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On a Structure Defined by a Tensor Field F of Type $(1,1)$ Satisfying $\prod_{j=1}^{k}[F^2+a(j)F+\lambda^2(j)I]=0$

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Abstract. The differentiable manifold with f – structure were studied by many authors, for example: K. Yano [7], Ishihara [8], Das [4] among others but thus far we do not know the geometry of manifolds which are endowed with special polynomial $F_{a(j)\times(j)}$ – structure satisfying

$$\prod_{j=1}^{K}\left[F^2+a\,(j)\,F+\lambda^2\,(j)\,I\right]=0$$

However, special quadratic structure manifold have been defined and studied by Sinha and Sharma [8]. The purpose of this paper is to study the geometry of differentiable manifolds equipped with such structures and define special polynomial structures for all values of $j=1,2,...,K\in N$, and obtain integrability conditions of the distributions π_m^j and $\widetilde{\pi}_m^j$.

1. Introduction

Let M^n be n – dimensional manifold of differentiability class C^∞. Suppose there exist on M^n, a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying

$$\prod_{j=1}^{k}[F^2+a(j)F+\lambda^2(j)I] \;=\; 0, \tag{1.1}$$

where $\lambda(j)$ are scalars not equal to zero and $a(j)$ are real numbers for $j=1,2,\ldots,k\in N$, the set of natural numbers. For arbitrary vector field X on M^n the

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above equation (1.1) can be put in the form

$$\prod_{j=1}^{k}[\bar{\bar{X}} + a(j)\bar{X} + \lambda^2(j)X] \;=\; 0, \tag{1.2}$$

where

$$\bar{X} \overset{def}{=} F(X). \tag{1.3}$$

Let us call the manifold M^n equipped with such a structure as the special $F_{a(j),\,\lambda(j)}$ – structure manifold.

Theorem 1.1. *The rank of F in the special polynomial $F_{a(j),\,\lambda(j)}$ – structure is equal to the dimension of the manifold.*

Proof. Assuming $\bar{X} = 0 \Rightarrow \bar{\bar{X}} = 0$. So from the equation (1.2) it follows that $\prod_{j=1}^{k}[\lambda^2(j)X] = 0 \Rightarrow X = 0$ as $\lambda(j) \neq 0$. So the Kernel of F is the trivial subspace $\{0\}$ of TM^n where TM^n denotes the tangent space of the manifold M^n. Hence if ν denotes the nullity of F, $\nu = 0$. If ρ be the rank of F, then from a well-known theorem of linear algebra

$$\rho + \nu \;=\; n. \tag{1.4}$$

Since $\nu = 0$, hence $\rho = n$. This proves the theorem. □

Theorem 1.2. *The dimension of manifold M^n equipped with the special polynomial $F_{a(j),\,\lambda(j)}$ – structure for $a^2(j) < 4\lambda^2(j)$ is even.*

Proof. Let δ be the eigen value of F and V be the corresponding eigen vector. Then

$$\bar{V} \;=\; \delta V$$

which yields

$$\bar{\bar{V}} \;=\; \delta^2 V.$$

Substituting these values of $\bar{V}$ and $\bar{\bar{V}}$ in (1.2), we obtain

$$\prod_{j=1}^{k}[\delta^2 V + a(j)\delta V + \lambda^2(j)V] \;=\; 0$$

which gives

$$\prod_{j=1}^{k}[\delta^2 + a(j)\delta + \lambda^2(j)I] \;=\; 0. \tag{1.5}$$

The roots of the above equation are given by

$$\delta \;=\; \frac{-a(j) \pm \sqrt{a^2(j) - 4\lambda^2(j)}}{2}; \quad j = 1, 2, \ldots, k \in N\ . \tag{1.6}$$

If $a^2(j) < 4\lambda^2(j)$, the eigen value of F are of the form $\alpha(j) \pm \beta(j)$, where

$$\alpha(j) \;=\; -\frac{a(j)}{2} \quad \text{and} \quad \beta(j) \;=\; \frac{\sqrt{4\lambda^2(j) - a^2(j)}}{2}\,.$$

Since the complex eigen values occur in pairs, therefore the dimension n of the manifold must be even. □

Theorem 1.3. *The special polynomial $F_{a(j),\,\lambda(j)}$ – structure is not unique.*

Proof. Let us put [5]

$$\mu(F'(X)) \;=\; F(\mu(X)), \tag{1.7}$$

where F' is a tensor field of type (1,1) and μ is a non-singular vector valued function on M^n. Thus

$$\begin{aligned} \mu(F'^2(X)) &= \mu F'(F'(X)) \\ &= \mu F'(F'(X)) \\ &= F(\mu(F'(X)) \\ &= F(F(\mu(X))) \\ &= F^2(\mu(X)). \end{aligned} \tag{1.8}$$

Thus we get

$$\prod_{j=1}^{k} \mu[F'^2(X)+a(j)F'(X)+\lambda^2(j)(X)] = \prod_{j=1}^{k}[F^2(\mu(X))+a(j)F(\mu(X))+\lambda^2(j)(\mu(X))] = 0\,.$$

By virtue of the equation (1.1). Thus we obtain

$$\prod_{j=1}^{k}[F'^2 + a(j)F' + \lambda^2(j)I] \;=\; 0$$

as μ is non singular. Hence F' gives the special polynomial $F_{a(j),\,\lambda(j)}$ – structure on the manifold M^n. □

2. Existence conditions

In this section, we shall prove the following:

Theorem 2.1. *In order that the even dimensional manifold M^{2km} may admit the special polynomial $F_{a(j),\,\lambda(j)}$ – structure for $a^2(j) < 4\lambda^2(j)$, it is necessary and sufficient that it contains k distributions π_m^j of dimensions m and k distributions $\tilde{\pi}_m^j$ conjugate to π_m^j such that they are mutually disjoint and span together a manifold of dimension $2km$.*

Proof. Suppose first that the manifold M^{2km} admits the special polynomial $F_{a(j),\,\lambda(j)}$ – structure for $a^2(j) < 4\lambda^2(j)$. Hence the tensor F has k sets of m eigen values each of the form $(\alpha(j) + i\beta(j))$ and other k sets of eigen values of the form $(\alpha(j) - i\beta(j))$, $j = 1, 2, \ldots, k \in N$. Let P^j_x, $x = 1, 2, \ldots, m$; $j = 1, 2 \ldots, k$ be m eigen vectors for the m eigen values $(\alpha(j) + i\beta(j))$ and Q^j_x, $x = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$ be m eigen vectors for the m eigen values $(\alpha(j) - i\beta(j))$ of F. Suppose

$$\prod_{j=1}^{k}[b^x_j P^j_x + c^x_j Q^j_x] \ = \ 0, \quad b^x_j,\, c^x_j \in R, \quad x = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, k \ . \tag{2.1}$$

Operating the above equation (2.1) by F and making use of the fact that P^l_x, Q^l_x are eigen vectors for the eigen values $(\alpha(l) + i\beta(l))$ and $(\alpha(l) - i\beta(l))$ of F, $1 < l < k \in N$, we get

$$[b^x_l P^l_x - c^x_l Q^l_x]\prod_{\substack{j=1\\ j\neq l}}^{k}[b^x_j P^j_x + c^x_j Q^j_x] \ = \ 0 \ . \tag{2.2}$$

Thus from equation (2.1) and (2.2), we get

$$b^x_l \ = \ 0 \ \text{ and } \ c^x_l \ = \ 0 \ , \ \ x = 1, 2, \cdots, m \ ; \ \ j \ = \ l. \tag{2.3}$$

Hence the set $\{P^l_x, Q^l_x\}$ is linearly independent. Similarly, we get $b^x_j = 0$ and $c^x_j = 0$, for all values of $j = 1, 2, \cdots, k \in N$; $x = 1, 2, \cdots, m$.

Hence the set $\{P^j_x, Q^j_x\}$ is linearly independent for all values of $x = 1, 2, \cdots, m$; $j = 1, 2, \cdots, k \in N$.

Let Lj and Mj be the linear transformation given by

$$Lj(X) \ = \ \bar{X} - (\alpha(j) - i\beta(j))X \tag{2.4}$$

and

$$Mj(X) \ = \ \bar{X} - (\alpha(j) + i\beta(j))X \ . \tag{2.5}$$

The results can be easily proved

$$\begin{aligned} Lj(P^j_x) &= 2i\beta P^j_x, \\ Lj(Q^j_x) &= 0, \\ Mj(P^j_x) &= 0, \\ Mj(Q^j_x) &= -2i\beta Q^j_x. \end{aligned} \tag{2.6}$$

Thus there exist k distributions π^j_m and k distributions $\tilde{\pi}^j_m$ each of dimension m such that they are mutually disjoint and span together a manifold of dimension $2km$. The projections Lj and Mj are given by (2.4) and (2.5).

Suppose conversely that there exist k distributions π_m^j and k distributions $\tilde{\pi}_m^j$ each of dimension m such that they have no common direction and span together a manifold of dimension $2km$.

Suppose in the k distributions π_m^j there are m linearly independent eigen vectors P_x^j and for the k distributions $\tilde{\pi}_m^j$ the m linearly independent eigen vectors are Q_x^j, $x = 1, 2, \cdots, m$; $j = 1, 2, \cdots, k \in N$. Then the set $\{P_x^j, Q_x^j\}$ is linearly independent.

Let $\{p_x^j, q_x^j\}$ be the set of 1-forms dual to the set $\{P_x^j, Q_x^j\}$. Then

$$\begin{aligned} p_j^x(P_y^j) &= \delta_y^x, \\ p_j^x(Q_y^j) &= 0, \\ q_j^x(P_y^j) &= 0, \\ q_j^x(Q_y^j) &= \delta_y^x, \end{aligned} \tag{2.7}$$

also let

$$\prod_{j=1}^{k} [p_j^x(X)P_x^j + q_j^x(X)Q_x^j] = X \tag{2.8}$$

Barring the equation (2.8) both sides and using the fact that P_x^l, Q_x^l are eigen vectors for the eigen values $\alpha(l) + i\beta(l)$ and $\alpha(l) - i\beta(l)$ of F we get

$$[(\alpha(l) + i\beta(l))\, p_l^x(X)P_x^l + (\alpha(l) - i\beta(l) q_l^x(X)Q_x^l] \prod_{\substack{j=1 \\ j\neq l}}^{k} [p_j^x(X)P_x^j + q_j^x(X)Q_x^j] = \bar{X}. \tag{2.9}$$

Thus from the equation (2.8) and (2.9), we get

$$\bar{X} = \alpha(l)X + [i\beta(l)\,(p_l^x(X)P_x^l - q_l^x(X)Q_x^l)] \prod_{\substack{j=1 \\ j\neq l}}^{k} [p_j^x(X)P_x^j + q_j^x(X)Q_x^j] \,. \tag{2.10}$$

Barring (2.9) again and using the same fact that P_x^l, Q_x^l are eigen vectors for the eigen values $\alpha(l) + i\beta(l)$ and $\alpha(l) - i\beta(l)$ of F, we get

$$\bar{\bar{X}} = [(\alpha(l) + i\beta(l))^2\,(p_l^x(X)P_x^l + (\alpha(l) - i\beta(l))^2 q_l^x(X)Q_x^l)] \prod_{\substack{j=1 \\ j\neq l}}^{k} [p_j^x(X)P_x^j + q_j^x(X)Q_x^j]. \tag{2.11}$$

In view of the equation (2.8) and (2.10) and (2.11), we get

$$\bar{\bar{X}} - 2\alpha(l)\bar{X} + (\alpha^2(l) + \beta^2(l))X = 0\,.$$

Since $\alpha(l) = -\frac{a(l)}{2}$ and $\beta(l) = \frac{\sqrt{4\lambda^2(l)-a^2(l)}}{2}$, where $1 \le l \le k$.

Similarly it follows that

$$\prod_{j=1}^{k}[\bar{\bar{X}} + a(j)\bar{X} + \lambda^2(j)X] = 0 \text{ for all } j = 1, 2, \cdots, k \in N .$$

Thus the manifold M^{2km} admits the special polynomial $F_{a(j),\,\lambda(j)}-$structure for $j = 1, 2, \cdots, k \in N$. □

Theorem 2.2. *We have*

$$\begin{aligned} L^2 j &= 2i\beta(j)Lj, \\ M^2 j &= -2i\beta(j)Mj, \\ LjMj &= MjLj = 0. \end{aligned} \tag{2.12}$$

Proof. We have in view of the equation (2.4)

$$Lj = F - (\alpha(j) - i\beta(j))I .$$

Thus

$$L^2 j = F^2 - 2[\alpha(j) - i\beta(j)]F + (\alpha(j) - i\beta(j))^2 I .$$

Since $\alpha(j) \pm \beta(j)$ is the root $\prod_{j=1}^{k}[F^2 + a(j)F + \lambda^2(j)I] = 0$, so

$$L^2 j = -a(j)F - \lambda^2(j)I - 2[\alpha(j) - i\beta(j)]F + (\alpha(j) - i\beta(j))^2 I$$

$$L^2 j = 2i\beta(j)[F - (\alpha(j) - i\beta(j))I]$$

$$L^2 j = 2i\beta(j)L(j) .$$

Similarly, it can be shown that

$$M^2 j = -2i\beta(j)M(j) .$$

Also,

$$LjMj = MjLj = [F - (\alpha(j) - i\beta(j))I][F - (\alpha(j) + i\beta(j))I]$$

or

$$LjMj = MjLj = F^2 + [\alpha^2(j) + \beta^2(j)]I - 2\alpha(j)F . \tag{2.13}$$

Since $\alpha(j) = -\frac{a(j)}{2}$ and $\alpha^2(j) + \beta^2(j) = \lambda^2(j)$.

Hence

$$LjMj \;=\; MjLj \;=\; F^2 + a(j)F + \lambda^2(j)I \;=\; 0, \tag{2.14}$$

Thus

$$LjMj \;=\; MjLj \;=\; 0\ .$$

Thus the theorem is proved. □

3. Nijenhuis Tensor $F_{a(j),\,\lambda(j)}$ – structure

The Nijenhuis Tensor $F_{a(j),\,\lambda(j)}$ – structure is the skew symmetric tensor of type (1,2) given by

$$N(X,Y) \;=\; [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]} \tag{3.1}$$

for arbitrary vector fields X, Y in M^n.

Theorem 3.1. *We have*

$$N(X,\overline{Y}) \;=\; N(\overline{X},Y)\ , \tag{3.2}$$

$$N(\overline{X},\overline{Y}) \;= -\lambda^2(j)N(X,Y) - a(j)N(X,\overline{Y})\ , \tag{3.3}$$

$$N(\overline{X},\overline{Y}) \;= -\lambda^2(j)N(X,Y) - a(j)N(\overline{X},Y)\ . \tag{3.4}$$

Proof. Barring X in (3.1), we have

$$N(\overline{X},Y) \;=\; [\overline{\overline{X}},\overline{Y}] + \overline{\overline{[\overline{X},Y]}} - \overline{[\overline{\overline{X}},Y]} - \overline{[\overline{X},\overline{Y}]}$$

which in view of (1.2) reduces to

$$N(\overline{X},Y) \;= -\lambda^2(j)[X,\overline{Y}] - a(j)[\overline{X},\overline{Y}] - \lambda^2(j)[\overline{X},Y] + \lambda^2(j)\overline{[X,Y]} - \overline{[\overline{X},\overline{Y}]}\ . \tag{3.5}$$

Barring Y in (3.1) and using (1.2), we have

$$N(X,\overline{Y}) \;= -\lambda^2(j)[\overline{X},Y] - a(j)[\overline{X},\overline{Y}] - \lambda^2(j)[X,\overline{Y}] + \lambda^2(j)\overline{[X,Y]} - \overline{[\overline{X},\overline{Y}]}\ . \tag{3.6}$$

From (3.5) and (3.6), we obtain (3.2). Barring X and Y in (3.1) and using (1.2), we have

$$\begin{aligned} N(\overline{X},\overline{Y}) = &-\lambda^4(j)[X,Y] + a(j)\lambda^2(j)[X,\overline{Y}] + a(j)\lambda^2(j)[\overline{X},Y] + a^2(j)[\overline{X},\overline{Y}] \\ &- \lambda^2(j)[\overline{X},\overline{Y}] + \lambda^2(j)\overline{[X,\overline{Y}]} + a(j)\overline{[\overline{X},\overline{Y}]} + \lambda^2(j)\overline{[\overline{X},Y]}\ . \end{aligned} \tag{3.7}$$

$$\begin{aligned}(3.8)\quad & \lambda^2(j)N(X,Y)\\ & = \lambda^2(j)[\overline{X},\overline{Y}] - \lambda^4(j)[X,Y] - a(j)\lambda^2(j)\overline{[X,Y]} - \lambda^2(j)\overline{[\overline{X},Y]} - \lambda^2(j)\overline{[X,\overline{Y}]}\end{aligned}$$

and

$$\begin{aligned}(3.9)\quad a(j)N(X,\overline{Y}) = & -a(j)\lambda^2[\overline{X},Y] - a^2(j)[\overline{X},\overline{Y}] - a(j)\lambda^2(j)[X,\overline{Y}]\\ & - a(j)\overline{[\overline{X},\overline{Y}]} + a(j)\lambda^2(j)\overline{[X,Y]}\end{aligned}$$

from (3.1), (3.7), (3.8) and (3.9), we get (3.3).

Equation (3.4) follows from (3.2) and (3.3). □

4. Integrability conditions

In this section, we shall establish some results on the integrability of the k distributions $\tilde{\pi}_m^j$ and π_m^j.

Theorem 4.1. *The necessary and sufficient condition that the k distributions π_m^l integrable is that*

$$(4.1)\qquad (dMj)(X,Y) = 0 \quad \textit{for all} \quad j = 1,2,\cdots,k \in N.$$

Proof. Suppose for particular value $j = l$, distribution π_m^l is integrable. Now

$$X, Y \in \pi_m^l \;\Rightarrow\; [X,Y] \in \pi_m^l\ .$$

Hence

$$(4.2)\qquad Ml(X) = 0, \quad Ml(Y) = 0 \text{ and } Ml([X,Y]) = 0,$$

we have [3]

$$(4.3)\qquad (dMl)(X,Y) = X.Ml(Y) - Y.Ml(X) - Ml([X,Y])\ .$$

Thus in view of equation (4.2), we have

$$(4.4)\qquad (dMl)(X,Y) = 0.$$

Similarly it follows that $(dMj)(X,Y) = 0$ for all $j = 1,2,\cdots,k$.

Hence the condition is necessary.

Suppose conversely that

$$(dMj)(X,Y) = 0 \text{ for all } X,Y \in k \text{ distributions } \pi_m^j$$

$$(dMj)(X,Y) = 0 \text{ for all } j = 1,2,\cdots,k.$$

Thus

$$Mj([X,Y]) = 0 \text{ as } Mj(X) = 0 = Mj(Y) \text{ for all } j = 1,2,\cdots,k .$$

Also

$$Lj([X,Y]) = \overline{[X,Y]} - (\alpha(j) - i\beta(j))[X,Y] \text{ for all } j = 1,2,\cdots,k$$

$$= (\alpha(j) + i\beta(j))[X,Y] - (\alpha(j) - i\beta(j))[X,Y] \text{ for all } j = 1,2,\cdots,k$$

or

$$Lj([X,Y]) = 2i\beta(j)[X,Y] \text{ for all } j = 1,2,\cdots,k .$$

Thus it follows that if $X, Y \in k$ distributions π_m^j then $[X,Y]$ also belongs to k distributions π_m^j. Thus the k distributions π_m^j is integrable. □

Theorem 4.2. *The necessary and sufficient condition for the k distributions $\tilde{\pi}_m^j$ to be integrable is that*

$$(dLj)(X,Y) = 0 \text{ for all } j = 1,2,\cdots,k .$$

Proof. Proof follows easily in a way similar to that of the Theorem 4.1. □

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