KYUNGPOOK Math. J. 50(2010), 389-397

On Some New Generalized Difference Statistically Convergent Sequence Spaces Defined by a Sequence of Orlicz Functions

Çiğdem Asma Bektaş*
Department of Mathematics, Firat University, Elazig, 23119, Turkey
e-mail : cigdemas78@hotmail.com

Gülcan Atici
Department of Mathematics, Muş Alparslan University, Muş, 49100, Turkey
e-mail : gatici23@hotmail.com

Abstract. In this paper we introduce the new generalized difference sequence spaces $\ell_\infty(\Delta_v^n, \mathcal{M}, p, q, s)$, $\bar{c}(\Delta_v^n, \mathcal{M}, p, q, s)$, $\overline{c_0}(\Delta_v^n, \mathcal{M}, p, q, s)$, $m(\Delta_v^n, \mathcal{M}, p, q, s)$ and $m_0(\Delta_v^n, \mathcal{M}, p, q, s)$ defined over a seminormed sequence space (X, q). We study some of its properties, like completeness, solidity, symmetricity etc. We obtain some relations between these spaces as well as prove some inclusion result.

1. Introduction

Throughout the article $w(X)$, $c(X)$, $c_0(X)$, $\bar{c}(X)$, $\overline{c_0}(X)$, $\ell_\infty(X)$, $m(X)$ and $m_0(X)$ will represent the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null X valued sequence spaces, where (X, q) is a seminormed space, seminormed by q. For $X = \mathbb{C}$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, ...)$, where θ is the zero element of X.

The idea of statistical convergence was introduced by Fast [8] and studied by various authors (see [2], [9], [17]).

The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists},$$

where χ_E is the characteristic function of E.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. We write $x_k \xrightarrow{stat} L$ or $stat - \lim x_k = L$.

* Corresponding Author.
Received February 8, 2010; revised May August 28, 2010; accepted September 3, 2010.
2000 Mathematics Subject Classification: 40C05, 40A05.
Key words and phrases: Difference sequence spaces, statistical convergence, seminorm.

The notion of difference sequence space was introduced by Kizmaz [11]. It was generalized by Et and Colak [3] as follows :

Let n be a non-negative integer. Then

$$X(\Delta^n) = \{x = (x_k) : \Delta^n x \in X\},$$

for $X = c_0$, c and ℓ_∞, where $n \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $(\Delta^n x_k) = \left(\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}\right)$.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Et and Esi [4] generalized the above sequence spaces to the following sequence spaces:

$$X(\Delta_v^n) = \{x = (x_k) : (\Delta_v^n x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $\Delta_v^0 x = (v_k x_k)$, $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and $(\Delta_v^n x_k) = \left(\Delta_v^{n-1} x_k - \Delta_v^{n-1} x_{k+1}\right)$ and so that

$$\Delta_v^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} v_{k+i} x_{k+i}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

If the convexity of an Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function, introduced and investigated by Nakano [14] and followed by Ruckle [16], Maddox [13], and many others.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define what is called an Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

which is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

The space ℓ_M with the norm closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Remark 1. *An Orlicz function satisfies the inequality* $M(\lambda x) \leq \lambda M(x)$ *for all* λ *with* $0 \leq \lambda \leq 1$.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \tag{1}$$

Definition 1.1([10]). *Let X be a sequence space. Then X is called:*

(i) *Solid (or normal), if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalar with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.*

(ii) *Symmetric if $(x_k) \in X$ implies $\left(x_{\pi(k)}\right) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.*

(iii) *Monotone provided X contains the canonical preimages of all its stepspace.*

Lemma 1.2([10]). *If a sequence space E is solid, then E is monotone.*

Lemma 1.3([18]). *For two sequences (p_k) and (t_k) we have $m_0(p) \supseteq m_0(t)$ if and only if $\liminf_{k\in K}(p_k/t_k) > 0$, where $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$.*

Lemma 1.4([18]). *Let $h = \inf p_k$ and $G = \sup p_k$, then the following are equivalent:*

(i) *$G < \infty$ and $h > 0$,*

(ii) *$m(p) = m$.*

Lemma 1.5([18]). *Let $K = \{n_1, n_2, ...\}$ be an infinite subset of $\mathbb{N}$ such that $\delta(K) = 0$. Let*

$$T = \{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k = n_i, i \in \mathbb{N} \text{ and } x_k = 0, \text{ otherwise}\}.$$

Then T is uncountable.

2. Main results

Definition 2.1. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be any sequence of strictly positive real numbers, s a non-negative real number and X be a seminormed space with the seminorm q. Then we define the following sequence spaces:

$$\overline{c}(\Delta_v^n,\mathcal{M},p,q,s)=\left\{(x_k)\in w(X) : k^{-s}[M_k(q(\frac{\Delta_v^n x_k - L}{\rho}))]^{p_k} \overset{stat}{\longrightarrow} 0, \text{ for some } \rho > 0, L \in X\right\},$$

$$\overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s)=\left\{(x_k)\in w(X) : k^{-s}[M_k(q(\frac{\Delta_v^n x_k}{\rho}))]^{p_k} \overset{stat}{\longrightarrow} 0, \text{ for some } \rho > 0\right\},$$

$$\ell_\infty(\Delta_v^n,\mathcal{M},p,q,s)=\left\{(x_k)\in w(X) : \sup_{k\geq 1} k^{-s}[M_k(q(\frac{\Delta_v^n x_k}{\rho}))]^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

We write

$$m(\Delta_v^n,\mathcal{M},p,q,s) = \overline{c}(\Delta_v^n,\mathcal{M},p,q,s) \cap \ell_\infty(\Delta_v^n,\mathcal{M},p,q,s),$$
$$m_0(\Delta_v^n,\mathcal{M},p,q,s) = \overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s) \cap \ell_\infty(\Delta_v^n,\mathcal{M},p,q,s).$$

For $p_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $\overline{c}(\Delta_v^n,\mathcal{M},q,s)$, $\overline{c_0}(\Delta_v^n,\mathcal{M},q,s)$, $m(\Delta_v^n,\mathcal{M},q,s)$, $m_0(\Delta_v^n,\mathcal{M},q,s)$ and $\ell_\infty(\Delta_v^n,\mathcal{M},q,s)$.

For $M_k = M$ for all $k \in \mathbb{N}$, we write these spaces as $\overline{c}(\Delta_v^n,M,p,q,s)$, $\overline{c_0}(\Delta_v^n,M,p,q,s)$, $m(\Delta_v^n,M,p,q,s)$, $m_0(\Delta_v^n,M,p,q,s)$ and $\ell_\infty(\Delta_v^n,M,p,q,s)$.

For $s=0$, $M_k(x)=x$ for all $k\in\mathbb{N}$ and $q(x)=|x|$, we obtain the results of Et, et.al [6].

Theorem 2.2. *$\overline{c}(\Delta_v^n,\mathcal{M},p,q,s)$, $\overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s)$, $m\left(\Delta_v^n,\mathcal{M},p,q,s\right)$ and $m_0\left(\Delta_v^n,\mathcal{M},p,q,s\right)$ are linear spaces.*

Proof. We shall prove only for $\overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s)$. The other cases can be proved similarly. Let $(x_k),(y_y)\in\overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s)$ and $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$k^{-s}[M_k(q(\frac{\Delta_v^n x_k}{\rho_1}))]^{p_k}\overset{stat}{\longrightarrow}0 \text{ as } k\to\infty$$

and

$$k^{-s}[M_k(q(\frac{\Delta_v^n y_k}{\rho_2}))]^{p_k}\overset{stat}{\longrightarrow}0 \text{ as } k\to\infty.$$

Define $\rho_3=\max\left(2\left|\alpha\right|\rho_1,2\left|\beta\right|\rho_2\right)$. Since M_k are non-decreasing and convex functions, q is a seminorm and Δ_v^n linear, we have

$$\begin{aligned}
&k^{-s}\left[M_k\left(q\left(\frac{\Delta_v^n\left(\alpha x_k+\beta y_k\right)}{\rho_3}\right)\right)\right]^{p_k}\\
\leq\quad & Dk^{-s}\left[M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho_1}\right)\right)\right]^{p_k}+Dk^{-s}\left[M_k\left(q\left(\frac{\Delta_v^n y_k}{\rho_2}\right)\right)\right]^{p_k}\overset{stat}{\longrightarrow}0,
\end{aligned}$$

as $k\to\infty$. This proves that $\overline{c_0}(\Delta_v^n,\mathcal{M},p,q,s)$ is linear space. □

Theorem 2.3. *The spaces $m_0\left(\Delta_v^n,\mathcal{M},p,q,s\right)$ and $m\left(\Delta_v^n,\mathcal{M},p,q,s\right)$ are paranormed space, paranormed by*

$$g_\Delta\left(x\right)=\sum_{k=1}^{n}q\left(x_k\right)+\inf\left\{\rho^{\frac{p_k}{H}}:\sup_k k^{-s}M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho}\right)\right)\leq1,\ \rho>0\right\},$$

where $H=\max(1,\sup p_k)$.

Proof. Clearly $g_\Delta\left(x\right)=g_\Delta\left(-x\right)$; $x=\bar{\theta}$ implies $\Delta_v^n x_k=\theta$ and as such $M_k\left(q(\theta)\right)=0$. Therefore $g_\Delta\left(\bar{\theta}\right)=0$. Now let (x_k) and (y_k) be in any of the spaces in the statement. Then we have ρ_1, $\rho_2>0$ such that

$$\sup_k k^{-s}M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho_1}\right)\right)\leq1$$

and

$$\sup_k k^{-s}M_k\left(q\left(\frac{\Delta_v^n y_k}{\rho_2}\right)\right)\leq1.$$

Let $\rho = \rho_1 + \rho_2$. Then by the convexity of M_k, we have

$$\begin{aligned} & \sup k^{-s} M_k\left(q\left(\frac{\Delta_v^n\left(x_k+y_k\right)}{\rho}\right)\right) \\ \leq\ & \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho_1}\right)\right)+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup k^{-s} M_k\left(q\left(\frac{\Delta_v^n y_k}{\rho_2}\right)\right) \\ \leq\ & 1. \end{aligned}$$

Hence we have,

$$\begin{aligned} & g_\Delta(x+y) \\ & =\sum_{k=1}^n q\left(x_k+y_k\right)+\inf\left\{\rho^{\frac{p_k}{H}}: \sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n\left(x_k+y_k\right)}{\rho}\right)\right) \leq 1,\ \rho>0\right\} \\ & \leq \sum_{k=1}^n q\left(x_k\right)+\inf\left\{\rho^{\frac{p_k}{H}}: \sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho}\right)\right) \leq 1,\ \rho>0\right\} \\ & \quad +\sum_{k=1}^n q\left(y_k\right)+\inf\left\{\rho^{\frac{p_k}{H}}: \sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n y_k}{\rho}\right)\right) \leq 1,\ \rho>0\right\} \\ & \leq g_\Delta(x)+g_\Delta(y). \end{aligned}$$

The continuity of scalar multiplication follows from the following equality:

$$\begin{aligned} g_\Delta(\lambda x) & =\sum_{k=1}^n q\left(\lambda x_k\right)+\inf\left\{\rho^{\frac{p_k}{H}}: \sup_{k \geq 1} k^{-s} M_k\left(q\left(\frac{\Delta_v^n(\lambda x_k)}{\rho}\right)\right) \leq 1,\ \rho>0\right\} \\ & =|\lambda| \sum_{k=1}^n q\left(x_k\right)+\inf\left\{(r|\lambda|)^{\frac{p_k}{H}}: \sup_{k \geq 1} k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k}{r}\right)\right) \leq 1,\ r>0\right\}, \end{aligned}$$

where $r=\frac{\rho}{|\lambda|}$. Hence the spaces $m\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ and $m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ are paranormed by g_Δ. □

Theorem 2.4. *Let (X, q) be complete seminormed space, then the spaces $m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ and $m\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ are complete.*

Proof. We prove it for the case $m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ and the other case can be established similarly. Let (x^i) be a Cauchy sequence in $m\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$. Let $\delta>0$ be fixed and $r>0$ be such that for a given $0<\varepsilon<1$, $\frac{\varepsilon}{r\delta}>0$ and $r\delta \geq 1$. Then $g_\Delta(x^i-x^s) \rightarrow 0$ as $i, s \rightarrow \infty$. Then there exists a positive integer n_0 such that $g_\Delta(x^i-x^s)<\frac{\varepsilon}{r\delta}$, for all $i, s \geq n_0$.

$$\sum_{k=1}^n q\left(x_k^i-x_k^s\right)+\inf\left\{\rho^{\frac{p_k}{H}}: \sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k^i-\Delta_v^n x_k^s}{\rho}\right)\right) \leq 1,\ \rho>0\right\}<\frac{\varepsilon}{r\delta}, \tag{2}$$

for all $i, s \geq n_0$.

From (2) we have $(x_k^i)_{i=1}^{\infty}$ is a Cauchy sequence in X for each $k = 1, 2, ..., n$. Hence $(x_k^i)_{i=1}^{\infty}$ is convergent in X, for each fixed $k = 1, 2, ..., n$. Let

$$\lim_{i\to\infty} x_k^i = x_k, \text{ for each } k = 1, 2, ..., n. \tag{3}$$

From (2) we have

$$k^{-s}\left[M_k\left(q\left(\frac{\Delta_v^n x_k^i - \Delta_v^n x_k^s}{\rho}\right)\right)\right] \le M(\frac{r\delta}{2}), \text{ for all } i, s \ge n_0 \text{ and all } k \in \mathbb{N},$$

$$\Rightarrow q\left(\Delta_v^n x_k^i - \Delta_v^n x_k^s\right) < \varepsilon, \text{ for all } i, s \ge n_0 \text{ and all } k \in \mathbb{N}.$$

Hence $(\Delta_v^n(x_k^i))_{i=1}^{\infty}$ for all $k \in \mathbb{N}$, is a Cauchy sequence in X and hence are convergent in X. Let $\lim_{i\to\infty} \Delta_v^n x_k^i = y_k$.

Now for $k = 1$, by (1) and (3) since $\lim_{i\to\infty} \Delta_v^n x_1^i = y_1$, we have $\lim_{i\to\infty} x_{k+1}^i = x_{k+1}$ exists. Proceeding in this way inductively we have $\lim_{i\to\infty} x_k^i = x_k$ for each $k \in \mathbb{N}$.

Now using the continuity of M_k and applying the standard techniques, we have for all $i \ge n_0$,

$$\lim_{s\to\infty}\sum_{k=1}^{n} q\big(x_k^i - x_k^s\big) + \inf\left\{\rho^{\frac{p_k}{H}} : \lim_{s\to\infty}\sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k^i - \Delta_v^n x_k^s}{\rho}\right)\right) \le 1,\ \rho > 0\right\} < \frac{2\varepsilon}{r\delta}$$

$$\Rightarrow \sum_{k=1}^{n} q\big(x_k^i - x_k\big) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_k k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k^i - \Delta_v^n x_k}{\rho}\right)\right) \le 1,\ \rho > 0\right\} < \frac{2\varepsilon}{r\delta}$$

$$\Rightarrow g_\Delta\left(x^i - x\right) < \frac{2\varepsilon}{r\delta}.$$

Hence $\left(x^i - x\right) \in m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$. Since $\left(x^i\right) \in m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ and $m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ is a linear space, so we have $x = x^i - \left(x^i - x\right) \in m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$. Hence $x \in m_0\left(\Delta_v^n, \mathcal{M}, p, q, s\right)$ is a closed subspace of $\ell_\infty(\Delta_v^n, \mathcal{M}, p, q, s)$. □

Theorem 2.5. *Let $n \ge 1$, then for all $0 < i \le n$, $Z\left(\Delta_v^i, \mathcal{M}, q, s\right) \subseteq Z\left(\Delta_v^n, \mathcal{M}, q, s\right)$ where $Z = \bar{c}$, $\overline{c_0}$, m and m_0. The inclusions are strict.*

Proof. We establish it for $\overline{c_0}\left(\Delta_v^{n-1}, \mathcal{M}, q, s\right) \subseteq \overline{c_0}\left(\Delta_v^n, \mathcal{M}, q, s\right)$. It follows from the following inequality

$$k^{-s} M_k\left(q\left(\frac{\Delta_v^n x_k}{\rho}\right)\right) \le D\left\{k^{-s} M_k\left(q\left(\frac{\Delta_v^{n-1} x_k}{\rho}\right)\right) + k^{-s} M_k\left(q\left(\frac{\Delta_v^{n-1} x_{k+1}}{\rho}\right)\right)\right\}$$

that $(x_k) \in \overline{c_0}\left(\Delta_v^{n-1}, \mathcal{M}, q, s\right)$ implies $(x_k) \in \overline{c_0}\left(\Delta_v^n, \mathcal{M}, q, s\right)$.

On applying the principle of induction it follows that $\overline{c_0}\left(\Delta_v^i, \mathcal{M}, q, s\right) \subseteq \overline{c_0}\left(\Delta_v^n, \mathcal{M}, q, s\right)$, for $i = 0, 1, 2, ..., n-1$. The proof for the rest of the cases will follow similarly. □

To show the inclusions are strict consider the following example.

Example 1. Let $M_k(x) = x$, $v_k = 1$, for all $k \in \mathbb{N}$, $s = 0$ and $q(x) = |x|$. Then the sequence $(x_k) = \left(k^{n-1}\right) \in Z(\Delta_v^n, \mathcal{M}, q, s)$ but $(x_k) \notin Z\left(\Delta_v^{n-1}, \mathcal{M}, q, s\right)$ for $Z = \overline{c_0}$ and m_0, since $\Delta^n x_k = 0$ and $\Delta^{n-1} x_k = (-1)^{n-1}(n-1)!$ for all $k \in \mathbb{N}$. Under the above restrictions, consider the sequence $(x_k) = (k^n)$. Then $(x_k) \in Z(\Delta_v^n, \mathcal{M}, q, s)$ but $(x_k) \notin Z\left(\Delta_v^{n-1}, \mathcal{M}, q, s\right)$ for $Z = \overline{c}$ and m.

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two seminorms q_1 and q_2 on X we have*

$$Z(\Delta_v^n, \mathcal{M}, p, q_1, s) \cap Z(\Delta_v^n, \mathcal{M}, t, q_2, s) \neq \varnothing,$$

where $Z = \overline{c}$, $\overline{c_0}$, m and m_0.

Proof. The proof follows from the fact that the zero sequence belongs to each of the classes the sequence spaces involved in the intersection. □

The proof of the following result is easy, so omitted.

Proposition 2.7. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, any sequence $p = (p_k)$ of strictly positive real numbers and seminorms q, q_1 and q_2 on X. Then*

(i) $\overline{c_0}(\Delta_v^n, \mathcal{M}, p, q, s) \subseteq \overline{c}(\Delta_v^n, \mathcal{M}, p, q, s)$,

(ii) $m_0(\Delta_v^n, \mathcal{M}, p, q, s) \subseteq m(\Delta_v^n, \mathcal{M}, p, q, s)$,

(iii) $Z(\Delta_v^n, \mathcal{M}, p, q_1, s) \cap Z(\Delta_v^n, \mathcal{M}, p, q_2, s) \subseteq Z(\Delta_v^n, \mathcal{M}, p, q_1 + q_2, s)$, where $Z = \overline{c}$, $\overline{c_0}$, m and m_0.

(iv) If q_1 is stronger than q_2, then $Z(\Delta_v^n, \mathcal{M}, p, q_1, s) \subseteq Z(\Delta_v^n, \mathcal{M}, p, q_2, s)$, where $Z = \overline{c}$, $\overline{c_0}$, m and m_0.

The proof of the following two theorems can be obtained from example 2, example 3, example 4 and example 5 of Tripathy [19], on taking $M_k = M$ and $v_k = 1$ for all $k \in \mathbb{N}$ and $s = 0$.

Theorem 2.8. *The sequence spaces $Z(\Delta_v^n, \mathcal{M}, p, q, s)$ are not solid for $n > 0$, where $Z = \overline{c}$, $\overline{c_0}$, m and m_0.*

Theorem 2.9. *The sequence spaces $Z(\Delta_v^n, \mathcal{M}, p, q, s)$ are not symmetric for $n > 0$, where $Z = \overline{c}$, $\overline{c_0}$, m and m_0.*

Proposition 2.10. *For two sequences (p_k) and (t_k) we have $m_0(\Delta_v^n, \mathcal{M}, t, q, s) \subseteq m_0(\Delta_v^n, \mathcal{M}, p, q, s)$ if and only if $\liminf_{k \in K} \frac{p_k}{t_k} > 0$, where $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$.*

Proof. The proof is obvious in view of Lemma 1.3. □

The following result is a consequence of the above result.

Corollary 2.11. *For two sequences (p_k) and (t_k) we have $m_0(\Delta_v^n, \mathcal{M}, t, q, s) =$*

$m_0(\Delta_v^n, \mathcal{M}, p, q, s)$ if and only if $\liminf_{k\in K} \frac{p_k}{t_k} > 0$ *and* $\liminf_{k\in K} \frac{t_k}{p_k} > 0$*, where $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$.*

The following result is obvious in view of Lemma 1.4.

Proposition 2.12. *Let $h = \inf p_k$ and $G = \sup p_k$, then the followings are equivalent:*
(i) $G < \infty$ and $h > 0$,
(ii) $m(\Delta_v^n, \mathcal{M}, p, q, s) = m(\Delta_v^n, \mathcal{M}, q, s)$.

Since the inclusion relations $m(\Delta_v^n, M, p, q, s) \subset \ell_\infty(\Delta_v^n, M, p, q, s)$ and $m_0(\Delta_v^n, M, p, q, s) \subset \ell_\infty(\Delta_v^n, M, p, q, s)$ are strict, we have the following result.

Corollary 2.13. *The spaces $m(\Delta_v^n, M, p, q, s)$ and $m_0(\Delta_v^n, M, p, q, s)$ are nowhere dense subsets of $\ell_\infty(\Delta_v^n, M, p, q, s)$.*

The following result is obvious in view of Lemma 1.5.

Proposition 2.14. *The spaces $m(\Delta_v^n, \mathcal{M}, p, q, s)$ and $m_0(\Delta_v^n, \mathcal{M}, p, q, s)$ are not separable.*

References

[1] Altın, Y., Et, M and Tripathy, B. C. *The sequence space $|\overline{N}_p|(M, r, q, s)$ on seminormed spaces*, Appl. Math. Comp., **154**(2004),423-430.

[2] Connor, J. S. *The statistical and strong p-Ces'aro convergence of sequences*, Analysis, **8**(1988), 47-63.

[3] Et, M. and Çolak, R. *On some generalized difference sequence spaces*, Soochow J. Math., **21**(1995), 377-386.

[4] Et, M. and Esi, A. *On Köthe-Toeplitz duals of generalized difference sequence spaces*, Bull Malaysian Math. Sc. Soc., **23**(2)(2000), 25-32.

[5] Et, M. and Nuray, F. *Δ^m-statistical convergence*, Indian J. Pure Appl. Math., **32**(6)(2001), 961-969.

[6] Et, M., Altınok, H. and Altın, Y. *On some generalized sequence spaces*, Appl. Math. Comp., **154**(2004), 167-173.

[7] Et, M., Altın, Y., Choudhary, B. and Tripathy, B.C. *On some classes of sequences defined by sequences of Orlicz functions*, Math. Inequalities and Appl., **9**(2)(2006), 335-342.

[8] Fast, H. *Sur la convergence statistique*, Colloq. Math., **2**(1951), 241-244.

[9] Fridy, J. A. *On statistical convergence*, Analysis, **5**(1985), 301-313.

[10] Kamthan, P. K. and Gupta, M. Sequence Spaces and Series, Marcel Dekkar Inc. New York, 1981.

[11] Kızmaz, H. *On certain sequence spaces*, Canad. Math.Bull., **24**(1981), 169-176.

[12] Lindenstrauss, J. and Tzafriri L. *On Orlicz sequence spaces*, Israel J. Math., **10**(1971), 379-390.

[13] Maddox, I. J. *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc., **104**(1988), 141-145.

[14] Nakano, H. *Concave modulars*, J. Math. Soc. Japan, **5**(1953), 29-49.

[15] Rath, D. and Tripathy, B.C. *Matrix maps on sequence spaces associated with sets of integers*, Indian J. Pure Appl. Math., **27**(2)(1996), 197-206.

[16] Ruckle, W. H. *FK spaces in which the sequences of coordinate vectors is bounded*, Canad. Jour. Math., **25**(1973), 973-978.

[17] Šalàt, T. *On statistically convergent sequences of real numbers*, Mathematica Slovaca, **2**(1980), 139-150.

[18] Tripathy, B. C. and Sen, M. *On generalized statistically convergent sequence spaces*, Indian J. Pure Appl. Math., **32**(11)(2001), 1689-1694.

[19] Tripathy, B. C. *Generalized difference paranormed statistically convergent sequences defined by Orlicz functions in a locally convex space*, Soochow J. Math., **30**(4)(2004), 431-446.

[20] Tripathy, B. C. *Matrix transformations between some classes of sequences*, J. Math. Anal. Appl., **206**(1997), 448-450.

[21] Tripathy, B. C. *On generalized difference paranormed statistically convergent sequences*, Indian J. Pure and Appl. Math., **35**(5) (2004), 655-663.

[22] Tripathy, B. C. and Mahanta, S. *On a class of generalized lacunary difference sequence spaces defined by Orlicz function*, Acta Math. Applicata Sinica, **20**(2) (2004), 231-238.

[23] Tripathy, B. C. and Sen, M. *Characterization of some matrix classes involving paranormed sequence spaces*, Tamkang J. Math., **37**(2)(2006), 155-162.

[24] Tripathy, B. C., Altın, Y. and Et, M. *Generalized difference sequences spaces on seminormed spaces defned by Orlicz functions*, Mathematica Slovaca, **58**(3)(2008), 315-324.

[25] Tripathy, B. C. and Sarma, B. *Sequence spaces of fuzzy real numbers defined by Orlicz functions*, Mathematica Slovaca, **58**(5)(2008), 621-628.

[26] Tripathy, B. C. and Borgogain, S. *The sequence space* $m(M,\phi,\Delta_m^n,p)^F$, Mathematical Modelling and Analysis, **13**(4) (2008), 577-586.

[27] Tripathy, B. C., Choudhary, B. and Sarma, B. *On some new type generalized difference sequence spaces*, Kyungpook Mathematical Journal, **48**(4)(2008), 613-622.