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The Hilbert-Type Integral Inequality with the System Kernel of -λ Degree Homogeneous Form

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Abstract. In this paper, the integral operator is used. We give a new Hilbert-type integral inequality, whose kernel is the homogeneous form with degree $-\lambda$ and with three pairs of conjugate exponents and the best constant factor and its reverse form are also derived. It is shown that the results of this paper represent an extension as well as some improvements of the earlier results.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1, f \in L^p(0, \infty)$, and $g \in L^q(0, \infty)$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then [1]:

$$(1.1) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Inequality (1.1) is named Hardy-Hilbert's integral inequality,which is important in analysis and applications. It has been studied and generalized in many directions by a number of people [2-21].

Let $(Tf)(y) := \int_0^\infty H(x, y)f(x)dx$, $\|f\|_p := (\int_0^\infty |f(x)|^p)^{1/p}$. We have $(Tf)(g) := \int_0^\infty (\int_0^\infty H(x, y)f(x)dx)g(y)dy$. If $H(x, y) = \frac{1}{x+y}$, in 2006,Yang [2] rewrote (1.1) as

$$(1.2) \qquad (Tf)(g) < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q$$

where $T : L^r(0, \infty) \to L^r(0, \infty) (r = p, q)$ is an integral operator.

For the purposes, we introduce some notations as follows:

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Let $p > 0, p \neq 1$ $\frac{1}{p} + \frac{1}{q} = 1$,$r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. A norm of f with the weight function $w(x)$ is defined by

$$\|f\|_{p,w} := \left\{ \int_0^\infty w(x)|f(x)|^p dx \right\}^{1/p}.$$

where $f(x)$, $w(x) \geq 0$ are measurable functions defined on $(0, \infty)$. If $\|f\|_{p,w} < \infty$, then it is marked by $f \in L^p_w(\mathbb{R}_+)$.

Supposing that $H(x, y) \geq 0$ is a real measurable in $(0, \infty) \times (0, \infty)$ and satisfies $H(ux, uy) = u^{-\lambda} H(x, y)(\lambda > 0, u > 0)$ for $(x, y) \in (0, \infty) \times (0, \infty)$, then $H(x, y)$ is called a homogeneous function of $-\lambda$-degree. Then we have the formal inner as follows:

$$(Tf, g) = (Tg, f) = \int_0^\infty \left(\int_0^\infty H(x, y) f(x) dx \right) g(y) dy = \int_0^\infty \int_0^\infty H(x, y) f(x) g(y) dx dy.$$

Define the integral operator T as:
for $f \in L^p(0, \infty)$,

$$(Tf)(y) := \int_0^\infty H(x, y) f(x) dx, y \in (0, \infty)$$

or for $g \in L^q(0, \infty)$,

$$(Tg)(x) := \int_0^\infty H(x, y) g(y) dy, x \in (0, \infty)$$

where T is call Hilbert-type integral operator, if T is bounded. $H(x, y)$ is call the kernel of T.

The main objective of this paper is to build a new Hilbert-type integral inequality, whose Kernel is the homogeneous form degree $-\lambda$ with three pairs of conjugate exponents and with the best constant factor. As applications,the equivalent forms and some particular results are given.

In the following, we always suppose that
1) $t \geq 0, p > 0, p \neq 1$ $\frac{1}{p} + \frac{1}{q} = 1.r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. $l > 1$, $\frac{1}{l} + \frac{1}{k} = 1$.
2) $H(x, y) \geq 0$ is a real measurable in $(0, \infty) \times (0, \infty)$ homogeneous function of $-\lambda$-degree$(\lambda > 0)$, and $0 < \int_0^\infty H(1, \sigma) \sigma^{-1+t-\frac{2th-s\lambda}{sh}} d\sigma < \infty$.
3) $w(x) = x^{p(1-t+\frac{2tl-r\lambda}{rl})-1}, \hat{w}(x) = x^{q(t-\frac{2tl-r\lambda}{rl})-1}; \widetilde{w}(y) = y^{q(1-t+\frac{2th-s\lambda}{sh})-1}, \bar{w}(y) = y^{p(t-\frac{2th-s\lambda}{sh})-1}$.

2. Lemma and main results

Lemma. *Define the weight functions:*

$$W(x) = \int_0^\infty H(x, y) \frac{x^{\frac{p}{q}(1-t+\frac{2tl-r\lambda}{rl})}}{y^{1-t+\frac{2th-s\lambda}{sh}}} dy, \quad \widetilde{W}(y) = \int_0^\infty H(x, y) \frac{y^{\frac{q}{p}(1-t+\frac{2th-s\lambda}{sh})}}{x^{1-t+\frac{2tl-r\lambda}{rl}}} dx.$$

then

$$W(x) = K_1 w(x), \qquad \widetilde{W}(y) = K_2 \widetilde{w}(y)$$

where

$$K_1 = \int_0^\infty H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}} d\sigma, \quad K_2 = \int_0^\infty H(1,\sigma)\sigma^{1-t+\frac{2tl-r\lambda}{rl}-2+\lambda} d\sigma$$

and $K := K_1 = K_2$.

Proof. Setting $y = x\sigma$ we have $W(x) = K_1 w(x)$.
on the other hand, we have

$$\widetilde{W}(y) = \int_0^\infty H(\frac{y}{\sigma}, y) \frac{y^{\frac{q}{p}(1-t+\frac{2th-s\lambda}{sh})}}{(\frac{y}{\sigma})^{1-t+\frac{2tl-r\lambda}{rl}}} d(\frac{y}{\sigma}) = K_2 \widetilde{w}(y)$$

easily$K_1 = K_2 = K$. the lemma is proved. □

Theorem 2.1. *If* $p > 1$, $f \in L^p_w(\mathbb{R}_+)$, $g \in L^q_{\widetilde{w}}(\mathbb{R}_+)$, *and* $\|f\|_{p,w} > 0$, $\|g\|_{q,\widetilde{w}} > 0$, *then*

$$(Tf, g) = (Tg, f) = \int_0^\infty \int_0^\infty H(x,y) f(x) g(y) dx dy < K \|f\|_{p,w} \|g\|_{q,\widetilde{w}}; \tag{2.1}$$

If $f \in L^p_w(\mathbb{R}_+)$ *and* $\|f\|_{p,w} > 0$, *then* $Tf \in L^p_{\widetilde{w}}(\mathbb{R}_+)$ *and*

$$\|Tf\|_{p,\bar{w}} = \left\{ \int_0^\infty y^{p(t-\frac{2th-s\lambda}{sh})-1} \left(\int_0^\infty H(x,y) f(x) dx \right)^p dy \right\}^{\frac{1}{p}} < K^p \|f\|_{p,w}. \tag{2.2}$$

K is defined by lemma, both constant factors, K and K^p are the best possible and inequalities (2.1) *and* (2.2) *are equivalent.*

Theorem 2.2. *If* $1 > p > 0$, $f(x), g(x) \geq 0$, *such that* $f \in L^p_w(\mathbb{R}_+)$, $g \in L^q_{\widetilde{w}}(\mathbb{R}_+)$, *and* $\|f\|_{p,w} > 0$, $\|g\|_{q,\widetilde{w}} > 0$, *then*

$$(Tf, g) = \int_0^\infty \int_0^\infty H(x,y) f(x) g(y) dx dy > K \|f\|_{p,w} \|g\|_{q,\widetilde{w}}. \tag{2.3}$$

If $f \in L^p_w(\mathbb{R}_+)$ *and* $\|f\|_{p,w} > 0$, *then* $Tf \in L^p_{\widetilde{w}}(\mathbb{R}_+)$ *and*

$$\|Tf\|_{p,\bar{w}} = \left\{ \int_0^\infty y^{p(t-\frac{2th-s\lambda}{sh})-1} \left(\int_0^\infty H(x,y) f(x) dx \right)^p dy \right\}^{\frac{1}{p}} > K^p \|f\|_{p,w}, \tag{2.4}$$

where both constant factors K and K^p are the best possible and inequalities (2.3) *and* (2.4) *are equivalent.*

We prove only Theorem 2.2, since the proof of Theorem 2.1 is the similar.

Proof of Theorem 2.2 By Hölder's inequality [22] and results of lemma we have,

$$(2.5)\quad (Tf,g)=\int_0^\infty\int_0^\infty H(x,y)f(x)\frac{x^{\frac{1}{q}(1-t+\frac{2tl-r\lambda}{rl})}}{y^{\frac{1}{p}(1-t+\frac{2th-s\lambda}{sh})}}g(y)\frac{y^{\frac{1}{p}(1-t+\frac{2th-s\lambda}{sh})}}{x^{\frac{1}{q}(1-t+\frac{2tl-r\lambda}{rl})}}dxdy$$

$$\geq\left\{\int_0^\infty\int_0^\infty H(x,y)\frac{x^{\frac{p}{q}(1-t+\frac{2tl-r\lambda}{rl})}}{y^{1-t+\frac{2th-s\lambda}{sh}}}f^p(x)dydx\right\}^{\frac{1}{p}}$$

$$\left\{\int_0^\infty\int_0^\infty H(x,y)\frac{y^{\frac{q}{p}(1-t+\frac{2th-s\lambda}{sh})}}{x^{1-t+\frac{2tl-r\lambda}{rl}}}g^q(y)dxdy\right\}^{\frac{1}{q}}$$

$$=\left\{\int_0^\infty W(x)f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty\widetilde{W}(y)g^q(y)dy\right\}^{\frac{1}{q}}=K\|f\|_{p,w}\|g\|_{q,\widetilde{w}}.$$

If (2.5) takes the form of equality, then there exist constants M and N, such that they are not all zero and

$$M\frac{x^{\frac{p}{q}(1-t+\frac{2tl-r\lambda}{rl})}}{y^{1-t+\frac{2th-s\lambda}{sh}}}f^p(x)=N\frac{y^{\frac{q}{p}(1-t+\frac{2th-s\lambda}{sh})}}{x^{1-t+\frac{2tl-r\lambda}{rl}}}g^q(y)\quad\text{a.e. in }(0,\infty)\times(0,\infty).$$

Hence there exists a constant C, such that

$$Mx^{p(1-t+\frac{2tl-r\lambda}{rl})}f^p(x)=Ny^{q(1-t+\frac{2th-s\lambda}{sh})}g^q(y)=C\qquad\text{a.e. in }(0,\infty),$$

Without loss of generality, suppose that $M\neq 0$, we my get $x^{p(1-t+\frac{2tl-r\lambda}{rl})-1}f^p(x)=C/(Mx)$ a.e. in $(0,\infty)$ which contradicts $f\in L^p_w(\mathbb{R}_+)$. Hence (2.6) takes a strict inequality and we have (2.3).

If the constant factor K in (2.3) is not the best possible, then there exists a positive constant $\widetilde{K}$ (with $\widetilde{K}>K$), such that (2.3) is still valid if we replace K by $\widetilde{K}$. Setting f_n and g_n as; $f_n(x)=g_n(x)=0$, for $x\in(0,1); f_n(x)=x^{-1+t-\frac{2tl-r\lambda}{rl}-\frac{1}{np}}$, $g_n(x)=x^{-1+t-\frac{2th-s\lambda}{sh}-\frac{1}{nq}}$, for $x\in[1,\infty)$. then for $n\in\mathbb{N}$,

$$(2.6)\qquad (Tf_n,g_n)>\widetilde{K}\|f_n\|_{p,w}\|g_n\|_{q,\widetilde{w}}=n\widetilde{K}.$$

Setting $x=\frac{y}{\sigma}$, then we obtain

$$\begin{aligned}J_n:=&(Tf_n,g_n)=\int_0^\infty\int_0^\infty H(x,y)f_n(x)g_n(y)dxdy\\&=\int_1^\infty\int_1^\infty H(x,y)x^{-1+t-\frac{2tl-r\lambda}{rl}-\frac{1}{np}}y^{-1+t-\frac{2th-s\lambda}{sh}-\frac{1}{nq}}dxdy\\&=\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^y H(1,\sigma)\sigma^{1-t+\frac{2tl-r\lambda}{rl}-2+\lambda+\frac{1}{np}}d\sigma\right)dy\\&=\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^1 H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma\right)dy\\&\qquad+\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_1^y H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma\right)dy\end{aligned}$$

$$
\begin{aligned}
=&n\int_0^1 H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma\\
&\qquad+\int_1^\infty H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}\left(\int_\sigma^\infty y^{-1-\frac{1}{n}}dy\right)d\sigma\\
=&n\left[\int_0^1 H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma+\int_1^\infty H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}-\frac{1}{nq}}d\sigma\right].
\end{aligned}
$$

In view of (2.6), we have $n\widetilde{K} < J_n$.

Secondly, by Fatou lemma, one has

$$
\begin{aligned}
\widetilde{K} &\le \underline{\min}_{n\to\infty}\frac{1}{n}J_n\\
&= \underline{\min}_{n\to\infty}\left[\int_0^1 H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma+\int_1^\infty H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}-\frac{1}{nq}}d\sigma\right]\\
&\le \left[\int_0^1 \underline{\min}_{n\to\infty} H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{np}}d\sigma+\int_1^\infty \underline{\min}_{n\to\infty} H(1,\sigma)\sigma^{-1+t-\frac{2th-s\lambda}{sh}-\frac{1}{nq}}d\sigma\right]\\
&= K.
\end{aligned}
$$

It follows that $\widetilde{K} \le K$, which contradicts the fact that $K < \widetilde{K}$. Hence the constant K in (2.3) is the best possible.

For $x > 0, n \in \mathbb{N}$, setting a bounded measurable function $\widetilde{f}_n$ as

$$
\widetilde{f}_n(x) = \begin{cases} n, & \text{if } f(x) > n,\\ f(x), & \text{if } \frac{1}{n} \le f(x) \le n,\\ \frac{1}{n}, & \text{if } f(x) < \frac{1}{n}, \end{cases}
$$

by the condition of $f \in L^p_w(\mathbb{R}_+)$, there exists $n_0 \in \mathbb{N}$, such that for $n \ge n_0 \in \mathbb{N}$, $0 < \int_{1/n}^n w(x)\widetilde{f}(x)^p dx < \infty$. Setting $\widetilde{g}_n(y) = y^{p(t-\frac{2th-s\lambda}{sh})-1}\left(\int_{1/n}^n H(x,y)\widetilde{f}_n(x)dx\right)^{p-1}$, $(\frac{1}{n} < x \le n; n \ge n_0)$ by Hölder's inequality [22], we have

$$
\begin{aligned}
\infty >& \int_{1/n}^n y^{q(1-t+\frac{2th-s\lambda}{sh})-1}\widetilde{g}_n^q(y)dy\\
=& \int_{1/n}^n y^{p(t-\frac{2th-s\lambda}{sh})-1}\left(\int_{1/n}^n H(x,y)\widetilde{f}_n(x)dx\right)^p dy\\
=& \int_{1/n}^n\int_{1/n}^n y^{p(t-\frac{2th-s\lambda}{sh})-1}\left(\int_{1/n}^n H(x,y)\widetilde{f}_n(x)dx\right)^p dxdy\\
=& \int_{1/n}^n\int_{1/n}^n H(x,y)\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy\\
>& K\left(\int_{1/n}^n x^{p(1-t+\frac{2tl-r\lambda}{rl})-1}\widetilde{f}_n^p(x)dx\right)^{1/p}\left(\int_{1/n}^n y^{q(1-t+\frac{2th-s\lambda}{sh})-1}\widetilde{g}_n(y)dy\right)^{1/q}.
\end{aligned}
$$

$$(2.7)\quad 0<K^p\int_{1/n}^{n} x^{p(1-t+\frac{2tl-r\lambda}{rl})-1}\widetilde{f}_n^p(x)dx<\int_{1/n}^{n} y^{q(1-t+\frac{2th-s\lambda}{sh})-1}\widetilde{g}_n^q(y)dy<\infty.$$

For $n\to\infty$, if $\int_0^\infty y^{q(1-t+\frac{2th-s\lambda}{sh})-1}\widetilde{g}_\infty^q(y)dy=\infty$, then we have (2.4); if $0<\int_0^\infty y^{q(1-t+\frac{2th-s\lambda}{sh})-1}\widetilde{g}_\infty^q(y)dy<\infty$, then by using (2.3), both (2.7) and (2.8) still take the form of strict inequalities, and we have (2.4).

On the other hand, if inequality (2.4) holds, then by Hölder's inequality, we have

$$\begin{aligned}(2.8)\quad (Tf,g) &= \int_0^\infty\int_0^\infty H(x,y)f(x)g(y)dxdy\\ &= \int_0^\infty\left(y^{-1+t-\frac{2th-s\lambda}{sh}+\frac{1}{q}}\int_0^\infty H(x,y)f(x)dx\right)\left(y^{1-t+\frac{2th-s\lambda}{sh}-\frac{1}{q}}g(y)\right)dy,\\ &\geq \left\{\int_0^\infty y^{p(t-\frac{2th-s\lambda}{sh})-1}\left(\int_0^\infty H(x,y)f(x)dx\right)^p dy\right\}^{\frac{1}{p}}\\ &\qquad \left\{\int_0^\infty y^{q(1-t+\frac{2th-s\lambda}{sh})-1}g(y)dy\right\}^{\frac{1}{q}}.\end{aligned}$$

Hence by (2.4), we have (2.3), and inequalities (2.4) and (2.3) are equivalent.

If the constant factor in (2.4) is not the best possible, by the inequality $(Tf,g)\leq\|Tf\|_{p,w}\|g\|_{q,\widetilde{w}}$ we may get a contradiction that the constant factor in (2.4) is not the best possible.

In the same way, (2.3) and (2.5) are equivalent and the constant factors are the best possible.

Remarks. The results of this paper include many other conclusions which have been published. For instance, in the following we suppose that the integrals in the right of the following inequalities converge to some positive numbers, $\frac{1}{p}+\frac{1}{q}=1$, and one omits the words that the constants factors are the best possible.

1) It is easy to see that for $t=0, l=r, p>1, r=s=2, \lambda=4, H(x,y)=\frac{1}{(x+ay)^2(x+ay)^2}$, the inequality (2.1) reduces to [9]

$$\int_0^\infty\int_0^\infty \frac{1}{(x+ay)^2(x+ay)^2}f(x)g(y)dxdy$$
$$<K\left\{\int_0^\infty x^{-p-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{-q-1}g^q(y)dy\right\}^{\frac{1}{q}},$$

where

$$K=\int_0^\infty\frac{dt}{(1+at)^2(1+at)^2}=\begin{cases}\frac{a+b}{(b-a)^2}\left[\frac{ln(b/a)}{b-a}-\frac{2}{a+b}\right], & \text{if } a\neq b,\\ \frac{1}{6a^2}, & \text{if } a=b.\end{cases}$$

2) Setting $t=0, l=r, p>1, H(x,y)=\frac{|\ln(x/y)|}{(\max\{x,y\})^{\lambda}}(\lambda>0)$, then we have [10]

$$\int_0^\infty\int_0^\infty \frac{|\ln(x/y)|}{(\max\{x,y\})^{\lambda}} f(x)g(y)dxdy$$

$$< \frac{r^2+s^2}{\lambda^2}\left\{\int_0^\infty x^{p(1-\lambda/s)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1-\lambda/r)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

3) Setting $t=0, l=r, p>1, H(x,y)=\frac{|\ln(x/y)|^{\beta}}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^{\alpha}}(\lambda>0, \alpha\in\mathbb{R}, \beta>-1)$, then we have [11]

$$\int_0^\infty\int_0^\infty \frac{|\ln(x/y)|^{\beta}}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^{\alpha}} f(x)g(y)dxdy$$

$$< \Gamma(\beta+1)\Sigma_{k=0}^{\infty}\binom{\alpha-\lambda}{k}\left[\frac{1}{(k+\frac{\lambda}{r})^{\beta+1}}+\frac{1}{(k+\frac{\lambda}{s})^{\beta+1}}\right]$$

$$\times\left\{\int_0^\infty x^{p(1-\lambda/s)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1-\lambda/r)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

4) Setting $t=0, l=r=2, p>1, H(x,y)=\frac{\arctan^{\beta}\sqrt{x^{\lambda}/y^{\lambda}}}{x^{\lambda}+y^{\lambda}}(\lambda>0, \beta>-1)$, then we have [12]

$$\int_0^\infty\int_0^\infty \frac{\arctan^{\beta}\sqrt{x^{\lambda}/y^{\lambda}}}{x^{\lambda}+y^{\lambda}} f(x)g(y)dxdy$$

$$< \frac{2}{\lambda(\beta+1)}(\frac{\pi}{2})^{\beta+1}\left\{\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1-\lambda/2)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

5) Setting $t=0, l=r, p>1, H(x,y)=\frac{(min\{x,y\})^{\lambda}}{|x+y|^{2\lambda}}, 0<\lambda<\frac{1}{2}$, then we have [13]

$$\int_0^\infty\int_0^\infty \frac{(min\{x,y\})^{\lambda}}{|x+y|^{2\lambda}} f(x)g(y)dxdy$$

$$< \left[B\left(1-2\lambda,(1+\frac{1}{s})\lambda\right)+B\left(1-2\lambda,(1+\frac{1}{r})\lambda\right)\right]$$

$$\times\left\{\int_0^\infty x^{p(1-\lambda/r)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1-\lambda/s)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

6) Setting $t=0, l=r, 1>p>0, H(x,y)=\frac{\ln(x/y)}{x^\alpha-y^\alpha}$,$\alpha>0$, then we have [14]

$$\int_0^\infty\int_0^\infty \frac{\ln(x/y)}{x^\alpha-y^\alpha}f(x)g(y)dxdy$$
$$>\left[\frac{\pi}{\alpha\sin(\frac{\pi}{r})}\right]^2\left\{\int_0^\infty x^{p(1-\lambda/s)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q(1-\lambda/r)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

7) Setting

a) $t=1, l=r=p, p>1, \lambda=1, H(x,y)=\frac{1}{|x-y|^{1-\alpha}\min\{x^\alpha,y^\alpha\}}$,$0<\alpha<\min\{\frac{1}{p},\frac{1}{q}\}$ then we have [15]

$$\int_0^\infty\int_0^\infty \frac{1}{|x-y|^{1-\alpha}\min\{x^\alpha,y^\alpha\}}f(x)g(y)dxdy$$
$$<\left[B\left(\frac{1}{p}-\alpha,\alpha\right)+B\left(\frac{1}{q}-\alpha,\alpha\right)\right]\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}},$$

b)$t=0, \lambda=1, l=p, p>1, H(x,y)=\frac{1}{|x-y|^{1-\alpha}\min\{x^\alpha,y^\alpha\}}$,$0<\alpha<\min\{\frac{1}{p},\frac{1}{q}\}$ then we have [15]

$$\int_0^\infty\int_0^\infty \frac{1}{|x-y|^{1-\alpha}\min\{x^\alpha,y^\alpha\}}f(x)g(y)dxdy$$
$$<\left[B\left(\frac{1}{p}-\alpha,\alpha\right)+B\left(\frac{1}{q}-\alpha,\alpha\right)\right]\left\{\int_0^\infty x^{p-2}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q-2}g^q(y)dy\right\}^{\frac{1}{q}}.$$

8) Setting $t=1, l=r=q, \lambda=1, p>1, H(x,y)=\frac{|x-y|^{\alpha-1}}{\max\{x^\alpha,y^\alpha\}}$, $\alpha>0$, then [16]

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\alpha-1}}{\max\{x^\alpha,y^\alpha\}}f(x)g(y)dxdy$$
$$<\left[B\left(\frac{1}{p},\alpha\right)+B\left(\frac{1}{q},\alpha\right)\right]\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.$$

9) Setting $t=1, l=r=p,\ \lambda=1-\alpha, p>1, H(x,y)=\frac{\min\{x^\alpha,y^\alpha\}}{\max\{x,y\}}$, $\alpha\geq 0$, then [17]

$$\int_0^\infty\int_0^\infty \frac{\min\{x^\alpha,y^\alpha\}}{\max\{x,y\}}f(x)g(y)dxdy<\frac{pq}{1+\alpha}\left\{\int_0^\infty x^\alpha f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^\alpha g^q(y)dy\right\}^{\frac{1}{q}}.$$

10) Setting $t=1$,$l=r=p, p>1,\ \lambda>2-\min\{p,q\}, H(x,y)=\frac{\ln(x/y)}{x^\lambda-y^\lambda}$, then we have [18]

$$\int_0^\infty\int_0^\infty \frac{\ln(x/y)}{x^\lambda-y^\lambda}f(x)g(y)dxdy$$

$$< \left[\frac{1}{\lambda}B\left(\frac{p+\lambda-2}{p\lambda},\frac{q+\lambda-2}{q\lambda}\right)\right]^2\left\{\int_0^\infty x^{1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{1-\lambda}g^q(y)dy\right\}^{\frac{1}{q}}$$

11) Setting $t=\alpha, l=r=p, 1>p>0,\ \lambda=\alpha\beta, 2-p<2-q, H(x,y)=\frac{1}{(x^\alpha+y^\alpha)^\beta}$, then we have[19]

$$\int_0^\infty\int_0^\infty \frac{1}{(x^\alpha+y^\alpha)^\beta}f(x)g(y)dxdy$$

$$> \frac{1}{\alpha}B\left(\frac{p+\beta-2}{p},\frac{q+\beta-2}{q}\right)\left\{\int_0^\infty \frac{(x^{1-\alpha}f(x))^p}{x^{1+\alpha(\beta-2)}}dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \frac{(y^{1-\alpha}g(y))^q}{y^{1+\alpha(\beta-2)}}dy\right\}^{\frac{1}{q}}.$$

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