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On Some Uniqueness Theorems of Meromorphic Functions Sharing Three Values

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Abstract. We prove some uniqueness theorems of meromorphic functions sharing three values which improves some results of Banerjee.

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities, we say that f and g share the value a CM(Counting Multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [6]. When we let r towards ∞ we will always assume that r may avoid a set I of finite linear measure, not necessarily the same every time in its approach to ∞.

Definition 1([2]). *Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r, a; f \mid= p)$ the counting function of those a-points of f whose multiplicities are excatly equal to p, where an a-point is counted only once.*

Definition 2([7], [8])**.** *Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r, a; f \mid\geq p)$ the counting function of those a-points of f whose multiplicities are greater than or equal to p, where an a-point is counted only once.*

Definition 3. *Let f and g share a value a IM. Let z be an a-point of f and g with multiplicities $m_f(z)$ and $m_g(z)$ respectivily. Let p be a positive integer.*
We put $\gamma_{(p}(z) = 1$ if both $m_f(z) > p$ and $m_g(z) > p$ and $\gamma_{(p}(z) = 0$, otherwise.
Let

$$\overline{n}(r, a; f \mid> p, g \mid> p) = \sum_{|z| \leq r} \gamma_{(p}(z).$$

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Clearly $\overline{n}(r,a;f\mid>p,g\mid>p)=\overline{n}(r,a;g\mid>p,f\mid>p)$. *We denote by* $\overline{N}(r,a;f\mid>p,g\mid>p)$ *the integrated counting function obtained by from* $\overline{n}(r,a;f\mid>p,g\mid>p)$. *Clearly* $\overline{N}(r,a;f\mid>p,g\mid>p)=\overline{N}(r,a;g\mid>p,f\mid>p)$.

Definition 4([15]). *Let* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *and* g *with multiplicities* p *and* q *respectivily. Let* s *be a positive integer. We denote by* $\overline{N}_{f>s}(r,1;g)$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p>q=s$. *Similarly* $\overline{N}_{f<s}(r,1;g)$ *will denote the reduced counting function of those* 1*-points of* f *and* g *where* $p<q=s$.

Definition 5([7], [8])**.** *Let* k *be a nonnegative integer or infinity. For* $a\in\mathbb{C}\cup\{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* m *is counted* m *times if* $m\leq k$ *and* $k+1$*times if* $m>k$. *If* $E_k(a;f)=E_k(a;g)$,*we say that* f *and* g *share the value* a *with weight* k.

The definition implies that if f,g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f-a$ of multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$ where m is not necessarily equal to n.
We write f,g share (a,k) to mean f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for all integers p , $0\leq p<k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

Definition 6([2]). *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share* (a,k) *where* $a\in\mathbb{C}\cup\{\infty\}$. *Let* z_0 *be an* a*-point of* f *with multiplicity* p, *an* a*-point of* g *of multiplicity* q. *We denote by* $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ *the counting function of those* a*-points of* f*and* g *where* $p>q(q>p)$, *by* $\overline{N}_E^{(k+1}(r,a;f)$ *the counting functions of those* a*-points of* f *and* g *where* $p=q\geq k+1$ *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_E^{(k+1}(r,a;g)$. *Clearly* $\overline{N}_E^{(k+1}(r,a;f)=\overline{N}_E^{(k+1}(r,a;g)$.

Definition 7([7], [8])**.** *Let* f,g *share the value* a *IM. We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the corresponding* a*-points of* g. *Clearly* $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

In 1998, 2003 H. X. Yi[12, 14] improved some results of Ueda[11], G.Brosch[4] and Lahiri[7] in the following theorems:

Theorem A([12]). *Let* f,g *share* $0,1,\infty$ *CM. If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2}\tag{1}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

Theorem B([14]). *Let* f, g *share* $(0,1)$, $(1,5)$, $(\infty,0)$. *If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+3\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{2}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

Theorem C([14]). *Let* f, g *share* $(0,1)$, $(1,3)$, $(\infty,0)$. *If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{3}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

Theorem D([14]). *et* f, g *share* $(0,1)$, $(1,6)$, $(\infty,2)$. *If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{4}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

In 2007 Banerjee[2] improved Theorem B and Theorem C by weakening the conditions (2) and (3) respectively as follows.

Theorem E. *Let* f, g *share* $(0,1)$, $(1,5)$, $(\infty,0)$. *If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+3\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{5}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

Theorem F. *Let* f, g *share* $(0,1)$, $(1,3)$, $(\infty,0)$. *If*

$$\limsup_{r\to\infty}\frac{N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{6}$$

then either $f\equiv g$ *or* $fg\equiv 1$.

Chen, Shen and Lin proved the following results that improved Theorem B, Theorem C and Theorem D in a manner different from Banerjee's.

Theorem G([5]). *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(0,1)$, $(1,m)$, $(\infty,0)$ *where* $m(\geq 2)$ *is an integer. If*

$$N(r,0;f\mid=1)+\frac{2(m+1)}{m-1}\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)<(\frac{1}{2}+0(1))T(r,f) \tag{7}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$.

Theorem H([5]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1),(1,m),(\infty,k)$ where m and k are positive integers or infinity satisfying $(m-1)(km-1)>(1+m)^2$. If*

$$N(r,0;f\mid=1)+\overline{N}(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)<(\frac{1}{2}+0(1))T(r,f) \tag{8}$$

for $r\notin I$ then either $f\equiv g$ or $fg\equiv 1$.

Very recently Banerjee proved the following theorem improving all previous results.

Theorem I([3]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k) where $m(\geq 2)$ is an integer. If*

$$\begin{aligned} N(r,0;f\mid=1)+\overline{N}(r,\infty;f)+\frac{m+3}{m-1}\overline{N}(r,\infty;f\mid\geq k+1)\\ -\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)<(\frac{1}{2}+0(1))T(r,f)\end{aligned} \tag{9}$$

for $r\notin I$ then either $f\equiv g$ or $fg\equiv 1$.

Improving a result of H. X. Yi[12], Lahiri[10] proved the following.

Theorem J. *Let f,g share $(0,0)$, $(1,2)$, $(\infty,0)$. If*

$$\limsup_{r\to\infty}\frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-m(r,1;g)}{T(r,f)}<1 \tag{10}$$

then either $f\equiv g$ or $fg\equiv 1$.

In 2006 Banerjee[1] improved the above result by weakening the condition (10) in the Theorem K and proved some supplementary results.

Theorem K. *Let f,g share $(0,0)$, $(1,2)$, $(\infty,0)$. If*

$$\limsup_{r\to\infty}\frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)-\overline{N}_L(r,1;g)-m(r,1;g)}{T(r,f)}<1 \tag{11}$$

then either $f\equiv g$ or $fg\equiv 1$.

Theorem L. *Let f,g share $(0,0)$, $(1,1)$, (∞,∞). If*

$$\limsup_{r\to\infty}\frac{3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)}{T(r,f)}<1 \tag{12}$$

then either $f\equiv g$ or $fg\equiv 1$.

Theorem M. *Let f, g share $(0,0)$, $(1,1)$, $(\infty,0)$. If*

$$\limsup_{r\to\infty} \frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)}{T(r,f)}<1 \tag{13}$$

then either $f\equiv g$ or $fg\equiv 1$.

Theorem N. *Let f, g share $(0,0)$, $(1,0)$, $(\infty,0)$. If*

$$\limsup_{r\to\infty} \frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)}{T(r,f)}<1 \tag{14}$$

then either $f\equiv g$ or $fg\equiv 1$, where $N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

Theorem O. *Let f, g share $(0,0)$, $(1,0)$, (∞,∞). If*

$$\limsup_{r\to\infty} \frac{3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)}{T(r,f)}<1 \tag{15}$$

then either $f\equiv g$ or $fg\equiv 1$, where $N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

The theorems stated so far evoke the following questions in our mind:
1. *Is it possible to reduce the nature of sharing of the value* 1 *in Theorem I, in particular what happens if f, g share the value* $(1,1)$?
2. *Is it possible to weaken the conditions* (11), (12), (13), (14), (15) *in Theorems K, L, M, N, O, respectively*?

Motivated by these questions we state our main results as follows.

Theorem 1. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(1,1)$, (∞,k). If*

$$\begin{aligned} N(r,0;f\mid=1)&+\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;f\mid\geq k+1)+2[\overline{N}_{g>2}(r,1;f)\\ &+\overline{N}_{f>2}(r,1;g)]-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)-\frac{1}{2}\overline{N}_E^{(3}(r,1;f)\\ &<(\frac{1}{2}+0(1))T(r,f)\end{aligned} \tag{16}$$

for $r\notin I$ then either $f\equiv g$ or $fg\equiv 1$.

Theorem 2. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(1,2)$, (∞,k). If*

$$\begin{aligned} N(r,0;f\mid=1)&+\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;f\mid\geq k+1)+\\ &-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)-\frac{1}{2}\overline{N}_E^{(3}(r,1;f)\\ &<(\frac{1}{2}+0(1))T(r,f)\end{aligned} \tag{17}$$

for $r\notin I$ then either $f\equiv g$ or $fg\equiv 1$.

Theorem 3. *Let* f, g *share* $(0,0)$, $(1,2)$, $(\infty,0)$. *If*

$$3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-\overline{N}_L(r,1;g)-m(r,1;g)-\overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)<(1+o(1))T(r,f) \tag{18}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$.

Theorem 4. *Let* f, g *share* $(0,0)$, $(1,1)$, (∞,∞). *If*

$$3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)-\overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(3}(r,1;f)<(1+o(1))T(r,f) \tag{19}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$.

Theorem 5. *Let* f, g *share* $(0,0)$, $(1,1)$, $(\infty,0)$. *If*

$$3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)-\overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)<(1+o(1))T(r,f) \tag{20}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$.

Theorem 6. *Let* f, g *share* $(0,0)$, $(1,0)$, $(\infty,0)$. *If*

$$3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)-\overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)<(1+o(1))T(r,f) \tag{21}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$, *where*
$N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

Theorem 7. *Let* f, g *share* $(0,0)$, $(1,0)$, (∞,∞). *If*

$$3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)-\overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(3}(r,1;f)<(1+o(1))T(r,f) \tag{22}$$

for $r\notin I$ *then either* $f\equiv g$ *or* $fg\equiv 1$, *where* $N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

Remark. (i) Theorem 1 answers the question 1 raised above.
(ii) Theorem 2 improves Theorem I when $m=2$.
(iii) Theorems 3, 4, 5, 6, 7 improve Theorems K, L, M, N, O respectively under weaker conditions as is asked in question 2 above.

2. Lemmas

In this section we present some lemmas which will be required to establish our theorems. In the lemmas several times we use the function H defined by $H = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}$.

Lemma 1([7]). *If f and g share $(0,0)$, $(1,0)$, $(\infty,0)$ then*
(i) $T(r,f) \leq 3T(r,g) + S(r,f)$
(ii) $T(r,g) \leq 3T(r,f) + S(r,g)$.

Lemma 1 shows that $S(r,f) = S(r,g)$ and we denote their common value by $S(r)$.

Lemma 2. *If f and g share $(1,1)$ then*

$$\begin{aligned} &N(r,1;g) - \overline{N}(r,1;g) \\ &\geq 2\overline{N}_L(r,1;g) + \overline{N}_L(r,1;f) + \overline{N}_E^{(2}(r,1;f) + \overline{N}_E^{(3}(r,1;f) + [\overline{N}_{f<4}(r,1;g) \\ &\quad + 2\overline{N}_{f<5}(r,1;g) + 3\overline{N}_{f<6}(r,1;g) + ...] + [\overline{N}_{f>3}(r,1;g) + 2\overline{N}_{f>4}(r,1;g) \\ &\quad + 3\overline{N}_{f>5}(r,1;g) + ...]. \end{aligned}$$

Proof. Let z_0 be a 1-point of f and g of respective multiplicities p and q. We denote by $N_1(r)$, $N_2(r)$ and $N_3(r)$ the counting functions of those 1-points of f and g when $2 \leq q < p$, $2 \leq q = p$ and $2 \leq p < q$ respectively where each point in these counting functions is counted $q-2$ times. Since f, g share $(1,1)$ we have

$$\begin{aligned} &N(r,1;g) - \overline{N}(r,1;g) \\ &= \overline{N}_E^{(2}(r,1;f) + N_2(r) + \overline{N}_L(r,1;g) + N_3(r) + \overline{N}_L(r,1;f) + N_1(r). \end{aligned}$$

Now $N_3(r) > \overline{N}_L(r,1;g) + [\overline{N}_{f<4}(r,1;g) + 2\overline{N}_{f<5}(r,1;g) + 3\overline{N}_{f<6}(r,1;g) + ...]$, $N_2(r) > \overline{N}_E^{(3}(r,1;f)$ and $N_1(r) > [\overline{N}_{f>3}(r,1;g) + 2\overline{N}_{f>4}(r,1;g) + ...]$ the lemma follows from above. □

Lemma 3. *If f and g share $(1,0)$ then*

$$\begin{aligned} &N(r,1;g) - \overline{N}(r,1;g) \\ &\geq 2\overline{N}_L(r,1;g) + \overline{N}_L(r,1;f) + \overline{N}_E^{(2}(r,1;f) + \overline{N}_E^{(3}(r,1;f) - \overline{N}_{f>1}(r,1;g) \\ &\quad - \overline{N}_{g>1}(r,1;f). \end{aligned}$$

Proof. Let z_0 be a 1-point of f and g of respective multiplicities p and q. We denote by $N_1(r)$, $N_2(r)$ and $N_3(r)$the counting functions of those 1-points of f and g when $1 \leq q < p$, $2 \leq q = p$ and $1 \leq p < q$ respectively where in the first counting function each point is counted $q-1$ times and in the remaining two $q-2$ times. Then observing that $N_2(r) \geq \overline{N}_E^{(3}(r,1;f)$ the proof follows in the line of proof of Lemma 2.4[1]. □

Lemma 4([8]). *If f and g share $(1,1)$ and $H \not\equiv 0$ then*

$$N(r,1;f\mid=1)=N(r,1;g\mid=1)\le N(r,H)+S(r,f)+S(r,g).$$

Lemma 5([13], [15])**.** *If f and g share $(1,0)$ and $H \not\equiv 0$ then*

$$N_E^{1)}(r,1;f)\le N(r,H)+S(r,f)+S(r,g),$$

where $N_E^{1)}(r,1;f)$ denotes the counting function of simple 1-points of f and g.

Lemma 6([10]). *If f and g share $(0,0)$, $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$ then*

$$N(r,H)\le \overline{N}_*(r,0;f,g)+\overline{N}_*(r,1;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Lemma 7. *If f and g share $(0,0)$,$(1,1)$, (∞,k) and $H \not\equiv 0$ then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)\\&\le \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\&\quad+T(r,g)-m(r,1;g)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ and $\overline{N}_0(r,0;g')$ are same as Lemma [6].

Proof. We have by Lemmas 4, 6 and 2,

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)\\&\le N(r,H)+\overline{N}(r,1;g)+\overline{N}(r,1;f\mid\ge 2)\\&\le \overline{N}_*(r,0;f,g)+\overline{N}_*(r,1;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\&\quad+\overline{N}(r,1;g)+\overline{N}(r,1;f\mid\ge 2)\\&\le \overline{N}_*(r,0;f,g)+\overline{N}_*(r,1;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\&\quad+N(r,1;g)-2\overline{N}_L(r,1;g)-\overline{N}_L(r,1;f)-\overline{N}_E^{(2}(r,1;f)-\overline{N}_E^{(3}(r,1;f)\\&\quad-[\overline{N}_{f<4}(r,1;g)+2\overline{N}_{f<5}(r,1;g)+3\overline{N}_{f<6}(r,1;g)+...]\\&\quad-[\overline{N}_{f>3}(r,1;g)+2\overline{N}_{f>4}(r,1;g)+....]+\overline{N}(r,1;f\mid\ge 2).\end{aligned}$$

Now we observe that $\overline{N}(r,1;f\mid\ge 2)=\overline{N}_L(r,1;g)+\overline{N}_L(r,1;f)+\overline{N}_E^{(2}(r,1;f)$,
and $\overline{N}_L(r,1;g)-[\overline{N}_{f<4}(r,1;g)+2\overline{N}_{f<5}(r,1;g)+3\overline{N}_{f<6}(r,1;g)+...]\le \overline{N}_{g>2}(r,1;f)$
and $\overline{N}_L(r,1;f)-[\overline{N}_{f>3}(r,1;g)+2\overline{N}_{f>4}(r,1;g)+....]\le \overline{N}_{f>2}(r,1;g)$.
Thus from above we obtain

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)\\&\le \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)-\overline{N}_L(r,1;g)+T(r,g)-m(r,1;g)\\&\quad+\overline{N}_0(r,0;g')+\overline{N}_0(r,0;f')-\overline{N}_E^{(3}(r,1;f)+\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g).\end{aligned}$$

This completes the proof. □

Lemma 8. *If f and g share $(0,1)$, $(1,1)$, (∞,k), then*
(i) $\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f \mid\ge 2) \le \overline{N}_*(r,1;f,g) + \overline{N}(r,\infty;f \mid\ge k+1)$,
(ii) $\overline{N}(r,1,f \mid> 2, g \mid> 2) \le 2\overline{N}(r,\infty;f \mid\ge k+1)$.

Proof. Let $\phi_1 = \frac{f'}{f-1} - \frac{g'}{g-1}$, $\phi_2 = \frac{f'}{f} - \frac{g'}{g}$ and $\phi_3 = \phi_1 - \phi_2$. Since $H \not\equiv 0$, we have $f \not\equiv g$ and hence it follows that $\phi_i \not\equiv 0$ for $i = 1,2,3$. Now

$$\begin{aligned}
\overline{N}_*(r,0;f,g) &\le \overline{N}(r,0;f \mid\ge 2)\\
&\le N(r,0;\phi_1)\\
&\le T(r,f) + O(1) = N(r,\infty;\phi_1) + S(r)\\
&\le \overline{N}_*(r,1;f,g) + \overline{N}(r,\infty;f \mid\ge k+1) + S(r)
\end{aligned}$$

which is (i).
Again

$$\begin{aligned}
\overline{N}(r,1;f \mid\ge 2) + \overline{N}(r,1;f \mid> 2, g \mid> 2) &\le N(r,0;\phi_2) \le T(r,\phi_2) + S(r)\\
&= N(r,\infty;\phi_2) + S(r)\\
&= \overline{N}(r,0;f \mid\ge 2) + \overline{N}(r,\infty;f \mid\ge k+1).
\end{aligned}$$

Hence from above we have

$$\begin{aligned}
&\overline{N}(r,1;f \mid\ge 2) + \overline{N}(r,1;f \mid> 2, g \mid> 2)\\
&\le \overline{N}_*(r,1;f,g) + \overline{N}(r,\infty;f \mid\ge k+1) + \overline{N}(r,\infty;f \mid\ge k+1)\\
&\le \overline{N}(r,1;f \mid\ge 2) + 2\overline{N}(r,\infty;f \mid\ge k+1),
\end{aligned}$$

which yields (ii). □

Lemma 9. *If f and g share $(0,1),(1,1),(\infty,k)$ and $H \not\equiv 0$, then*
$T(r,f) \le 2\overline{N}(r,0;f \mid= 1) + 2\overline{N}(r,\infty;f) + 10\overline{N}(r,\infty;f \mid\ge k+1)$
$+ 4[\overline{N}_{g>2}(r,1;f) + \overline{N}_{f>2}(r,1;g)] - m(r,1;g) - \overline{N}_L(r,1;g) - \overline{N}_E^{(3}(r,1;f) + S(r)$.

Proof. We denote by $N_0(r,o;f')$ the counting function of those zeros of f' which are not the zeros of $f(f-1)$. Similarly we define $N_0(r,0;g')$. Then by the Second

Fundamental Theorem and Lemma 7 we have

$$\begin{aligned}
&T(r,f)+T(r,g)\\
&\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;g)\\
&\quad -N_0(r,0;g')-N_0(r,o;f')+S(r)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,0;f\mid\ge 2)+2\overline{N}(r,\infty;f)+[\overline{N}_*(r,0;f,g)\\
&\quad +\overline{N}_*(r,\infty;f,g)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+\overline{N}_0(r,0;g')+\overline{N}_0(r,0;f')+T(r,g)\\
&\quad -m(r,1;g)+\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)]-N_0(r,0;g')-N_0(r,o;f')+S(r)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+2\overline{N}(r,0;f\mid\ge 2)+\overline{N}_*(r,0;f,g)\\
&\quad +\overline{N}(r,\infty;f\mid\ge k+1)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+T(r,g)-m(r,1;g)\\
&\quad +\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)+S(r)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+3\overline{N}(r,0;f\mid\ge 2)+\overline{N}(r,\infty;f\mid\ge k+1)\\
&\quad -\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+T(r,g)-m(r,1;g)+\overline{N}_{g>2}(r,1;f)\\
&\quad +\overline{N}_{f>2}(r,1;g)+S(r).
\end{aligned}$$

Now using (i) of Lemma 8 we obtain

$$\begin{aligned}
&T(r,f)+T(r,g)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+3[\overline{N}_*(r,1;f,g)+\overline{N}(r,\infty;f\mid\ge k+1)]\\
&\quad +\overline{N}(r,\infty;f\mid\ge k+1)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+T(r,g)-m(r,1;g)\\
&\quad +\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)+S(r).
\end{aligned}$$

Thus we obtain from above using (ii) of Lemma 8

$$\begin{aligned}
&T(r,f)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+4\overline{N}(r,\infty;f\mid\ge k+1)+\overline{N}_{g>2}(r,1;f)\\
&\quad +\overline{N}_{f>2}(r,1;g)+3[\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)+\overline{N}(r,1;f\mid>2,g\mid>2)]\\
&\quad -m(r,1;g)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+S(r)\\
&\le 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+4\overline{N}(r,\infty;f\mid\ge k+1)+6\overline{N}(r,\infty;f\mid\ge k+1)\\
&\quad +4\overline{N}_{g>2}(r,1;f)+4\overline{N}_{f>2}(r,1;g)-m(r,1;g)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+S(r)\\
&= 2\overline{N}(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+10\overline{N}(r,\infty;f\mid\ge k+1)+4\overline{N}_{g>2}(r,1;f)\\
&\quad +4\overline{N}_{f>2}(r,1;g)-m(r,1;g)-\overline{N}_L(r,1;g)-\overline{N}_E^{(3}(r,1;f)+S(r).
\end{aligned}$$

This proves the lemma. □

Lemma 10. *If f and g share $(0,0)$, $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$, then*

$$T(r,f) \le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)] + \overline{N}_L(r,1;f) - \overline{N}_E^{(1}(r,1;f) - \overline{N}_E^{(1}(r,\infty;f) - m(r,1;g) - \overline{N}_E^{(3}(r,1;f) + S(r).$$

Proof. By the Second Fundamental Theorem we have

$$\begin{aligned}
&T(r,f) + T(r,g)\\
&\le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;g) - N_0(r,0;g')\\
&\quad - N_0(r,o;f') + S(r) \text{ [where } N_0(r,0;g'), N_0(r,o;f') \text{ are same as Lemma 9]}\\
&= 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}(r,1;f) + \overline{N}(r,1;g) - N_0(r,0;g') - N_0(r,o;f') + S(r).
\end{aligned}$$

Now by Lemma 3, 5, and 6 we see that

$$\begin{aligned}
&\overline{N}(r,1;f) + \overline{N}(r,1;g)\\
&= \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_E^{1)}(r,1;f) + \overline{N}_E^{(2}(r,1;f) + \overline{N}(r,1;g)\\
&\le \overline{N}_E^{1)}(r,1;f) + \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) + N(r,1;g) - \overline{N}_E^{(2}(r,1;f)\\
&\quad - \overline{N}_E^{(3}(r,1;f) - \overline{N}_L(r,1;f) - 2\overline{N}_L(r,1;g) + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)]\\
&\le N(r,H) + T(r,g) - m(r,1;g) - \overline{N}_L(r,1;g) - \overline{N}_E^{(3}(r,1;f)\\
&\quad + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)]\\
&\le \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g')\\
&\quad + T(r,g) - m(r,1;g) - \overline{N}_L(r,1;g) - \overline{N}_E^{(3}(r,1;f) + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)]\\
&= \overline{N}(r,0;f) - \overline{N}_E^{(1}(r,0;f) + \overline{N}(r,\infty;f) - \overline{N}_E^{(1}(r,\infty;f) + \overline{N}_L(r,1;f) + T(r,g)\\
&\quad + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') - m(r,1;g) - \overline{N}_E^{(3}(r,1;f)\\
&\quad + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)].
\end{aligned}$$

Therefore from above we obtain

$$\begin{aligned}
&T(r,f) + T(r,g)\\
&\le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) + \overline{N}_L(r,1;f) + T(r,g) - m(r,1;g) - \overline{N}_E^{(1}(r,0;f)\\
&\quad - \overline{N}_E^{(1}(r,\infty;f) - \overline{N}_E^{(3}(r,1;f) + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)]
\end{aligned}$$

and hence,

$$\begin{aligned}
&T(r,f)\\
&\le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) + \overline{N}_L(r,1;f) - m(r,1;g) - \overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f)\\
&\quad - \overline{N}_E^{(3}(r,1;f) + [\overline{N}_{g>1}(r,1;f) + \overline{N}_{f>1}(r,1;g)].
\end{aligned}$$

This proves the lemma. □

Lemma 11([12]). *If f and g share $(0,0),(1,0),(\infty,0)$ and $H \equiv 0$, then f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞).*

Lemma 12([9]). *If f and g be two nonconstant meromorphic functions sharing $(0,\infty)$, $(1,\infty)$, (∞,∞) and $f \equiv g$, then*

$$N(r,0;f\mid\geq 2)+N(r,1;f\mid\geq 2)+N(r,\infty;f\mid\geq 2)=S(r).$$

Lemma 13([10], [14]). *Let f and g be two nonconstant meromorphic functions sharing 0, 1, ∞, CM.*
If

$$\limsup_{r\to\infty}\frac{2\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)-m(r,1;g)}{T(r,f)}<1 \tag{23}$$

then either $f \equiv g$ or $fg \equiv 1$.

3. Proofs of the Theorems

Proof of Theorem 1: Suppose that $H \not\equiv 0$. Then by Lemma 9 we obtain a contradiction to (16). Hence $H \equiv 0$. Therefore by Lemma 11, f and g share $(0,\infty),(1,\infty),(\infty,\infty)$. Therefore by Lemma 12

$$\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)=S(r).$$

Now by Theorem A our theorem follows. □

Proof of Theorem 2: Since f and g share $(1,2)$ it follows that

$$\overline{N}_{g>2}(r,1;f)+\overline{N}_{f>2}(r,1;g)=S(r).$$

Again since f, g share $(1,2)$, f, g share $(1,1)$ and Theorem 2 follows from Theorem 1. □

Proof of Theorem 6: Suppose that $H \not\equiv 0$. Then by Lemma 10 we obtain a contradiction of (21). Therefore $H \equiv 0$ and hence by Lemma 11, f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞). Therefore, $\overline{N}_E^{(1}(r,\infty;f)=\overline{N}(r,\infty;f)$ and $\overline{N}_E^{(1}(r,0;f)=\overline{N}(r,0;f)$. Then by Lemma 12 we obtain $N_{\otimes}(r,1;f,g)+\overline{N}_E^{(3}(r,1;f)=S(r)$. Thus by (21) and Lemma 13 the proof follows. This completes the proof. □

Proof of Theorem 7: Since f,g share (∞,∞), we see that $\overline{N}_E^{(1}(r,\infty;f)=\overline{N}(r,\infty;f)$ and therefore the theorem follows easily from Theorem 6, remembering that sharing (∞,∞) implies sharing $(\infty,0)$. This proves the theorem. □

Proof of Theorem 5: Suppose that $H \not\equiv 0$. Then by the Second Fundamental Theorem and by Lemma 7 with $k = 0$ we obtain

$$\begin{aligned}
&T(r,f) + T(r,g) \\
&\le 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') \\
&\quad + \overline{N}_0(r,0;g') + T(r,g) - m(r,1;g) - \overline{N}_L(r,1;g) - \overline{N}_E^{(3}(r,1;f) \\
&\quad + \overline{N}_{g>2}(r,1;f) + \overline{N}_{f>2}(r,1;g) - N_0(r,0;g') - N_0(r,o;f') \\
&\quad + S(r), \text{ [where } N_0(r,0;g'),\ N_0(r,o;f') \text{ are same as Lemma 9]} \\
&\le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) + T(r,g) - m(r,1;g) - \overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f) \\
&\quad - \overline{N}_E^{(3}(r,1;f) + \overline{N}_{f>2}(r,1;g) + S(r).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&T(r,f) \\
&\le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) - m(r,1;g) - \overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f) - \overline{N}_E^{(3}(r,1;f) \\
&\quad + \overline{N}_{f>2}(r,1;g) + S(r),
\end{aligned}$$

which contradicts (20). Hence $H \equiv 0$ and the theorem follows from Lemmas 11, 12 and 13. This completes the proof. □

Proof of Theorem 4: Since f, g share (∞,∞), $\overline{N}_E^{(1}(r,\infty;f) = \overline{N}(r,\infty;f)$ and our theorem follows easily from Theorem 5. This proves the theorem. □

Proof of Theorem 3: Suppose that $H \not\equiv 0$. Since f, g share $(1, 2)$,

$$\overline{N}_{f>2}(r,1;g) + \overline{N}_{g>2}(r,1;f) = S(r).$$

Therefore proceeding as in the proof of Theorem 5, we obtain

$$\begin{aligned}
&T(r,f) \\
&\le 3\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) - m(r,1;g) - \overline{N}_E^{(1}(r,0;f) - \overline{N}_E^{(1}(r,\infty;f) - \overline{N}_E^{(3}(r,1;f) \\
&\quad - \overline{N}_L(r,1;g) + S(r),
\end{aligned}$$

which contradicts (18). Hence $H \equiv 0$ and the theorem follows from Lemmas 11, 12 and 13. This completes the proof. □

References

[1] A. Banerjee, *Uniqueness of meromorphic functions that share three values*, The Australian J. Math. Annal. Appl.(http://ajmaa.org)3.2(1)(2006), 1-10.

[2] A. Banerjee, *Meromorohic functions that share three values with finite weight*, Ann. Polon. Math., **91**(1)(2007), 57-69.

[3] A. Banerjee, *Some further results on Lahiri*, Filomat., **23**(2009), 37-49.

[4] G. Brosch, *Eindeutigkeitssätzefür meromorphic Funktionen*, Thesis, Technical University of Aachen, 1989.

[5] J. Chen, S. Shen and W. C. Lin, *Meromorphic functions sharing three values with weight*, J. Inequal. Pure Appl. Math., **8**(1)(2007), Ary.19, [http://jipam.vu.edu.au/].

[6] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford(1964).

[7] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**(2001), 193-206.

[8] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var., **46**(2001), 241-253.

[9] I. Lahiri, *Weighted sharing of three values and uniqueness of meromorphic functions*, Kodai Math. J., **24**(2001), 421-435.

[10] I. Lahiri, *Meromorphic functions sharing three values*, Southeast Asian Bull. Math., **26**(2003), 961-966.

[11] H. Ueda, *Unicity theorems for meromorphic or entire functions II*, Kodai Math. J., **6**(1983), 26-36.

[12] H. X. Yi, *Meromorphic functions that share three values*, Bull. Hong Kong Math. Soc., **2**(1998), 679-688.

[13] H. X. Yi, *Meromorphic functions that share two or three values II*, Kodai Math. J., **22**(1999), 264-272.

[14] H. X. Yi, *Meromorphic functions that share three values*, Bull. Hong Kong Math. Soc., **2**(1998), 679-688.

[15] H. X. Yi and Thamir C Alzahari, *Weighted value sharing and a question of I. Lahiri*, Complex Var Theory Appl., **49**(15)(2004), 1063-1078.