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Ostrowski's Type Inequalities for (α, m) −Convex Functions

Muhamet Emin Özdemir, Havva Kavurmaci *and* Erhan Set*
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
e-mail: emos@atauni.edu.tr, havva.kvrmc@yahoo.com *and* erhanset@yahoo.com

Abstract. In this paper, we establish new inequalities of Ostrowski's type for functions whose derivatives in absolute value are (α, m) −convex.

1. Introduction

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [1]).

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1.1}$$

This inequality is well known in the literature as the *Ostrowski inequality.* For some results which generalize, improve and extend the inequality (1.1) see ([1], [4], [6], [8]) and the references therein.

In [9], G. Toader defined $m-$convexity as the following:

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. Denote by $K_m(b)$ the set of the $m-$convex functions on $[0, b]$ for which $f(0) \leq 0$.

In [7], V.G. Miheşan defined $(\alpha, m)-$ convexity as the following :

Definition 2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m) −convex,*

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where $(\alpha, m) \in [0,1]^2$, *if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ *and* $t \in [0,1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m) = (1, m)$, $(\alpha, m)-$convexity reduces to $m-$convexity; $(\alpha, m) = (\alpha, 1)$, $(\alpha, m)-$convexity reduces to $\alpha-$convexity and for $(\alpha, m) = (1, 1)$, $(\alpha, m)-$ convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$. For recent results and generalizations concerning $(\alpha, m)-$ convex functions, see ([2] and [3]).

The following theorem contains the Hadamard type integral inequality (see for example [5]).

Theorem 1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be an M-Lipschitzian mapping on* I *and* $a, b \in I$ *with* $a < b$. *Then we have the inequality;*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M(b-a)}{4}. \tag{1.2}$$

In [1], in order to prove some inequalities related to Ostrowski inequality, M. Alomari, M. Darus, S.S. Dragomir and P. Cerone used the following lemma.

Lemma 1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L[a, b]$, *then the following equality holds:*

$$\begin{aligned} f(x) - \frac{1}{b-a}\int_a^b f(u)du &= \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx + (1-t)a)\,dt \\ &- \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx + (1-t)b)\,dt \end{aligned}$$

for each $x \in [a, b]$.

The main purpose of this paper is to establish several Ostrowski's type inequalities for functions whose derivatives in absolute value are $(\alpha, m)-$convex.

2. Main results

In order to prove our results we need the following equality:

$$\begin{aligned} mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du &= \frac{(x-ma)^2}{b-a}\int_0^1 tf'(tx + m(1-t)a)dt \\ &-\frac{(mb-x)^2}{b-a}\int_0^1 tf'(tx + m(1-t)b)dt \end{aligned} \tag{2.1}$$

which is a special case of Lemma 1 with $ma \to a$ and $mb \to b$.

Theorem 2. *Let I be an open real interval such that $[0,\infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([ma, mb])$, where $ma, mb \in I$ with $a < b$. If $|f'|^q$ is $(\alpha, m)-$ convex on $[ma, mb]$ for $(\alpha, m) \in [0,1] \times (0,1]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \le M$, $x \in [ma, mb]$, then the following inequality holds :*

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \le M \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{\alpha m + 1}{\alpha + 1} \right)^{\frac{1}{q}} \frac{(x-ma)^2 + (mb-x)^2}{b-a}$$

for each $x \in [ma, mb]$.

Proof. From (2.1) and using the Hölder's inequality for $q > 1$, we have

$$\begin{aligned}
&\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \\
\le\ & \frac{(x-ma)^2}{b-a} \int_0^1 t \left| f'(tx + m(1-t)a) \right| dt \\
&+ \frac{(mb-x)^2}{b-a} \int_0^1 t \left| f'(tx + m(1-t)b) \right| dt \\
\le\ & \frac{(x-ma)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx + m(1-t)a) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \frac{(mb-x)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx + m(1-t)b) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is (α, m) −convex and $|f'(x)| \le M$, then we have

$$\begin{aligned}
&\int_0^1 \left| f'(tx + m(1-t)a) \right|^q dt \\
\le\ & \int_0^1 \left[t^\alpha \left| f'(x) \right|^q + m(1-t^\alpha) \left| f'(a) \right|^q \right] dt \\
\le\ & \frac{M^q}{\alpha+1} (1 + \alpha m)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left| f'(tx + m(1-t)b) \right|^q dt \\
\le\ & \int_0^1 \left[t^\alpha \left| f'(x) \right|^q + m(1-t^\alpha) \left| f'(b) \right|^q \right] dt \\
\le\ & \frac{M^q}{\alpha+1} (1 + \alpha m).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|mf(x)-\frac{1}{b-a}\int_{ma}^{mb}f(u)du\right| \\
\leq\ & \frac{(x-ma)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\alpha+1}(1+\alpha m)\right)^{\frac{1}{q}} \\
&+\frac{(mb-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\alpha+1}(1+\alpha m)\right)^{\frac{1}{q}} \\
=\ & M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}}\frac{(x-ma)^2+(mb-x)^2}{b-a}.
\end{aligned}$$

This completes the proof. □

Remark 1. *Since for $p\in(1,\infty)$ we have*

$$\frac{1}{2}\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\leq 1,$$

if in Theorem 2 we put $m=1$, we obtain

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\leq M\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \tag{2.2}$$

Now, if we choose in (2.2), $x=\frac{a+b}{2}$, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\leq\frac{M(b-a)}{2}.$$

Theorem 3. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([ma,mb])$, where $ma,mb\in I$. If $|f'|^q$ is $(\alpha,m)-$convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]\times(0,1]$ and $|f'(x)|\leq M$, $q\in[1,\infty)$, $x\in[ma,mb]$, then the following inequality holds:*

$$\left|mf(x)-\frac{1}{b-a}\int_{ma}^{mb}f(u)du\right|\leq M\left(\frac{2+\alpha m}{\alpha+2}\right)^{\frac{1}{q}}\frac{(x-ma)^2+(mb-x)^2}{2(b-a)}$$

for each $x\in[ma,mb]$.

Proof. Suppose that $q = 1$. From (2.1) we have

$$\begin{aligned} &\left| mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du \right| \\ \leq\ & \frac{(x-ma)^2}{b-a}\int_0^1 t\left|f'\left(tx+m\left(1-t\right)a\right)\right|dt \\ &+\frac{(mb-x)^2}{b-a}\int_0^1 t\left|f'\left(tx+m\left(1-t\right)b\right)\right|dt. \end{aligned}$$

Since $|f'|$ is (α, m) −convex on $[ma, mb]$ we know that for any $t \in [0, 1]$

$$\left|f'\left(tx+m\left(1-t\right)y\right)\right| \leq t^{\alpha}\left|f'(x)\right| + m(1-t^{\alpha})\left|f'(y)\right|$$

so

$$\begin{aligned} &\left| mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du \right| \\ \leq\ & \frac{(x-ma)^2}{b-a}\int_0^1 t\left[t^{\alpha}\left|f'(x)\right| + m\left(1-t^{\alpha}\right)\left|f'(a)\right|\right]dt \\ &+\frac{(mb-x)^2}{b-a}\int_0^1 t\left[t^{\alpha}\left|f'(x)\right| + m\left(1-t^{\alpha}\right)\left|f'(b)\right|\right]dt \\ =\ & \frac{(x-ma)^2}{b-a}\int_0^1 [t^{\alpha+1}\left|f'(x)\right| + m\left(t-t^{\alpha+1}\right)\left|f'(a)\right|]dt \\ &+\frac{(mb-x)^2}{b-a}\int_0^1 \left[t^{\alpha+1}\left|f'(x)\right| + m\left(t-t^{\alpha+1}\right)\left|f'(b)\right|\right]dt \\ \leq\ & \frac{(x-ma)^2}{b-a}\frac{M}{\alpha+2}\left[1+\frac{\alpha m}{2}\right] \\ &+\frac{(mb-x)^2}{b-a}\frac{M}{\alpha+2}\left[1+\frac{\alpha m}{2}\right] \\ =\ & \left(\frac{2+\alpha m}{\alpha+2}\right)\frac{M}{b-a}\left[\frac{(x-ma)^2+(mb-x)^2}{2}\right], \end{aligned}$$

where we have used the fact that

$$\int_0^1 t^{\alpha+1}dt = \frac{1}{\alpha+2}$$

and

$$\int_0^1 \left(t-t^{\alpha+1}\right)dt = \frac{\alpha}{2\left(\alpha+2\right)}.$$

The proof is completed for this case. Suppose now that $q > 1$. From (2.1) and using the well-known power-mean inequality, we obtain

$$\begin{aligned}
&\left| mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du \right| \\
\leq\ & \frac{(x-ma)^2}{b-a}\int_0^1 t\left|f'\left(tx+m\left(1-t\right)a\right)\right|dt \\
&+\frac{(mb-x)^2}{b-a}\int_0^1 t\left|f'\left(tx+m\left(1-t\right)b\right)\right|dt \\
\leq\ & \frac{(x-ma)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+m\left(1-t\right)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(mb-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+m\left(1-t\right)b\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is (α, m) −convex on $[ma, mb]$, we know that for every $t \in [0, 1]$

$$\left|f'\left(tx+m\left(1-t\right)y\right)\right|^q \leq t^\alpha \left|f'(x)\right|^q + m(1-t^\alpha)\left|f'(y)\right|^q,$$

so we obtain

$$\begin{aligned}
&\int_0^1 t\left|f'\left(tx+m\left(1-t\right)a\right)\right|^q dt \\
\leq\ & \int_0^1 t\left[t^\alpha\left|f'(x)\right|^q + m\left(1-t^\alpha\right)\left|f'(a)\right|^q\right]dt \\
\leq\ & \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t\left|f'\left(tx+m\left(1-t\right)b\right)\right|^q dt \\
\leq\ & \int_0^1 t\left[t^\alpha\left|f'(x)\right|^q + m\left(1-t^\alpha\right)\left|f'(b)\right|^q\right]dt \\
\leq\ & \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left| mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du \right| \\
\leq\ & M\left(\frac{2+\alpha m}{\alpha+2}\right)^{\frac{1}{q}}\frac{(x-ma)^2+(mb-x)^2}{2\left(b-a\right)}
\end{aligned}$$

which completes the proof. □

Remark 2. *Since for* $p \in (1, \infty)$ *we have*

$$\frac{1}{2} \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \le 1,$$

if in Theorem 3 we put $m = 1$ *and* $x = \frac{a+b}{2}$*, we obtain*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \frac{M(b-a)}{4}$$

which is the inequality in (1.2)*.*

Remark 3. *In Theorem 3, if we choose* $(\alpha, m) = (\alpha, 1)$*, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

which is the inequality in (1.1)*.*

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