_0 := \int_0^1 \langle u(t), v(t)\rangle dt$. It is easy to show that the decomposition $H^0([0,1], \mathfrak{g}) = H^0([0,1], \mathfrak{g}_-) \oplus H^0([0,1], \mathfrak{g}_+)$ is an orthogonal time-space decomposition with respect to $\langle\ ,\ \rangle_0$. For simplicity, set $H^0_\pm := H^0([0,1], \mathfrak{g}_\pm)$ and $\langle\ ,\ \rangle_{0,H^0_\pm} := -\pi_{H^0_-}^*\langle\ ,\ \rangle_0 + \pi_{H^0_+}^*\langle\ ,\ \rangle_0$, where $\pi_{H^0_-}$ (resp. $\pi_{H^0_+}$) is the projection of $H^0([0,1], \mathfrak{g})$ onto H^0_- (resp. H^0_+). It is clear that $\langle u, v\rangle_{0,H^0_\pm} = \int_0^1 \langle u(t), v(t)\rangle_{\mathfrak{g}_\pm} dt$ $(u,\ v \in H^0([0,1], \mathfrak{g}))$. Hence $(H^0([0,1], \mathfrak{g}), \langle\ ,\ \rangle_{0,H^0_\pm})$ is a Hilbert space, that is, $(H^0([0,1], \mathfrak{g}), \langle\ ,\ \rangle_0)$ is a pseudo-Hilbert space. Let $H^1([0,1], G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G$ such that the weak derivative g' of g is squared integrable (with respect to $\langle\ ,\ \rangle_{\mathfrak{g}_\pm}$), that is, $g_*^{-1}g' \in H^0([0,1], \mathfrak{g})$. Define a map $\phi : H^0([0,1], \mathfrak{g}) \to G$ by $\phi(u) = g_u(1)$ $(u \in H^0([0,1], \mathfrak{g}))$, where g_u is the element of $H^1([0,1], G)$ satisfying $g_u(0) = e$ and $g_{u*}^{-1}g_u' = u$. We call this map the *parallel transport map* (from 0 to 1). This submersion ϕ is a pseudo-Riemannian submersion of $(H^0([0,1], \mathfrak{g}), \langle\ ,\ \rangle_0)$ onto $(G, \langle\ ,\ \rangle)$. Let $\pi : G \to G/K$ be the natural projection. It follows from Theorem A of [16] (resp. Theorem 1 of [17]) that, in the case where M is curvature adapted (resp. of class C^ω), M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. In particular, if components of $(\pi \circ \phi)^{-1}(M)$ are proper complex isoparametric, then we call M a *proper complex equifocal submanifold*. Let M be a proper complex equifocal

C^ω-submanifold in G/K, $\widetilde{M_0}$ be a component of $\widetilde{M} := (\pi\circ\phi)^{-1}(M)$. Denote by $W_{\widetilde{M_0}}$ the group defined as above for this proper complex isoparametric submanifold $\widetilde{M_0}$, where we take u_0 as the base point.

Let $N = G/K$ be a symmetric space of non-compact type, $(\mathfrak{g},\sigma)$ be the orthogonal symmetric Lie algebra associated with a symmetric pair (G,K) and $\mathfrak{g} = \mathfrak{f}+\mathfrak{p}$ be the Cartan decomposition. Note that $\mathfrak{f}$ is the Lie algebra of K and $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$, where e is the identity element of G. Let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}(G)$-invariant non-degenerate inner product of $\mathfrak{g}$ inducing the Riemannian metric of N. Let $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{f}^{\mathbf{c}}$, $\mathfrak{p}^{\mathbf{c}}$ and $\langle\ ,\ \rangle^{\mathbf{c}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{f}$, $\mathfrak{p}$ and $\langle\ ,\ \rangle$, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{a} + \sum\limits_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. Then $(\mathfrak{g}^{\mathbf{c}},\mathfrak{f}^{\mathbf{c}})$ is a semi-simple symmetric pair, $\mathfrak{a}$ is a maximal split abelian subspace of $\mathfrak{p}^{\mathbf{c}}$ and $\mathfrak{p}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum\limits_{\alpha\in\triangle_+}\mathfrak{p}^{\mathbf{c}}_\alpha$ is the root space decomposition with respect to $\mathfrak{a}$, where $\mathfrak{a}^{\mathbf{c}}$ and $\mathfrak{p}^{\mathbf{c}}_\alpha$ are the complexifications of $\mathfrak{a}$ and $\mathfrak{p}_\alpha$, respectively. Note that $\mathfrak{a}^{\mathbf{c}}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{p}^{\mathbf{c}}$. See [25] and [23] about the general theory of a semi-simple symmetric pair. Let $G^{\mathbf{c}}$ (resp. $K^{\mathbf{c}}$) be the complexification of G (resp. K). The 2-multiple of the real part $\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}$ of $\langle\ ,\ \rangle^{\mathbf{c}}$ is the Killing form of $\mathfrak{g}^{\mathbf{c}}$ regarded as a real Lie algebra. The restriction $2\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}}\times\mathfrak{p}^{\mathbf{c}}}$ is an $\mathrm{Ad}(K^{\mathbf{c}})$- invariant non-degenerate inner product of $\mathfrak{p}^{\mathbf{c}}$ $(= T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}))$. Denote by $\langle\ ,\ \rangle'$ the $G^{\mathbf{c}}$-invariant pseudo-Riemannian metric on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from $2\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}}\times\mathfrak{p}^{\mathbf{c}}}$. Define an almost complex structure J_0 of $\mathfrak{p}^{\mathbf{c}}$ by $J_0(X+\sqrt{-1}Y) = -Y+\sqrt{-1}X$ $(X,Y\in\mathfrak{p})$. It is clear that J_0 is $\mathrm{Ad}(K^{\mathbf{c}})$-invariant. Denote by $\widetilde{J}$ the $G^{\mathbf{c}}$-invariant almost complex structure on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from J_0. It is shown that $(G^{\mathbf{c}}/K^{\mathbf{c}},\langle\ ,\ \rangle',\widetilde{J})$ is an anti-Kaehlerian manifold and a (semi-simple) pseudo-Riemannian symmetric space. We call this anti-Kaehlerian manifold an *anti-Kaehlerian symmetric space associated with* G/K and simply denote it by $G^{\mathbf{c}}/K^{\mathbf{c}}$. Let $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection and $\phi^{\mathbf{c}} : H^0([0,1],\mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$. This map $\phi^{\mathbf{c}}$ is defined in similar to ϕ (see Section 6 of [17] in detail). Let M be a complete C^ω-submanifold in G/K and $M^{\mathbf{c}}$ be the extrinsic complexification of M. Let $\widetilde{M^{\mathbf{c}}_0}$ be a component of $\widetilde{M^{\mathbf{c}}} := (\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$. In [19], we called the group generated by complex reflections of order two with respect to complex focal hyperplanes constructing the focal set of $\widetilde{M^{\mathbf{c}}_0}$ at an arbitrary fixed point u_1 *the complex Coxeter group associated with* M. Denote by W_M this group, which is discrete (see Proposition 3.7 of [19]). Since the complex focal hyperplanes of $\widetilde{M_0}$ at u_0 coincides with those of $\widetilde{M^{\mathbf{c}}_0}$ at u_1 under some identification of $(T^\perp_{u_0}\widetilde{M_0})^{\mathbf{c}}$ with $T^\perp_{u_1}(\widetilde{M^{\mathbf{c}}_0})$, we see that $W_{\widetilde{M_0}}$ is isomorphic to W_M.

At the end of this section, we recall the notions of the Weyl group and the affine Weyl group associated with a root system. Let $\triangle$ be a subset of the dual space E^* of a Euclidean space E consisting of non-zero vectors. We consider the following three conditions:

(i) If $\alpha,\,\beta\in\triangle$, then $s_\alpha(\beta)\in\triangle$, where s_α is the reflection with respect to

$\alpha^{-1}(0)$,

(ii) If $\alpha,\ \beta \in \triangle$, then $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbf{Z}$,

(iii) If $\alpha,\ a\alpha \in \triangle$ $(a \in \mathbf{R})$, then $a = \pm 1$.

If $\triangle$ satisfies the conditions (i) and (ii), then it is called a *root system* and furthermore, if $\triangle$ satisfies the condition (iii), then it is said to be *reduced* (see [13]). Here we note that, if $\triangle$ satisfies the conditions (i) and (iii), then it is called a root system in [15]. In this paper, if $\triangle$ satisfies only the condition (i), then we call it a *weakly root system*. For a weakly root system $\triangle$, we denote by $W_\triangle$ the group generated by the reflection's with respect to $\alpha^{-1}(0)$'s $(\alpha \in \triangle)$ and denote by $W_\triangle^A$ the affine transformation group generated by the reflections with respect to $\alpha^{-1}(j)$'s $(\alpha \in \triangle,\ j \in \mathbf{Z})$. We call $W_\triangle$ the *linear transformation group associated with* $\triangle$ and $W_\triangle^A$ the *affine transformation group associated with* $\triangle$. If $W_\triangle$ is finite, then we call $W_\triangle$ the *Coxeter group associated with* $\triangle$ and, if $\triangle$ is a root system, then $W_\triangle$ is called the *Weyl group associated with* $\triangle$. Also, if $W_\triangle^A$ is discrete, then we call $W_\triangle^A$ the *affine Weyl group associated with* $\triangle$.

3. Focal points on the ideal boundary

In this section, we introduce the notion of a focal point on the ideal boundary $N(\infty)$ for a submanifold M in a Hadamard manifold N. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of N and A the shape tensor of M. Let $\gamma_v : [0,\infty) \to N$ be the normal geodesic of M of direction $v \in T_x^\perp M$. If there exists a M-Jacobi field (resp. strongly M-Jacobi field) Y along γ_v satisfying $\lim\limits_{t\to\infty} \frac{||Y_t||}{t} = 0$, then we call $\gamma_v(\infty)\,(\in N(\infty))$ a *focal point* (resp. *strongly focal point*) *on the ideal boundary* $N(\infty)$ *of* M *along* γ_v, where $\gamma_v(\infty)$ is the asymptotic class of γ_v (see Fig. 1). Here a M-Jacobi field along γ_v implies a Jacobi field Y along γ_v satisfying $Y(0) \in T_xM$ and $Y'(0)_T = -A_vY(0)$ and a strongly M-Jacobi field along γ_v implies a Jacobi field Y along γ_v satifying $Y(0) \in T_xM$ and $Y'(0) = -A_vY(0)$, where $Y'(0) = \widetilde{\nabla}_vY$ and $Y'(0)_T$ is the tangential (to M) component of $Y'(0)$. We call $\mathrm{Span}\{Y_0\,|\,Y\ :\ \text{a } M\text{−Jacobi field along } \gamma_v \text{ s.t. } \lim_{t\to\infty} \frac{||Y_t||}{t} = 0\}$ the *nullity space* of the focal point $\gamma_v(\infty)$. Also, if there exists a M-Jacobi field Y along γ_v satisfying $\lim\limits_{t\to\infty} \dfrac{||Y_t||}{t} = 0$ and $\mathrm{Sec}(v, Y(0)) < 0$, then we call $\gamma_v(\infty)$ a *focal point of non-Euclidean type on* $N(\infty)$ *of* M *along* γ_v, where $\mathrm{Sec}(v, Y(0))$ is the sectional curvature for the 2-plane spanned by v and $Y(0)$. If $\exp^\perp(T_x^\perp M)$ is totally geodesic for each $x \in M$, M is called a *submanifold with section*. This notion has been recently defined in [11]. For a submanifold with section in a symmetric space of non-compact type, we have the following fact.

Proposition 3.1. *Let M be a submanifold with section in a symmetric space $N := G/K$ of non-compact type and v be a normal vector of M at x. Then the following conditions* (i) *and* (ii) *are equivalent:*

(i) *$\gamma_v(\infty)$ is a focal point on $N(\infty)$ of M along γ_v,*

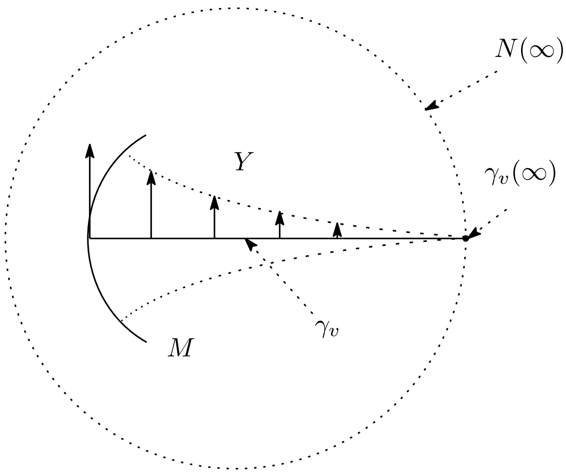

Fig. 1.

(ii) *$\gamma_v(\infty)$ is a strongly focal point on $N(\infty)$ of M along γ_v.*

Furthermore, if M is homogeneous (hence it is a principal orbit of a polar action H on N), then these conditions are equivalent to the following conditions:

(iii) *there exists a normal geodesic variation $\delta : [0,\infty) \times (-\varepsilon,\varepsilon) \to N$ such that $\delta(\cdot,0) = \gamma_v(\cdot)$, the variational vector field $\frac{\partial\delta}{\partial s}|_{s=0}$ is a strongly M-Jacobi field and that $\delta(\cdot,s)(\infty) = \gamma_v(\infty)$ for any $s \in (-\varepsilon,\varepsilon)$.*

(iv) *the action on $N(\infty)$ induced from the H-action posseses a non-trivial subaction having $\gamma_v(\infty)$ as a fixed point.*

Proof. First we shall show (i) $\Rightarrow$ (ii). Assume that $\gamma_v(\infty)$ is a focal point on $N(\infty)$ along γ_v. Hence there exists an M-Jacobi field Y along γ_v such that $\lim_{t\to\infty} \frac{||Y(t)||}{t} = 0$. The Jacobi field Y is described as

$$Y(t) = P_{\gamma_v|_{[0,t]}}\left(D^{co}_{tv}(Y(0)) + D^{si}_{tv}(-A_{tv}Y(0) + Y'(0)_\perp)\right),$$

where $P_{\gamma_v|_{[0,t]}}$ is the parallel translation along $\gamma_v|_{[0,t]}$, D^{co}_{tv} and D^{si}_{tv} are as in the previous section, A (resp. $\nabla^\perp$) is the shape tensor (resp. the normal connection) of M and $Y'(0)_\perp$ is the normal component of $Y'(0)(=\widetilde{\nabla}_v Y)$. Since M has section, we have $D^{co}_{tv}(Y(0)), D^{si}_{tv}(A_{tv}Y(0)) \in T_xM$. Hence we have $||Y(t)|| \geq ||(D^{co}_{tv} - D^{si}_{tv} \circ A_{tv})(Y(0))||$. The strongly M-Jacobi field Y^S along γ_v with $Y^S(0) = Y(0)$ is described as

$$Y^S(t) = P_{\gamma_v|_{[0,t]}}\left((D^{co}_{tv} - D^{si}_{tv} \circ A_{tv})(Y(0))\right). \tag{3.1}$$

Hence we have $||Y(t)|| \geq ||Y^S(t)||$ and hence $\lim_{t\to 0} \frac{||Y^S(t)||}{t} = 0$. Thus $\gamma_v(\infty)$ is a strongly focal point on $N(\infty)$ along γ_v. Thus we have (i) $\Rightarrow$ (ii). The converse (ii) $\Rightarrow$ (i) is trivial.

Next we shall show that (ii) $\Rightarrow$ (iii) holds if M is homogeneous. Assume that $\gamma_v(\infty)$ is a strongly focal point on $N(\infty)$ along γ_v. Hence there exists a strongly M-Jacobi field Y^S along γ_v with $\lim_{t\to 0}\frac{\|Y^S(t)\|}{t}=0$. Since Y^S is described as in (3.1), we have $\|Y^S(t)\|=\|(D^{co}_{tv}-D^{si}_{tv}\circ A_{tv})(Y^S(0))\|$. Since M is a homogeneous submanifold with section, it is catched as a principal orbit of some complex polar action $H\times G/K\to G/K$ $(H\subset G)$. See [17] about the definition of a complex polar action. Let $\{\exp sX\mid s\in\mathbf{R}\}$ be a one-parameter subgroup of H such that $\frac{d(\exp sX)(x)}{ds}|_{s=0}=Y(0)$. Set $\alpha(s):=(\exp sX)(x)$. Let $\widetilde{v}$ be the parallel normal vector field along α with $\widetilde{v}_0=v$. Define $\delta:[0,\infty)\times(-\varepsilon,\varepsilon)\to N$ by $\delta(t,s):=\exp^{\perp}(t\widetilde{v}_s)$. We have $\frac{\partial\delta}{\partial s}|_{s=0}=Y^S$. Set $Y^S_{s_0}:=\frac{\partial\delta}{\partial s}|_{[0,\infty)\times\{s_0\}}$ for each $s_0\in(-\varepsilon,\varepsilon)$. Since $Y^S_{s_0}$ is a strongly M-Jacobi field along $\delta(\cdot,s_0)$, it is described as in (3.1). Hence we have $\|Y^S_{s_0}(t)\|=\|(D^{co}_{t\widetilde{v}_s}-D^{si}_{t\widetilde{v}_s}\circ A_{t\widetilde{v}_s})(Y^S_{s_0}(0))\|$. Since M is a principal orbit of the H-action, we have $\widetilde{v}_s=(\exp sX)_*(v)$ $(s\in(-\varepsilon,\varepsilon))$. From this fact, we have $\|(D^{co}_{t\widetilde{v}_{s_0}}-D^{si}_{t\widetilde{v}_{s_0}}\circ A_{t\widetilde{v}_{s_0}})(Y^S_{s_0}(0))\|=\|(D^{co}_{tv}-D^{si}_{tv}\circ A_{tv})(Y^S(0))\|$ $((t,s_0)\in[0,\infty)\times(-\varepsilon,\varepsilon))$. Therefore, we have $\|Y^S_{s_0}(t)\|=\|Y^S(t)\|$ $((t,s_0)\in[0,\infty)\times(-\varepsilon,\varepsilon))$. Hence we have

$$\begin{aligned}\lim_{t\to\infty}\frac{d(\delta(t,s_0),\gamma_v(t))}{t}&\leq\lim_{t\to\infty}\frac{1}{t}\int_0^{s_0}\|Y^S_s(t)\|ds\\&=\lim_{t\to\infty}\frac{s_0\|Y^S(t)\|}{t}=0,\end{aligned}$$

that is, $\delta(\cdot,s_0)(\infty)=\gamma_v(\infty)$. Thus (ii) $\Rightarrow$ (iii) is shown. The converse is trivial. Also, (iii) $\Leftrightarrow$ (iv) is trivial. This completes the proof. □

Remark 3.1. Let γ be a normal geodesic of a princial orbit M of a polar action H on $N=G/K$. If $\gamma(\infty)$ is a fixed point of the action on $N(\infty)$ induced from the H-action, then $\gamma(\infty)$ is a focal point of M along γ having $T_{\gamma(0)}M$ as the nullity space.

Now we shall illustrate that the second-half of the statement in Proposition 3.1 does not hold without the assumption of the homogeneity of M. Let S be a horosphere in a symmetric space $N=G/K$ of non-compact type and M be a non-homogeneous hypersurface in N through $x\in S$ such that $j^2_x(\iota_M)=j^2_x(\iota_S)$ but $j^3_x(\iota_M)\neq j^3_x(\iota_S)$ and that M positions outside or inside S (see Fig. 2), where ι_M (resp. ι_S) is the inclusion map of M (resp. S) into N and $j^2_x(\cdot)$ is the 2-jet of $\cdot$ at x. Since M is a hypersurface, it has sections. Let v be the inward unit normal vector of S at x. Then $\gamma_v(\infty)$ is a focal point on $N(\infty)$ of M along γ_v but there does not exist a normal (to M) geodesic variation $\delta:[0,\infty)\times(-\varepsilon,\varepsilon)\to N$ of γ_v such that $\delta(\cdot,0)=\gamma_v$, the variational vector field $\frac{\partial\delta}{\partial s}|_{s=0}$ is a strongly M-Jacobi field and that $\delta(\cdot,s_0)(\infty)=\gamma_v(\infty)$ for each $s_0\in(-\varepsilon,\varepsilon)$. Thus the second-half of the statement in Proposition 3.1 does not hold without the assumption of the homogeneity of M. Since S is complex equifocal, $j^2_x(\iota_M)=j^2_x(\iota_S)$ and $j^3_x(\iota_M)\neq j^3_x(\iota_S)$, we see that M is not complex equifocal. In more general, it is conjectured that the second-half of the statement in Proposition 3.1 holds if M is complex equifocal.

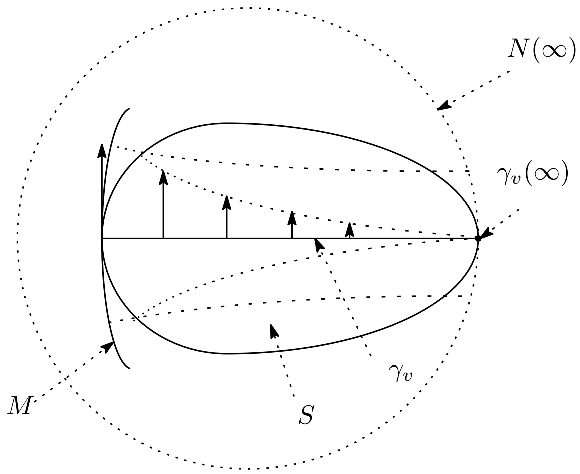

Fig. 2.

4. Proof of Theorem A

In this section, we shall prove Theorem A stated in Introduction. First we prepare the following lemma.

Lemma 4.1. *Let M be a curvature-adapted submanifold with section in a symmetric space $N = G/K$ of non-compact type. Then the following conditions* (i) *and* (ii) *are equivalent:*

(i) *M has no focal point of non-Euclidean type on $N(\infty)$,*

(ii) *for each unit normal vector v of M and each $\mu \in \operatorname{Spec} R(\cdot, v)v \setminus \{0\}$, $\pm\sqrt{-\mu}$ are not eigenvalues of $A_v|_{\operatorname{Ker}(R(\cdot,v)v-\mu\,\mathrm{id})}$, where R is the curvature tensor of G/K and A is the shape tensor of M.*

Proof. First we note that the condition (i) is equivalent to the following condition:

(i′) M has no strongly focal point of non-Euclidean type on $N(\infty)$.

In fact, this fact follows from Proposition 3.1 because M has sections. Fix a unit normal vector v of M at any $x = gK \in M$. Since M is curvature-adapted, we have

$$T_xM = \bigoplus_{\lambda\in\operatorname{Spec} A_v} \bigoplus_{\mu\in\operatorname{Spec} R(\cdot,v)v} (\operatorname{Ker}(R(\cdot,v)v - \mu\,\mathrm{id}) \cap \operatorname{Ker}(A_v - \lambda\,\mathrm{id})), \tag{4.1}$$

where $\operatorname{Spec}(\cdot)$ is the spectrum of $(\cdot)$. A strongly M-Jacobi field Y along γ_v with $Y(0) \in \operatorname{Ker}(R(\cdot,v)v - \mu\,\mathrm{id}) \cap \operatorname{Ker}(A_v - \lambda\,\mathrm{id})$ $(\mu \neq 0)$ is described as

$$\begin{aligned} Y(t) &= P_{\gamma_v|_{[0,t]}}\left((D^{co}_{tv} - D^{si}_{tv} \circ A_{tv})(Y(0))\right) \\ &= \left(\cosh(t\sqrt{-\mu}) - \frac{\lambda\sinh(t\sqrt{-\mu})}{\sqrt{-\mu}}\right) P_{\gamma_v|_{[0,t]}}(Y(0)). \end{aligned}$$

If $\lambda = \pm\sqrt{-\mu}$, then we have $||Y(t)|| = ||Y(0)||e^{\pm t\sqrt{-\mu}}$. Hence we have $\lim_{t\to\infty} \frac{||Y(t)||}{t} = 0$ or $\lim_{t\to-\infty} \frac{||Y(t)||}{t} = 0$. Also, from $Y(0) \in \operatorname{Ker}(R(\cdot,v)v - \mu\,\mathrm{id})$ and $\mu \neq 0$, we have $\mathrm{Sec}(v, Y(0)) < 0$. Hence, either $\gamma_v(\infty)$ or $\gamma_{-v}(\infty)$ is a strongly focal point of non-Euclidean type on $N(\infty)$ of M. Thus not (ii) $\Rightarrow$ not (i), that is, (i) $\Rightarrow$ (ii) is shown. Assume that (ii) holds. Take an arbitrary $X(\neq 0) \in T_xM$ with $\mathrm{Sec}(v, X) < 0$. Let $S := \{(\lambda, \mu) \in \mathrm{Spec}\, A_v \times \mathrm{Spec}\, R(\cdot, v)v \,|\, \mathrm{Ker}(R(\cdot, v)v - \mu\,\mathrm{id}) \cap \mathrm{Ker}(A_v - \lambda\,\mathrm{id}) \neq \{0\}\}$ and $S_0 := \{(\lambda, \mu) \in S \,|\, \mu = 0\}$. Let $X = \sum_{(\lambda,\mu)\in S} X_{\lambda,\mu}$, where $X_{\lambda,\mu} \in \mathrm{Ker}(R(\cdot, v)v - \mu\,\mathrm{id}) \cap \mathrm{Ker}(A_v - \lambda\,\mathrm{id})$. The strongly M-Jacobi field Y along γ_v with $Y(0) = X$ is described as

$$
\begin{aligned}
Y(t) &= P_{\gamma_v|_{[0,t]}}\left((D^{co}_{tv} - D^{si}_{tv} \circ A_{tv})(X)\right) \\
&= \sum_{(\lambda,\mu)\in S\setminus S_0} \left(\cosh(t\sqrt{-\mu}) - \frac{\lambda \sinh(t\sqrt{-\mu})}{\sqrt{-\mu}}\right) P_{\gamma_v|_{[0,t]}}(X_{\lambda,\mu}) \\
&\quad + \sum_{(\lambda,\mu)\in S_0} (1 - t\lambda) P_{\gamma_v|_{[0,t]}}(X_{\lambda,\mu}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
||Y(t)||^2 = &\sum_{(\lambda,\mu)\in S\setminus S_0} \left(\cosh(t\sqrt{-\mu}) - \frac{\lambda \sinh(t\sqrt{-\mu})}{\sqrt{-\mu}}\right)^2 ||X_{\lambda,\mu}||^2 \\
&+ \sum_{(\lambda,\mu)\in S_0} (1 - t\lambda)^2 ||X_{\lambda,\mu}||^2.
\end{aligned}
$$

Since $\mathrm{Sec}(v, X) < 0$, there exists $(\lambda_0, \mu_0) \in S \setminus S_0$ with $X_{\lambda_0,\mu_0} \neq 0$. Then we have

$$
\begin{aligned}
\frac{||Y(t)||^2}{t^2} &\geq \frac{1}{t^2}\left(\cosh(t\sqrt{-\mu_0}) - \frac{\lambda_0 \sinh(t\sqrt{-\mu_0})}{\sqrt{-\mu_0}}\right)^2 ||X_{\lambda_0,\mu_0}||^2 \\
&= \frac{1}{2t^2}\left((1 - \frac{\lambda_0}{\sqrt{-\mu_0}})e^{t\sqrt{-\mu_0}} + (1 + \frac{\lambda_0}{\sqrt{-\mu_0}})e^{-t\sqrt{-\mu_0}}\right)^2 ||X_{\lambda_0,\mu_0}||^2.
\end{aligned}
$$

Hence, since $\lambda_0 \neq \pm\sqrt{-\mu_0}$ by the assumption, we have $\lim_{t\to\infty} \frac{||Y(t)||}{t} = \infty$. From the arbitrariness of $X \in T_xM$, it follows that $\gamma_v(\infty)$ is not a focal point of non-Euclidean type (on $N(\infty)$) of M. Furthermore, from the arbitrarinesses of v and x, we see that M has no focal point of non-Euclidean type on $N(\infty)$. Thus (ii) $\Rightarrow$ (i) is shown. □

Remark 4.1. By imitating the proof of this lemma, it is shown that the following conditions (i′) and (ii′) are equivalent:

(i′) M has no focal point on $N(\infty)$,

(ii′) for each unit normal vector v of M and each $\mu \in \mathrm{Spec}\, R(\cdot, v)v$, $\pm\sqrt{-\mu}$ are not eigenvalues of $A_v|_{\mathrm{Ker}(R(\cdot,v)v - \mu I)}$.

By using this lemma, the statement of Theorem A is proved.

Proof of Theorem A. According to the proof of the statement (ii) of Theorem 1 in

[17], M is proper complex equifocal if and only if the following condition $(*)$ holds:

$(*)$ for each unit normal vector v of $M^{\mathbf{c}}$ and each $\mu \in \mathrm{Spec}_J R^{\mathbf{c}}(\cdot, v)v \setminus \{0\}$, $\sqrt{\mu}$ (2-values) are not J-eigenvalues of $A^{\mathbf{c}}_v|_{\mathrm{Ker}(R^{\mathbf{c}}(\cdot,v)v-\mu\,\mathrm{id})}$, where $A^{\mathbf{c}}$ is the shape tensor of $M^{\mathbf{c}}$, $R^{\mathbf{c}}$ is the curvature tensor of $G^{\mathbf{c}}/K^{\mathbf{c}}$, J is the complex structures of $M^{\mathbf{c}}$ and $\mathrm{Spec}_J(\cdot)$ is the J-spectrum of $(\cdot)$.

It is easy to show that this condition $(*)$ is equivalent to the condition (ii) of Lemma 4.1. Hence the statement of Theorem A follows from Lemma 4.1. □

5. Proofs of Theorems B∼E

In this section, we shall prove Theorems B∼E. For its purpose, we prepare a lemma. Let M be a curvature-adapted and proper complex equifocal C^ω-submanifold in a symmetric space G/K of non-compact type, where we may assume that $eK \in M$ (e : the identity element of G) by operating an element of G to M if necessary and hence the constant path $\hat{0}$ at the zero element 0 of $\mathfrak{g}$ is contained in $\widetilde{M} := (\pi\circ\phi)^{-1}(M)$. Denote by $\widetilde{M}_0$ the component of $\widetilde{M}$ containing $\hat{0}$. Fix a unit normal vector v of M at eK. Set $\mathfrak{p} := T_{eK}(G/K)$ and $\mathfrak{b} := T^\perp_{eK}M$. Let $\mathfrak{p} = \mathfrak{a} + \sum\limits_{\alpha\in\triangle_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to a lexicographically ordered maximal abelian subspace $\mathfrak{a}$ containing $\mathfrak{b}$. Let $\overline{\triangle} := \{\alpha|_{\mathfrak{b}} \,|\, \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$ and $\mathfrak{p} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) + \sum_{\beta\in\overline{\triangle}_+} \mathfrak{p}_\beta$ be the root space decomposition with respect to $\mathfrak{b}$, where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p}$. For convenience, we denote $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ by $\mathfrak{p}_0$. Then we have $\mathfrak{p}_\beta = \sum_{\alpha\in\triangle_+ \text{ s.t. } \alpha|_{\mathfrak{b}}=\pm\beta} \mathfrak{p}_\alpha$ $(\beta \in \overline{\triangle}_+)$ and $\mathfrak{p}_0 = \mathfrak{a} + \sum_{\alpha\in\triangle_+ \text{ s.t. } \alpha|_{\mathfrak{b}}=0} \mathfrak{p}_\alpha$. Let v^L be the horizontal lift of v to $\hat{0}$. Denote by A (resp. $\widetilde{A}$) the shape tensor of M (resp. $\widetilde{M}_0$). According to Theorem 5.9 of [16], we have the following fact.

Lemma 5.1. *If the spectrum of A_v is equal to $\{\lambda_1, \cdots, \lambda_g\}$, then the spectrum of $\widetilde{A}^{\mathbf{c}}_{v^L}$ is given by*

$$\begin{aligned}
&\{0\} \cup \{\lambda_i \,|\, i \in I_0\} \\
&\cup \left(\bigcup_{\mu\in\mathrm{Spec}\,R(\cdot,v)v\setminus\{0\}} \left\{ \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\sqrt{-\mu}}{\lambda_i} + j\pi\sqrt{-1}} \,\Big|\, i \in I^+_\mu,\ j \in \mathbf{Z} \right\} \right) \\
&\cup \left(\bigcup_{\mu\in\mathrm{Spec}\,R(\cdot,v)v\setminus\{0\}} \left\{ \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\lambda_i}{\sqrt{-\mu}} + (j+\frac{1}{2})\pi\sqrt{-1}} \,\Big|\, i \in I^-_\mu,\ j \in \mathbf{Z} \right\} \right),
\end{aligned}$$

where $I_0 = \{i \,|\, \mathrm{Ker}\,R(\cdot,v)v \cap \mathrm{Ker}(A_v - \lambda_i\,\mathrm{id}) \neq \{0\}\}$, $I^+_\mu := \{i \in I_\mu \,|\, |\lambda_i| > \sqrt{-\mu}\}$ and $I^-_\mu := \{i \in I_\mu \,|\, |\lambda_i| < \sqrt{-\mu}\}$ as $I_\mu := \{i \,|\, \mathrm{Ker}(R(\cdot,v)v - \mu\,\mathrm{id}) \cap \mathrm{Ker}(A_v - \lambda_i\,\mathrm{id}) \neq \{0\}\}$.

Now we shall prove Theorems B and C in terms of this lemma.

Proof of Theorems B and C. Let $m_A := \max\limits_{v\in\mathfrak{b}\setminus\{0\}} \sharp\mathrm{Spec}\,A_v$ and $m_R := \max\limits_{v\in\mathfrak{b}\setminus\{0\}} \sharp\mathrm{Spec}\,R(\cdot,v)v$.

Let $U := \{v \in \mathfrak{b} \setminus \{0\} \,|\, \sharp\mathrm{Spec}A_v = m_A,\ \sharp\mathrm{Spec}\,R(\cdot,v)v = m_R\}$, which is an open dense subset of $\mathfrak{b} \setminus \{0\}$. Note that $\mathrm{Spec}\,R(\cdot,v)v = \{-\beta(v)^2 \,|\, \beta \in \overline{\triangle}_+\}$ and, if $v \in U$, then $\beta(v)^2$'s ($\beta \in \overline{\triangle}_+$) are mutually distinct (i.e., $m_R = \sharp\overline{\triangle}_+$). Let $\mathrm{Spec}A_v = \{\lambda_1^v, \cdots, \lambda_{m_A}^v\}$ ($\lambda_1^v > \cdots > \lambda_{m_A}^v$) ($v \in U$). Then it follows from Lemma 5.1 that

$$
\begin{aligned}
&\mathrm{Spec}\widetilde{A}^{\mathbf{c}}_{v^L}\\
&= \{0\} \cup \{\lambda_i^v \,|\, i \in I_0^v\}\\
&\cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \{ \frac{\beta(v)}{\mathrm{arctanh}\frac{\beta(v)}{\lambda_i^v} + j\pi\sqrt{-1}} \,|\, i \in (I_\beta^v)^+,\ j \in \mathbf{Z}\} \right)\\
&\cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \{ \frac{\beta(v)}{\mathrm{arctanh}\frac{\lambda_i^v}{\beta(v)} + (j+\frac{1}{2})\pi\sqrt{-1}} \,|\, i \in (I_\beta^v)^-,\ j \in \mathbf{Z}\} \right)
\end{aligned} \tag{5.1}
$$

for any $v \in U$, where $I_0^v := \{i \,|\, \mathfrak{p}_0 \cap \mathrm{Ker}(A_v - \lambda_i^v \mathrm{id}) \neq \{0\}\}$, $(I_\beta^v)^+ := \{i \in I_\beta^v \,|\, |\lambda_i^v| > |\beta(v)|\}$ and $(I_\alpha^v)^- := \{i \in I_\beta^v \,|\, |\lambda_i^v| < |\beta(v)|\}$ as $I_\beta^v := \{i \,|\, \mathfrak{p}_\beta \cap \mathrm{Ker}(A_v - \lambda_i^v \mathrm{id}) \neq \{0\}\}$. Let F be the sum of all complex focal hyperplanes of $(\widetilde{M}_0, \widehat{0})$. From (5.1), the set

$$
\bigcup_{v \in U} \left(\begin{array}{l}
\{\frac{1}{\lambda_i^v} v^L \,|\, i \in I_0^v \text{ s.t. } \lambda_i^v \neq 0\} \cup \\
\left(\bigcup_{\beta \in \overline{\triangle}_+} \{ \frac{\mathrm{arctanh}\frac{\beta(v)}{\lambda_i^v} + j\pi\sqrt{-1}}{\beta(v)} v^L \,|\, i \in (I_\beta^v)^+, j \in \mathbf{Z}\} \right) \cup \\
\left(\bigcup_{\beta \in \overline{\triangle}_+} \{ \frac{\mathrm{arctanh}\frac{\lambda_i^v}{\beta(v)} + (j+\frac{1}{2})\pi\sqrt{-1}}{\beta(v)} v^L \,|\, i \in (I_\beta^v)^-, j \in \mathbf{Z}\} \right)
\end{array} \right) \tag{5.2}
$$

is contained in F. Fix $v_0 \in U$. Since the set (5.2) is contained in F and F consists of infinitely many complex hyperplanes of $(T^\perp_{\hat{0}}\widetilde{M}_0)^{\mathbf{c}}$, it is shown by delicate argument that there exist the complex linear functions ϕ_i ($i \in I_0^{v_0}$ s.t. $\lambda_i^{v_0} \neq 0$), $\phi^1_{\beta,i,j}$ ($\beta \in \overline{\triangle}_+$, $i \in (I_\beta^{v_0})^+$, $j \in \mathbf{Z}$) and $\phi^2_{\beta,i,j}$ ($\beta \in \overline{\triangle}_+$, $i \in (I_\beta^{v_0})^-$, $j \in \mathbf{Z}$) on $(T^\perp_{\hat{0}}\widetilde{M}_0)^{\mathbf{c}}(= \mathfrak{b}^{\mathbf{c}})$ satisfying $\phi_i(v) = \lambda_i^v$ ($v \in U'$), $\phi^1_{\beta,i,j}(v) = \dfrac{\beta(v)}{\mathrm{arctanh}\frac{\beta(v)}{\lambda_i^v} + j\pi\sqrt{-1}}$ ($v \in U'$) and $\phi^2_{\beta,i,j}(v) = \dfrac{\beta(v)}{\mathrm{arctanh}\frac{\lambda_i^v}{\beta(v)} + (j+\frac{1}{2})\pi\sqrt{-1}}$ ($v \in U'$), respectively, where U' is a sufficiently small neighborhood of v_0 in U. Since $\phi^k_{\beta,i,j}(v) = \frac{\beta(v)\phi^k_{\beta,i,0}(v)}{\beta(v) + j\pi\phi^k_{\beta,i,0}(v)\sqrt{-1}}$ for all $v \in U'$ and all $j \in \mathbf{Z}$ and $\phi^k_{\beta,i,j}$'s are complex linear, we see that $\frac{\beta(v)}{\phi^k_{\beta,i,j}(v)}$ is independent of the choice of $v \in U'$, where $\beta \in \overline{\triangle}_+$ and $(k,i) \in (\{1\} \times (I_\beta^{v_0})^+) \cup (\{2\} \times (I_\beta^{v_0})^-)$. That is, $\dfrac{\beta(v)}{\lambda_i^v}$ ($i \in (I_\beta^{v_0})^+$ ($\beta \in \overline{\triangle}_+$)) and $\dfrac{\lambda_i^v}{\beta(v)}$ ($i \in (I_\beta^{v_0})^-$ ($\beta \in \overline{\triangle}_+$)) are independent of the choices of $v \in U'$. Set $c^+_{\beta,i} := \dfrac{\beta(v_0)}{\lambda_i^{v_0}}$ $(i \in (I^{v_0}_\beta)^+ \ (\beta \in \overline{\triangle}_+))$ and $c^-_{\beta,i} := \dfrac{\lambda_i^{v_0}}{\beta(v_0)}$ $(i \in (I^{v_0}_\beta)^- \ (\beta \in \overline{\triangle}_+))$. Hence we have $\phi^1_{\beta,i,j} = \dfrac{\beta^{\mathbf{c}}|_{\mathfrak{b}^{\mathbf{c}}}}{\mathrm{arctanh} c^+_{\beta,i} + j\pi\sqrt{-1}}$ and $\phi^2_{\beta,i,j} = \dfrac{\beta^{\mathbf{c}}|_{\mathfrak{b}^{\mathbf{c}}}}{\mathrm{arctanh} c^-_{\beta,i} + (j+\frac12)\pi\sqrt{-1}}$. Clearly we have

$$
\begin{aligned}
F = & \left(\bigcup_{i \in I_0^{v_0} \text{ s.t. } \lambda_i^{v_0} \neq 0} \phi_i^{-1}(1)\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^+ \times \mathbf{Z}} (\phi^1_{\beta,i,j})^{-1}(1)\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^- \times \mathbf{Z}} (\phi^2_{\beta,i,j})^{-1}(1)\right) \\
= & \left(\bigcup_{i \in I_0^{v_0} \text{ s.t. } \lambda_i^{v_0} \neq 0} \phi_i^{-1}(1)\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^+ \times \mathbf{Z}} (\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^+_{\beta,i} + j\pi\sqrt{-1})\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^- \times \mathbf{Z}} (\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^-_{\beta,i} + (j+\frac{1}{2})\pi\sqrt{-1})\right).
\end{aligned}
\tag{5.3}
$$

The group $W_{\widetilde{M_0}}$ is generated by the complex reflections of order two with respect to the complex hyperplanes in (5.3) constructing F. This group is isomorphic to the complex Coxeter group W_M associated with M and hence it is discrete and, according to Lemma 3.5 of [19], F is $W_{\widetilde{M_0}}$-invariant. On the other hand, it is easy to show that the complex reflection group generated by the complex reflections of order two with respect to $(\beta^{\mathbf{c}})^{-1}(0)$'s $(\beta \in \overline{\triangle}_+)$ is of rank r, where $r := \mathrm{codim}\, M$. Therefore, we have

$$
\begin{aligned}
F = & \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^+ \times \mathbf{Z}} (\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^+_{\beta,i} + j\pi\sqrt{-1})\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{(i,j) \in (I^{v_0}_\beta)^- \times \mathbf{Z}} (\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^-_{\beta,i} + (j+\frac{1}{2})\pi\sqrt{-1})\right),
\end{aligned}
\tag{5.4}
$$

where we note that $\{i \in I_0^{v_0} \,|\, \lambda_i^{v_0} \neq 0\}$ is not necessarily empty set. Denote by $\mathrm{proj}_{\mathbf{R}}$ the natural projection of $\mathfrak{b}^{\mathbf{c}}$ onto $\mathfrak{b}$ and set $F_{\mathbf{R}} := \mathrm{proj}_{\mathbf{R}}(F)$. Then we have

$$
\begin{aligned}
F_{\mathbf{R}} = & \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{i \in (I^{v_0}_\beta)^+} \beta^{-1}(\mathrm{arctanh} c^+_{\beta,i})\right) \\
& \cup \left(\bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{i \in (I^{v_0}_\beta)^-} \beta^{-1}(\mathrm{arctanh} c^-_{\beta,i})\right).
\end{aligned}
\tag{5.5}
$$

Let $W'_{\widetilde{M}_0}$ be the group generated by the reflections (in $\mathfrak{b}$) with respect to the hyperplanes constructing $F_{\mathbf{R}}$. Since F is $W_{\widetilde{M}_0}$-invariant, $F_{\mathbf{R}}$ is $W'_{\widetilde{M}_0}$-invariant. Therefore, since $F_{\mathbf{R}}$ consists of finite pieces of (real) hyperplanes (in $\mathfrak{b}$), the intersection of all the hyperplanes constructing $F_{\mathbf{R}}$ is non-empty. Take an element Z of the intersection. Then complex hyperplanes in (5.4) constructing F are rewritten as

$$(5.6)\qquad \begin{aligned}&(\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^{+}_{\beta,i}+j\pi\sqrt{-1})=Z+(\beta^{\mathbf{c}})^{-1}(j\pi\sqrt{-1}),\\ &(\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c^{-}_{\beta,i}+(j+\frac{1}{2})\pi\sqrt{-1})=Z+(\beta^{\mathbf{c}})^{-1}((j+\frac{1}{2})\pi\sqrt{-1}),\end{aligned}$$

respectively. Hence we see that $W_{\widetilde{M}_0}$ is isomorphic to the group generated by the (real) reflections with respect to the hyperplanes $\widehat{\beta}^{-1}(j\pi)$'s ($\beta\in\overline{\triangle}^{+}_{+}$, $j\in\mathbf{Z}$) and $\widehat{\beta}^{-1}((j+\frac{1}{2})\pi)$'s ($\beta\in\overline{\triangle}^{-}_{+}$, $j\in\mathbf{Z}$) in $\sqrt{-1}\mathfrak{b}$, where $\widehat{\beta}:=-\sqrt{-1}\beta^{\mathbf{c}}|_{\sqrt{-1}\mathfrak{b}}$ and $\overline{\triangle}^{\pm}_{+}:=\{\beta\in\overline{\triangle}_{+}\,|\,(I^{v_0}_{\beta})^{\pm}\neq\emptyset\}$. Thus $W_{\widetilde{M}_0}$ is isomorphic to the affine transformation group associated with $\overline{\triangle}$. Hence, since F is $W_{\widetilde{M}_0}$-invariant, we see that $\overline{\triangle}$ is a weakly root system. This completes the proof of Theorem B. According to (5.6), for each fixed $\beta\in\overline{\triangle}_{+}$, $c^{+}_{\beta,i}$'s ($i\in(I^{v_0}_{\beta})^{+}$) coincide and so are $c^{-}_{\beta,i}$'s ($i\in(I^{v_0}_{\beta})^{-}$) also. In particular, we have $\sharp(I^{v_0}_{\beta})^{+}\leq 1$ and $\sharp(I^{v_0}_{\beta})^{-}\leq 1$. This fact implies that $\sharp\mathrm{Spec}\,A_{v_0}$ is evaluated from above as in the statement of Theorem C. From $v_0\in U$ and the definition of U, it follows that $\sharp\mathrm{Spec}\,A_v\leq\sharp\mathrm{Spec}\,A_{v_0}$ for any normal vector v. Therefore the statement of Theorem C follows. □

Remark 5.1. In the case where M is curvature-adapted equifocal submanifold in a symmetric space G/K of compact type, we have

$$\begin{aligned}&\mathrm{Spec}\widetilde{A}^{\mathbf{c}}_{v^L}=\mathrm{Spec}\widetilde{A}_{v^L}\\ =&\{0\}\cup\{\lambda^v_i\,|\,i\in I^v_0\}\\ &\cup\left(\bigcup_{\beta\in\overline{\triangle}_{+}}\{\frac{\beta(v)}{\mathrm{arctanh}\frac{\beta(v)}{\lambda^v_i}+j\pi}\,|\,i\in I^v_{\beta},\ j\in\mathbf{Z}\}\right),\end{aligned}$$

where $\widetilde{A}_{v^L}$, $\overline{\triangle}_{+}, I^v_0$ and I^v_{β} are as in the above proof. Also, we have

$$F=\bigcup_{\beta\in\overline{\triangle}_{+}}\ \bigcup_{(i,j)\in I^{v_0}_{\beta}\times\mathbf{Z}}(\beta^{\mathbf{c}})^{-1}(\mathrm{arctanh} c_{\beta,i}+j\pi)$$

and hence

$$(5.7)\qquad F_{\mathbf{R}}=\bigcup_{\beta\in\overline{\triangle}_{+}}\ \bigcup_{(i,j)\in I^{v_0}_{\beta}\times\mathbf{Z}}\beta^{-1}(\mathrm{arctanh} c_{\beta,i}+j\pi),$$

where $F, F_{\mathbf{R}}$ and v_0 are as in the above proof and $c_{\beta,i}:=\frac{\beta(v_0)}{\lambda^{v_0}_i}$. Furthermore, it is shown that $F_{\mathbf{R}}$ is $W'_{\widetilde{M}_0}$-invariant, where $W'_{\widetilde{M}_0}$ is as in the above proof. Note that $W'_{\widetilde{M}_0}$ is the affine Coxeter group asociated with the isoparametric submanifold $\widetilde{M}_0$.

However, it does not follow from these facts that, for each fixed β, $c_{\beta,i}$'s ($i \in I_\beta^{v_0}$) coincide because of the existenceness of the term $j\pi$ in the right-hand side of the relation (5.7). Thus we cannot evaluate $\sharp\mathrm{Spec}\,A_v$ from above for curvature-adapted equifocal submanifolds in a symmetric space of compact type.

Next we shall prove Corollary B.1.

Proof of Corollary B.1. According to Theorem 2 in [19], it follows that M is decomposed into the (non-trivial) product of two curvature-adapted and proper complex equifocal submanifolds if and only if the complex Coxeter group associated with M is decomposable. Hence the statement of Corollary B.1 follows from Theorem B. □

Next we shall prove Corollary B.2 in terms of Corollary B.1.

Proof of Corollary B.2. Since $\mathrm{codim}\,M = \mathrm{rank}\,G/K$, we have $\overline{\triangle} = \triangle$, that is, $W_{\overline{\triangle}}$ is equal to the Weyl group associated with the symmetric space G/K. Hence, since G/K is reducible, $W_{\overline{\triangle}}$ is decomposable. Therefore, the statement of Corollary B.2 follows from Corollary B.1. □

Next we shall prove Theorem D.

Proof of Theorem D. Without loss of generality, we may assume $x_0 = eK$. According to the proof of Theorem B, the sum F of all complex focal hyperplanes of $(\widetilde{M}, \hat{0})$ is as in (5.4). The intersection of $F\,(\subset (T^\perp_{\hat{0}}\widetilde{M})^{\mathbf{c}} = \mathfrak{b}^{\mathbf{c}})$ with $\mathfrak{b}$ is as follows:

$$F \cap \mathfrak{b} = \bigcup_{\beta \in \overline{\triangle}_+} \bigcup_{i \in (I_\beta^{v_0})^+} \beta^{-1}(\mathrm{arctanh} c^+_{\beta,i}). \tag{5.8}$$

Since $\beta^{-1}(\mathrm{arctanh} c^+_{\beta,i})$ ($i \in (I_\beta^{v_0})^+$ ($\beta \in \overline{\triangle}_+$)) are (real) hyperplanes in $\mathfrak{b}$ through Z in the proof of Theorems B and C and $\mathfrak{b}\,(\subset \mathfrak{p} \subset \mathfrak{g})$ is abelian, $\exp^\perp(\beta^{-1}(\mathrm{arctanh} c^+_{\beta,i}))$ ($i \in (I_\beta^{v_0})^+$ ($\beta \in \overline{\triangle}_+$)) are totally geodesic hypersurfaces through $\exp^\perp(Z)$ in the section $\Sigma := \exp^\perp(\mathfrak{b})$. On the other hand, it is clear that $\exp^\perp(F \cap \mathfrak{b})$ is the focal set of (M, eK). Hence, the statement of Theorem D follows. □

Next we shall prove Theorem E.

Proof of Theorem E. From (5.4) and (5.8), the statement of Theorem E follows. □.

Appendix 1

In this appendix, we shall first calculate the complex Coxeter group W_M and the real Coxeter group $W_{M,\mathbf{R}}$ associated with a principal orbit M of a Hermann type action $H \times G/K \to G/K$ without use of Theorems B and E. Let θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ and σ be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$. Without loss of generality, we may assume that $\sigma \circ \theta = \theta \circ \sigma$. Denote by A the shape tensor of M. Then $H(eK)$ is a totally geodesic singular orbit of the H-action and M is catched as a partial tube over $H(eK)$. Let $L := \mathrm{Fix}(\sigma\circ\theta)$. The submanifold $\exp^\perp(T^\perp_{eK}(H(eK)))$ is totally geodesic and it is isometric to the symmetric space $L/H\cap K$, where $\exp^\perp$ is the normal exponential map of $H(eK)$. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H. Denote by the same symbols the involutions of $\mathfrak{g}$ associated with θ and σ. Set $\mathfrak{p} := \mathrm{Ker}(\theta+\mathrm{id})\,(\subset\mathfrak{g})$ and $\mathfrak{q} := \mathrm{Ker}(\sigma+\mathrm{id})\,(\subset\mathfrak{g})$. Take $x := \exp^\perp(\xi) = \exp_G(\xi)K \in M\cap\exp^\perp(T^\perp_{eK}(H(eK)))$, where $\xi\in\mathfrak{p}$. For simplicity, set $g := \exp_G(\xi)$. Let Σ be the section of M through x, which pass through eK. Let $\mathfrak{b} := T_{eK}\Sigma$, $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} = T_{eK}(G/K)$ containing $\mathfrak{b}$, $\triangle$ be the root system with respect to $\mathfrak{a}$ and $\mathfrak{p} = \mathfrak{a}+\sum\limits_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. Set $\mathfrak{p}' := \mathfrak{p}\cap\mathfrak{q}(= T^\perp_{eK}(H(eK)))$. The orthogonal complement ${\mathfrak{p}'}^\perp$ of $\mathfrak{p}'$ in $\mathfrak{p}$ is equal to $\mathfrak{p}\cap\mathfrak{h}$. Set $\overline{\triangle} := \{\alpha|_{\mathfrak{b}}\,|\,\alpha\in\triangle \text{ s.t. } \alpha|_{\mathfrak{b}}\neq 0\}$, $\mathfrak{p}_\beta := \sum\limits_{\alpha\in\triangle_+ \text{ s.t. } \alpha|_{\mathfrak{b}}=\pm\beta}\mathfrak{p}_\alpha$ for $\beta\in\overline{\triangle}_+$, $\overline{\triangle}^H_+ := \{\beta\in\overline{\triangle}_+\,|\,{\mathfrak{p}'}^\perp\cap\mathfrak{p}_\beta\neq\{0\}\}$ and $\overline{\triangle}^V_+ := \{\beta\in\overline{\triangle}_+\,|\,\mathfrak{p}'\cap\mathfrak{p}_\beta\neq\{0\}\}$. Since both $\mathfrak{p}'$ and ${\mathfrak{p}'}^\perp$ are Lie triple systems of $\mathfrak{p}$ and $\mathfrak{b}$ is contained in $\mathfrak{p}'$, we have ${\mathfrak{p}'}^\perp = \mathfrak{z}_{{\mathfrak{p}'}^\perp}(\mathfrak{b}) + \sum\limits_{\beta\in\overline{\triangle}^H_+}({\mathfrak{p}'}^\perp\cap\mathfrak{p}_\beta)$ and $\mathfrak{p}' = \mathfrak{b}+\sum\limits_{\beta\in\overline{\triangle}^V_+}(\mathfrak{p}'\cap\mathfrak{p}_\beta)$, where $\mathfrak{z}_{{\mathfrak{p}'}^\perp}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in ${\mathfrak{p}'}^\perp$. Take $\eta\in T^\perp_x M$. For each $X\in{\mathfrak{p}'}^\perp\cap\mathfrak{p}_\beta$ $(\beta\in\overline{\triangle}^H_+)$, we can show

$$A_\eta\widetilde{X}_\xi = -\beta(\bar\eta)\tanh\beta(\xi)\widetilde{X}_\xi \tag{A.1}$$

(see the proof of Theorem B of [18]), where $\widetilde{X}_\xi$ is the horizontal lift of X to ξ (see Section 3 of [18] about this definition) and $\bar\eta$ is the element of $\mathfrak{b}$ with $\exp^\perp_{*\xi}(\bar\eta) = \eta$ (where $\bar\eta$ is regarded as an element of $T_\xi\mathfrak{p}'$ under the natural identification of $\mathfrak{p}'$ with $T_\xi\mathfrak{p}'$). Also, for each $Y\in T_x(M\cap\exp^\perp(\mathfrak{p}'))\cap g_*\mathfrak{p}_\beta$ $(\beta\in\overline{\triangle}^V_+)$, we can show

$$A_\eta Y = -\frac{\beta(\bar\eta)}{\tanh\beta(\xi)}Y \tag{A.2}$$

(see the proof of Theorem B of [18]). Let $\widetilde{M_0}$ be a component of $\widetilde{M} := (\pi\circ\phi)^{-1}(M)$ and $\widetilde{A}$ be the shape tensor of $\widetilde{M_0}$. From (A.1), (A.2) and Lemma 5.1, we have

$$\begin{aligned}\mathrm{Spec}\widetilde{A}^{\mathbf{c}}_{\eta^L} = \{0\}&\cup\{\frac{\beta(\bar\eta)}{-\beta(\xi)+j\pi\sqrt{-1}}\,|\,\beta\in\overline{\triangle}^V_+,\ j\in\mathbf{Z}\}\\ &\cup\{\frac{\beta(\bar\eta)}{-\beta(\xi)+(j+\frac12)\pi\sqrt{-1}}\,|\,\beta\in\overline{\triangle}^H_+,\ j\in\mathbf{Z}\}.\end{aligned} \tag{A.3}$$

Denote by F the sum of all complex focal hyperplanes of $(\widetilde{M_0},u)$, where $u\in(\pi\circ$

$\phi)^{-1}(gK)\cap\widetilde{M_0}$. From (A.3), we have

$$
\begin{aligned}
g_*^{-1}F &= \left(\bigcup_{\beta\in\overline{\triangle}_+^V}\bigcup_{j\in\mathbf{Z}}(\beta^{\mathbf{c}})^{-1}(-\beta(\xi)+j\pi\sqrt{-1})\right)\\
&\quad\cup\left(\bigcup_{\beta\in\overline{\triangle}_+^H}\bigcup_{j\in\mathbf{Z}}(\beta^{\mathbf{c}})^{-1}(-\beta(\xi)+(j+\frac{1}{2})\pi\sqrt{-1})\right)\\
&=\left(\bigcup_{\beta\in\overline{\triangle}_+^V}\bigcup_{j\in\mathbf{Z}}(-\xi+(\beta^{\mathbf{c}})^{-1}(j\pi\sqrt{-1}))\right)\\
&\quad\cup\left(\bigcup_{\beta\in\overline{\triangle}_+^H}\bigcup_{j\in\mathbf{Z}}(-\xi+(\beta^{\mathbf{c}})^{-1}((j+\frac{1}{2})\pi\sqrt{-1}))\right),
\end{aligned}
\tag{A.4}
$$

where we regard F as a subspace of $(T^\perp_{gK}M)^{\mathbf{c}}$ under the natural identification of $(T^\perp_u\widetilde{M_0})^{\mathbf{c}}$ with $(T^\perp_{gK}M)^{\mathbf{c}}$. From (A.4), it follows that the complex Coxeter group W_M associated with M is isomorphic to the affine Weyl group associated with the root system $\overline{\triangle}$. Also, we have $g_*^{-1}(F\cap T^\perp_{gK}M)=\bigcup_{\beta\in\overline{\triangle}_+^V}(-\xi+\beta^{-1}(0))$. Hence the real Coxeter group $W_{M,\mathbf{R}}$ associated with M is isomorphic to the group generated by the reflections with respect to $\beta^{-1}(0)$'s ($\beta\in\overline{\triangle}_+^V$). Since $\overline{\triangle}_+^V$ is the positive root system associated with the symmetric space $\exp^\perp(\mathfrak{p}')=L/H\cap K$, $W_{M,\mathbf{R}}$ is isomorphic to the Weyl group associated with $L/H\cap K$.

Next we shall list up the numbers $\max_{v\in T^\perp M}\sharp\mathrm{Spec}\,A_v$ for principal orbits M's of Hermann type actions H's on irreducible symmetric spaces G/K's of non-compact type satisfying $\mathrm{cohom}\,H=\mathrm{rank}(G/K)$. We shall use the notations of the last paragraph. Since $\mathfrak{p}_\beta=\mathfrak{p}_\beta\cap\mathfrak{p}'+\mathfrak{p}_\beta\cap\mathfrak{p}'^\perp$ for each $\beta\in\overline{\triangle}_+$, we have $\overline{\triangle}_+=\overline{\triangle}_+^V\cup\overline{\triangle}_+^H$. Hence, from $(A.1)$ and $(A.2)$, we have the following equality:

$$
\max_{v\in T^\perp M}\sharp\mathrm{Spec}\,A_v=\sharp\overline{\triangle}_++\sharp(\overline{\triangle}_+^V\cap\overline{\triangle}_+^H).
\tag{$A.5$}
$$

In case of $\mathrm{cohom}\,H=\mathrm{rank}(G/K)$, then we have $\mathfrak{a}=\mathfrak{b}$ and hence $\overline{\triangle}_+=\triangle_+$. Hence we can list up the numbers $\max_{v\in T^\perp M}\sharp\mathrm{Spec}\,A_v$ for the principal orbits M's in the case (see Tables 2∼4). The symbol $\widetilde{SO_0(1,8)}$ in Table 4 denotes the universal covering of $SO_0(1,8)$ and the symbol α in Table 4 denotes an outer automorphism of G_2^2.

H	G/K	$\max_{v\in T^\perp M}\sharp\mathrm{Spec}\,A_v$
$SO(n)$	$SL(n,\mathbf{R})/SO(n)$	$\frac{n(n-1)}{2}$
$SO_0(p,n-p)$	$SL(n,\mathbf{R})/SO(n)$	$\frac{n(n-1)}{2}$
$Sp(n)$	$SU^*(2n)/Sp(n)$	$\frac{n(n-1)}{2}$
$SO^*(2n)$	$SU^*(2n)/Sp(n)$	$n(n-1)$
$Sp(p,n-p)$	$SU^*(2n)/Sp(n)$	$\frac{n(n-1)}{2}$
$S(U(p)\times U(q))\ (p\leq q)$	$SU(p,q)/S(U(p)\times U(q))$	$\begin{cases} p(p+1) & (p<q) \\ p^2 & (p=q)\end{cases}$
$SO_0(p,q)\ (p\leq q)$	$SU(p,q)/S(U(p)\times U(q))$	$p(2p+1)$
$SO^*(2p)$	$SU(p,p)/S(U(p)\times U(p))$	$p(2p-1)$
$SL(p,\mathbf{C})\cdot U(1)$	$SU(p,p)/S(U(p)\times U(p))$	p^2
$SU(n)$	$SL(n,\mathbf{C})/SU(n)$	$\frac{n(n-1)}{2}$
$SO(n,\mathbf{C})$	$SL(n,\mathbf{C})/SU(n)$	$n(n-1)$
$SO(p)\times SO(q)\ (p\leq q)$	$SO_0(p,q)/SO(p)\times SO(q)$	$\begin{cases} p^2 & (p<q) \\ p(p-1) & (p=q)\end{cases}$
$SO(p,\mathbf{C})$	$SO_0(p,p)/SO(p)\times SO(p)$	$p(p-1)$
$U(n)$	$SO^*(2n)/U(n)$	$\begin{cases} \frac{n^2-1}{4} & (n:\mathrm{odd}) \\ \frac{n^2}{4} & (n:\mathrm{even})\end{cases}$
$SO(n,\mathbf{C})$	$SO^*(2n)/U(n)$	$\frac{n(n-1)}{2}$
$SU(2i,2n-2i)\cdot U(1)$	$SO^*(4n)/U(2n)$	n^2
$SU(i,2n-i+1)\cdot U(1)$	$SO^*(4n+2)/U(2n+1)$	n^2+n
$SO_0(i,2n-i+1)$	$SO(2n+1,\mathbf{C})/SO(2n+1)$	$2n^2$
$SO_0(2i,2n-2i)$	$SO(2n,\mathbf{C})/SO(2n)$	$\frac{(2n-1)^2}{2}$
$U(n)$	$Sp(n,\mathbf{R})/U(n)$	n^2
$SU(i,n-i)\cdot U(1)$	$Sp(n,\mathbf{R})/U(n)$	n^2
$Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$\begin{cases} p^2+p & (p<q) \\ p^2 & (p=q)\end{cases}$
$SU(p,q)\cdot U(1)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$\frac{1}{2}p(3p+5)$
$SU^*(2p)\cdot U(1)$	$Sp(p,p)/Sp(p)\times Sp(p)$	$2p^2$
$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$Sp(n,\mathbf{C})/Sp(n)$	$2n^2$
$Sp(n,\mathbf{R})$	$Sp(n,\mathbf{C})/Sp(n)$	n^2
$Sp(i,n-i)$	$Sp(n,\mathbf{C})/Sp(n)$	n^2

Table 2.

H	G/K	$\max_{v\in T^\perp M}\sharp\mathrm{Spec}\,A_v$
$Sp(4)/\{\pm 1\}$	$E_6^6/(Sp(4)/\{\pm 1\})$	36
$Sp(4,\mathbf{R})$	$E_6^6/(Sp(4)/\{\pm 1\})$	36
$Sp(2,2)$	$E_6^6/(Sp(4)/\{\pm 1\})$	36
$SU(6)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	24
$Sp(1,3)$	$E_6^2/SU(6)\cdot SU(2)$	36
$Sp(4,\mathbf{R})$	$E_6^2/SU(6)\cdot SU(2)$	34
$SU(2,4)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	30
$SU(3,3)\cdot SL(2,\mathbf{R})$	$E_6^2/SU(6)\cdot SU(2)$	24
$Spin(10)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	6
$Sp(2,2)$	$E_6^{-14}/Spin(10)\cdot U(1)$	10
$SU(2,4)\cdot SU(2)$	$E_6^{-14}/Spin(10)\cdot U(1)$	10
$SU(1,5)\cdot SL(2,\mathbf{R})$	$E_6^{-14}/Spin(10)\cdot U(1)$	10
$SO^*(10)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	7
$SO_0(2,8)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	10
F_4	E_6^{-26}/F_4	3
$Sp(1,3)$	E_6^{-26}/F_4	6
F_4^{-20}	E_6^{-26}/F_4	3
E_6	$E_6^{\mathbf{c}}/E_6$	36
E_6^2	$E_6^{\mathbf{c}}/E_6$	36
E_6^{-14}	$E_6^{\mathbf{c}}/E_6$	36
$Sp(4,\mathbf{C})$	$E_6^{\mathbf{c}}/E_6$	72
$SU(8)/\{\pm 1\}$	$E_7^7/(SU(8)/\{\pm 1\})$	63
$SL(8,\mathbf{R})$	$E_7^7/(SU(8)/\{\pm 1\})$	63
$SU^*(8)$	$E_7^7/(SU(8)/\{\pm 1\})$	63
$SU(4,4)$	$E_7^7/(SU(8)/\{\pm 1\})$	63
$SO'(12)\cdot SU(2)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	24
$SU(4,4)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	36
$SU(2,6)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	36
$SO^*(12)\cdot SL(2,\mathbf{R})$	$E_7^{-5}/SO'(12)\cdot SU(2)$	24
$SO_0(4,8)\cdot SU(2)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	24
$E_6\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	9
$SU^*(8)$	$E_7^{-25}/E_6\cdot U(1)$	15
$SU(2,6)$	$E_7^{-25}/E_6\cdot U(1)$	15
$E_6^{-14}\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	9
E_7	$E_7^{\mathbf{c}}/E_7$	63
E_7^7	$E_7^{\mathbf{c}}/E_7$	63
E_7^{-5}	$E_7^{\mathbf{c}}/E_7$	63
E_7^{-25}	$E_7^{\mathbf{c}}/E_7$	63
$SL(8,\mathbf{C})$	$E_7^{\mathbf{c}}/E_7$	126

Table 3.

H	G/K	$\max_{v\in T^\perp M} \sharp\mathrm{Spec}\, A_v$
$SO'(16)$	$E_8^8/SO'(16)$	120
$SO_0(8,8)$	$E_8^8/SO'(16)$	120
$E_7\cdot Sp(1)$	$E_8^{-24}/E_7\cdot Sp(1)$	24
$SO^*(16)$	$E_8^{-24}/E_7\cdot Sp(1)$	36
$SO_0(4,12)$	$E_8^{-24}/E_7\cdot Sp(1)$	36
$E_7^{-5}\cdot Sp(1)$	$E_8^{-24}/E_7\cdot Sp(1)$	24
$E_7^{-25}\cdot SL(2,\mathbf{R})$	$E_8^{-24}/E_7\cdot Sp(1)$	24
E_8	$E_8^{\mathbf{c}}/E_8$	120
E_8^8	$E_8^{\mathbf{c}}/E_8$	120
E_8^{-24}	$E_8^{\mathbf{c}}/E_8$	120
$SO(16,\mathbf{C})$	$E_8^{\mathbf{c}}/E_8$	240
$Sp(3)\cdot Sp(1)$	$F_4^4/Sp(3)\cdot Sp(1)$	24
$Sp(1,2)\cdot Sp(1)$	$F_4^4/Sp(3)\cdot Sp(1)$	24
$Sp(3,\mathbf{R})\cdot SL(2,\mathbf{R})$	$F_4^4/Sp(3)\cdot Sp(1)$	24
$Spin(9)$	$F_4^{-20}/Spin(9)$	2
$Sp(1,2)\cdot Sp(1)$	$F_4^{-20}/Spin(9)$	2
$\widetilde{SO_0(1,8)}$	$F_4^{-20}/Spin(9)$	4
F_4	$F_4^{\mathbf{C}}/F_4$	24
F_4^4	$F_4^{\mathbf{C}}/F_4$	24
F_4^{-20}	$F_4^{\mathbf{C}}/F_4$	24
$Sp(3,\mathbf{C})\cdot SL(2,\mathbf{C})$	$F_4^{\mathbf{C}}/F_4$	48
$SO(4)$	$G_2^2/SO(4)$	6
$SL(2,\mathbf{R})\times SL(2,\mathbf{R})$	$G_2^2/SO(4)$	6
$\alpha(SO(4))$	$G_2^2/SO(4)$	6
G_2	$G_2^{\mathbf{c}}/G_2$	6
G_2^2	$G_2^{\mathbf{c}}/G_2$	6
$SL(2,\mathbf{C})\times SL(2,\mathbf{C})$	$G_2^{\mathbf{c}}/G_2$	12

Table 4.

Appendix 2

In this appendix, we prove the following important fact for a curvature-adapted submanifold with globally flat and abelian normal bundle in a symmetric space.

Proposition A.1. *Let M be a curvature-adapted submanifold with globally flat and abelian normal bundle in a symmetric space G/K, A be the shape tensor of M and R be the curvature tensor of G/K. Then, for any $x\in M$,*

$$\{R(\cdot,v)v|_{T_xM}\,|\,v\in T_x^\perp M\}\cup\{A_v\,|\,v\in T_x^\perp M\}$$

is a commuting family of linear transformations of T_xM.

Proof. We shall show this statement in the case where G/K is of non-compact type.

Let $v_i \in T^\perp_x M$ $(i=1,2)$. Since M has abelian normal bundle, $R(\cdot, v_1)v_1|_{T_xM}$ and $R(\cdot, v_2)v_2|_{T_xM}$ commute with each other. Since M has globally flat and abelian normal bundle, A_{v_1} and A_{v_2} commute with each other. We shall show that $R(\cdot, v_1)v_1|_{T_xM}$ and A_{v_2} commute with each other. Let $x = gK$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} := T_{eK}(G/K)$ containing $\mathfrak{b} := g_*^{-1}(T^\perp_x M)$. Let $\triangle$ be the root system with respect to $\mathfrak{a}$ and set $\overline{\triangle} := \{\alpha|_{\mathfrak{b}} \,|\, \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$. For each $\beta \in \overline{\triangle}$, we set $\mathfrak{p}_\beta := \{X \in \mathfrak{p} \,|\, \mathrm{ad}(b)^2(X) = \beta(b)^2 X \ (\forall\, b \in \mathfrak{b})\}$. Then we have $\mathfrak{p} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) + \sum_{\beta \in \overline{\triangle}_+} \mathfrak{p}_\beta$, where $\overline{\triangle}_+$ is the positive root system under some lexicographic ordering and $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p}$. Consider

$$D := \{v \in T^\perp_x M \,|\, \mathrm{Span}\{v\} \cap \left(\bigcup_{(\beta_1,\beta_2) \in \overline{\triangle}_+ \times \overline{\triangle}_+ \text{ s.t. } \beta_1 \neq \beta_2} (l_{\beta_1} \cap l_{\beta_2}) \right) = \emptyset\},$$

where $l_{\beta_i} := \beta_i^{-1}(1)$ $(i=1,2)$. It is clear that D is open and dense in $T^\perp_x M$. Take $v \in D$. Then, since $\beta(v)$'s $(\beta \in \overline{\triangle}_+)$ are mutually distinct, the decomposition $T_xM = g_*(\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) \ominus \mathfrak{b}) + \sum_{\beta \in \overline{\triangle}_+} g_* \mathfrak{p}_\beta$ is the eigenspace decomposition of $R(\cdot, v)v|_{T_xM}$. Since M is curvature-adapted by the assumption and hence $[R(\cdot, v)v|_{T_xM}, A_v] = 0$, we have

$$\text{(A.6)} \quad T_xM = \sum_{\lambda \in \mathrm{Spec}\, A_v} \left((g_*(\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) \ominus \mathfrak{b}) \cap \mathrm{Ker}(A_v - \lambda \mathrm{id})) + \sum_{\beta \in \overline{\triangle}_+} (g_* \mathfrak{p}_\beta \cap \mathrm{Ker}(A_v - \lambda\, \mathrm{id})) \right).$$

Suppose that (A.6) does not hold for some $v_0 \in T^\perp_x M \setminus D$. Then it is easy to show that there exists a neighborhood U of v_0 in $T^\perp_x M$ such that (A.6) does not hold for any $v \in U$. Clearly we have $U \cap D = \emptyset$. This contradicts the fact that D is dense in $T^\perp_x M$. Hence (A.6) holds for any $v \in T^\perp_x M \setminus D$. Therefore, (A.6) holds for any $v \in T^\perp_x M$. In particular, (A.6) holds for v_2. On the other hand, the decomposition $T_xM = g_* \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) + \sum_{\beta \in \overline{\triangle}_+} g_* \mathfrak{p}_\beta$ is the common eigenspace decomposition of $R(\cdot, v)v|_{T_xM}$'s $(v \in T^\perp_x M)$. From these facts, we have

$$T_xM = \sum_{\lambda \in \mathrm{Spec}\, A_{v_2}} \sum_{\mu \in \mathrm{Spec}\, R(\cdot, v_1)v_1|_{T_xM}} (\mathrm{Ker}(R(\cdot, v_1)v_1|_{T_xM} - \mu\, \mathrm{id}) \cap \mathrm{Ker}(A_{v_2} - \lambda\, \mathrm{id})),$$

which implies that $R(\cdot, v_1)v_1|_{T_xM}$ and A_{v_2} commute with each other. This completes the proof. □

Remark A.1. O. Goertsches and G. Thorbergsson [10] have already shown that the statement of this proposition holds for principal orbits of Heremann actions on symmetric spaces of compact type.

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