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Numerical Plank Problem

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Abstract. Parallel to the plank problem, we investigate the numerical plank problem.

1. Introduction

In this paper we assume that E and F are Banach spaces. Let S_E be the unit sphere of E. For a natural number k, a mapping $P : E \to F$ is called a continuous *k-homogeneous polynomial* if there is a continuous k-linear mapping $A : E \times \cdots \times E \to F$ such that $P(x) = A(x, \ldots, x)$ for all $x \in E$. We let $\mathcal{P}(^kE : F)$ denote the Banach space of continuous k-homogeneous polynomials from E into F, endowed with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \le 1\}$. If $F = \mathbb{R}$, we denote $\mathcal{P}(^kE : \mathbb{R}) = \mathcal{P}(^kE)$. See [7] for details about polynomials on an infinite dimensional Banach space. By a convex body in Euclidean space $\mathbb{R}^n$ we shall mean a compact convex subset K. If u is a unit vector, we shall mean the width of K in the direction u is the distance between supporting hyperplanes of K orthogonal to u. A plank in $\mathbb{R}^n$ is the region between two parallel hyperplanes. In 1930 Tarski posed the plank problem:

Tarski's conjecture. *If a convex body of minimum width 1 is covered by a collection of planks in $\mathbb{R}^n$, then the sum of the widths of these planks is at least 1.*

Tarski proved this if the body is an Euclidean ball in 2 or 3 dimensions. This problem was solved in general by T. Bang in 1951. Given a convex body K, the relative width of a plank S is the width of S divided by the width of K in the direction perpendicular to S. Bang asked a more general question:

Question [3]. *If a convex body is covered by a union of planks, must the relative widths of the planks add up to at least 1?*

The general case of this affine plank problem is still open. If K is a centrally symmetric convex body, then it may be regarded as the unit ball of some finite

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dimensional normed space. K. Ball proved in [2] that if $t_1,\dots,t_n>0$, $t_1+\cdots+t_n=1$ and $\Phi_k\in S_{E^*}$ $(k=1,2,\dots,n)$, then there exists an $x\in S_E$ with $|\Phi_k(x)|\ge t_k$ for all k. Thus the union of planks of relative width summing up to less than 1 can not cover the unit ball. As a corollary, the formulated result obtained:

Theorem A. *If E is a finite dimensional real Banach space and $\Phi_k\in S_{E^*}$ $(k=1,2,\dots,n)$, then there exists an $x\in S_E$ such that $|\Phi_k(x)|\ge\frac{1}{n}$ for all k and the constant $\frac{1}{n}$ is best.*

It is natural that Theorem A gives rise to the definition of the corresponding plank constant. Révész and Sarantopoulos [9] studied plank problem for complex Banach spaces and in particular for the classical $L_p(\mu)$ spaces. Contrary to the linear case, the author [8] recently study the polynomial plank problem as follows: For $n,k\in\mathbb{N}$ and a Banach space E, we denote

$$\begin{aligned} c(n,k:E) \;&:=\; sup\{c>0:\forall P_1,\dots,P_n\in\mathcal{P}(^kE) \text{ with } \|P_j\|=1, \text{there exists}\\ &\qquad x\in E \text{ with } \|x\|=1 \text{ such that } |P_j(x)|\ge c, \text{ for all } j=1,\dots,n\}.\end{aligned}$$

We call $c(n,k:E)$ the *polynomial plank constant* of E with order n,k. Clearly $0\le c(n,k:E)\le 1$. Among other results, we showed that $c(2,2:H)=\frac{1}{3}$ for every real Hilbert space with $\dim(H)\ge 2$. We also investigated the polynomial plank constant $c(n,k:E)$.

Parallel to the polynomial plank problem, we investigate the numerical polynomial plank problem. Let

$$\Pi(E)=\{\ (x,x^*)\ :\ x\in S_E,\ x^*\in S_{E^*}, x^*(x)=1\ \}.$$

The numerical radius of $P\in\mathcal{P}(^kE:E)$ is defined by

$$v(P):=\sup\ \{\ |x^*(P(x))|\ :\ (x,x^*)\in\Pi(E)\}.$$

For $n,k\in\mathbb{N}$ and a Banach space E, we denote

$$\begin{aligned} &c_{\text{num}}(n,k:E):=sup\{c>0:\forall P_1,\dots,P_n\in\mathcal{P}(^kE:E) \text{ with } v(P_j)=1,\\ &\text{there exists } (x,x^*)\in\Pi(E) \text{ such that } |x^*(P_j(x))|\ge c, \text{ for all } j=1,\dots,n\}.\end{aligned}$$

We call $c_{\text{num}}(n,k:E)$ the *numerical polynomial plank constant* of E with order n,k. Clearly $0\le c_{\text{num}}(n,k:E)\le 1$.

In this paper we show:

-$c_{\text{num}}(n,k:H)=c(n,k+1:H)$ for every Hilbert space H. In particular, we show that $c_{\text{num}}(2,1:H)=\frac{1}{3}$, where H is a *real* Hilbert space with $\dim(H)\ge 2$.

-$c_{\text{num}}(2,k:l_1)=c_{num}(2,k:l_\infty)=0$.

For $n,k\in\mathbb{N}$ and a Banach space E,

-$c_{\text{num}}(n,k+1:E)\le c_{num}(n,k:E)$;

-$\lim_{n,k\to\infty} c_{\text{num}}(n,k:E)=\lim_{n\to\infty}\lim_{k\to\infty} c_{\text{num}}(n,k:E)$
$=\lim_{k\to\infty}\lim_{n\to\infty} c_{\text{num}}(n,k:E)$;

-$c_{\text{num}}(n, k : E^{**}) \leq c_{\text{num}}(n, k : E)$.

2. Results

Theorem 2.1. *Let $n, k \in \mathbb{N}$ and H be a Hilbert space. Then $c_{num}(n, k : H) = c(n, k+1 : H)$.*

Proof. ($\leq$): Let $Q_1, \ldots, Q_n \in \mathcal{P}(^{k+1}H)$ with $\|Q_j\| = 1$ for all $j = 1, \ldots, n$. By the Riesz representation theorem for H^*, there exist $P_1, \ldots, P_n \in \mathcal{P}(^k H : H)$ such that

$$Q_j(x) =< x, P_j(x) > \quad (x \in H, \quad \forall j).$$

By definition of $c_{\text{num}}(n, k : H)$, given $\epsilon > 0$, there exists $x_0 \in S_H$ such that

$$|Q_j(x_0)| = |< x_0, P_j(x_0) >| \geq c_{\text{num}}(n, k : H) - \epsilon, \quad \forall j.$$

Thus $c_{\text{num}}(n, k : H) - \epsilon \leq c(n, k+1 : H)$, so $c_{\text{num}}(n, k : H) \leq c(n, k+1 : H)$.

($\geq$): Let $P_1, \ldots, P_n \in \mathcal{P}(^k H : H)$ with $v(P_j) = 1$ for all $j = 1, 2, \cdots, n$. Let $Q_j \in \mathcal{P}(^{k+1}H)$ be such that

$$Q_j(x) :=< x, P_j(x) > \quad (x \in H, \quad \forall j).$$

Then we have

$$\|Q_j\| = sup_{x \in S_H} \ |< x, P_j(x) >| = v(P_j) = 1, \quad \forall j.$$

By definition of $c(n, k+1 : H)$, given $\epsilon > 0$, there exists $x_0 \in S_H$ such that

$$|< x_0, P_j(x_0) >| = |Q_j(x_0)| \geq c(n, k+1 : H) - \epsilon, \quad \forall j.$$

Thus $c_{\text{num}}(n, k : H) \geq c(n, k+1 : H) - \epsilon$, so $c_{\text{num}}(n, k : H) \geq c(n, k+1 : H)$. □

Corollary 2.2. *We have $c_{num}(2, 1 : H) = \frac{1}{3}$, where H is a real Hilbert space.*

Proof. Theorem 3.2 in [8] asserts that $c(2, 2 : H) = \frac{1}{3}$. Thus $c_{\text{num}}(2, 1 : H) = c(2, 2 : H) = \frac{1}{3}$. □

By the definition of $c_{\text{num}}(n, k : E)$, the following is obvious.

Proposition 2.3. *For $n, k \in \mathbb{N}$ and a Banach space E, we have*

$$c_{num}(n+1, k : E) \leq c_{num}(n, k : E).$$

Proposition 2.4. *For $n, k \in \mathbb{N}$ and a Banach space E, we have*

$$c_{num}(n, k+1 : E) \leq c_{num}(n, k : E).$$

Proof. Let $0 < \epsilon < 1$ and $P_1, \ldots, P_n \in \mathcal{P}(^k E : E)$ with $v(P_j) = 1$ for all $j = 1, \ldots, n$. We can find $(x_j, x_j^*) \in \Pi(E)$ such that $|x_j^*(P_j(x_j))| > 1 - \epsilon$. Note that $v(x_j^* P_j) > 1 - \epsilon$ for all j. Indeed, it follows that

$$\begin{aligned} v(x_j^* P_j) &= \sup\{ \ |x_j^*(x)| \ |x^*(P(x))| \ : \ (x, x^*) \in \Pi(E) \ \} \\ &\geq |x_j^*(x_j)| \ |x_j^*(P(x_j))| = |x_j^*(P(x_j))| \\ &> 1 - \epsilon. \end{aligned}$$

Define $Q_j(x) := \frac{x_j^*(x)P_j(x)}{v(x_j^*P_j)}$ for all $x \in E$. Then $Q_j \in \mathcal{P}({}^{k+1}E : E)$ with $v(Q_j) = 1$ for all $j = 1, 2, \cdots, n$. We can find $(x_0, x_0^*) \in \Pi(E)$ such that $|x_0^*(Q_j(x_0))| = \frac{|x_j^*(x_0)|\ |x_0^*(P_j(x_0))|}{v(x_j^*P_j)} > c_{num}(n, k+1 : E) - \epsilon$ for all j. We have

$$\begin{aligned} |x_0^*(P_j(x_0))| &= \frac{v(x_j^*P_j)}{|x_j^*(x_0)|}(c_{num}(n, k+1 : E) - \epsilon) \\ &> \frac{1-\epsilon}{|x_j^*(x_0)|}(c_{num}(n, k+1 : E) - \epsilon) \\ &\geq (1-\epsilon)(c_{num}(n, k+1 : E) - \epsilon), \end{aligned}$$

showing $(1-\epsilon)(c_{num}(n, k+1 : E) - \epsilon) \leq c_{num}(n, k : E)$. Since $\epsilon > 0$ was arbitrary, we have $c_{num}(n, k+1 : E) \leq c_{num}(n, k : E)$. □

Theorem 2.5. *For the real spaces l_1, l_∞, we have $c_{num}(2, k : l_1) = c_{num}(2, k : l_\infty) = 0$ for every $k \in \mathbb{N}$.*

Proof. First we will show that $c_{\rm num}(2, k : l_1) = 0$. Let $T_1, T_2 \in \mathcal{P}({}^1 l_1 : l_1)$ be such that

$$T_1((x_n)) := (\frac{1}{2}x_1, \frac{1}{2}x_1, 0, 0, \ldots),\ T_2((x_n)) := (\frac{1}{2}x_2, -\frac{1}{2}x_2, 0, 0, \ldots)$$

for $(x_n) \in l_1$. Then $v(T_j) = 1$ for all $j = 1, 2$. Let $c \geq 0$ such that there exists $(\ (w_n),\ (\alpha_n)\ ) \in \Pi(l_1)$ satisfying

$$| < (\alpha_n),\ T_j((w_n)) > | \geq c \ \text{ for all } j = 1, 2.$$

We will show that $c = 0$.

Case 1: $w_1 w_2 = 0$

If $w_1 = 0, |w_2| = 1$, then $\alpha_1 = t, \alpha_2 = \pm 1$ for some $t \in [-1, 1]$. Thus

$$c \leq | < (\alpha_n),\ T_1((w_n)) > | = \frac{1}{2}|t \pm 1|\ |w_1| = 0.$$

Thus $c = 0$.

If $|w_1| = 1, w_2 = 0$, then $\alpha_1 = \pm 1, \alpha_2 = t$ for some $t \in [-1, 1]$. By a similar argument as in the above, $c = 0$.

Case 2: $w_1 w_2 \neq 0$

If $w_1 w_2 > 0$, then $\alpha_1 = \alpha_2 = 1$ or $\alpha_1 = \alpha_2 = -1$. Thus

$$c \leq | < (\alpha_n),\ T_2((w_n)) > | = 0.$$

Thus $c = 0$.

If $w_1 w_2 < 0$, then $\alpha_1 = 1, \alpha_2 = -1$ or $\alpha_1 = -1, \alpha_2 = 1$. By a similar argument as in the above, $c = 0$, which shows $c_{\rm num}(2, k : l_1) = 0$. Some similar argument as in the above shows that $c_{\rm num}(2, k : l_\infty) = 0$. Therefore, we complete the proof. □

Proposition 2.6. *Suppose E is an infinite dimensional Banach space. Then*

$$\lim_{n,k\to\infty} c_{num}(n,k:E) = \lim_{n\to\infty}\lim_{k\to\infty} c_{num}(n,k:E) = \lim_{k\to\infty}\lim_{n\to\infty} c_{num}(n,k:E).$$

Proof. By Proposition 2.4, for each $n \in \mathbb{N}$, $(c_{\text{num}}(n,k : E))_{k=1}^{\infty}$ is a decreasing sequence in $[0,1]$. So $\lim_{k\to\infty} c_{\text{num}}(n,k : E)$ exists in $[0,1]$. Let $a_n := \lim_{k\to\infty} c_{\text{num}}(n,k : E) \quad (n \in \mathbb{N})$. By Proposition 2.3, $(a_n)_{n=1}^{\infty}$ is a decreasing sequence in $[0,1]$. So $\lim_{n\to\infty} a_n$ exists in $[0,1]$. Let $a := \lim_{n\to\infty} a_n$. Let $\epsilon > 0$ be given. There is an $n_0 \in \mathbb{N}$ such that $|a_{n_0} - a| < \frac{\epsilon}{2}$. Since $a_{n_0} = \lim_{k\to\infty} c_{\text{num}}(n_0, k : E)$, there is a $k_0 \in \mathbb{N}$ such that $|c_{\text{num}}(n_0, k_0 : E) - a_{n_0}| < \frac{\epsilon}{2}$. By Propositions 2.3 and 2.4,we have, for $n \geq n_0, k \geq k_0$,

$$\begin{aligned} |c_{\text{num}}(n,k:E) - a| &\leq |c_{num}(n_0,k:E) - a| \leq |c_{num}(n_0,k_0:E) - a| \\ &= |c_{num}(n_0,k_0:E) - a_{n_0}| + |a_{n_0} - a| < \epsilon, \end{aligned}$$

showing $\lim_{n,k\to\infty} c_{\text{num}}(n,k : E) = a = \lim_{n\to\infty}\lim_{k\to\infty} c_{\text{num}}(n,k : E)$. Since a coordinate-wise nonincreasing double sequence $(a_{n,k})$ always has a limit, in any order, and it is always the $\inf a_{n,k}$, we complete the proof. □

Let E and F be Banach spaces. A bounded k-homogeneous polynomial P has an extension $\overline{P} \in \mathcal{P}({}^kE^{**} : F^{**})$ to the bidual E^{**} of E, which is called the Aron-Berner extension of P in [1]. In fact, $\overline{P}$ is defined in the following way: We first start with the complex-valued bounded k-homogeneous polynomial $P \in \mathcal{P}({}^kE)$. Let A be the bounded symmetric k-linear form on E corresponding to P. We can extend A to an k-linear form $\overline{A}$ on the bidual E^{**} in such a way that for each fixed j, $1 \leq j \leq k$ and for each fixed $x_1, \ldots, x_{j-1} \in E$ and $z_{j+1}, \ldots, z_m \in E^{**}$, the linear form

$$z \to \overline{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_k), \quad z \in E^{**},$$

is weak-star continuous. By this weak-star continuity A can be extended to an k-linear form $\overline{A}$ on E^{**}, beginning with the last variable and working backwards to the first. Then the restriction

$$\overline{P}(z) = \overline{A}(z, \ldots, z)$$

is called the Aron-Berner extension of P. In particular, Davie and Gamelin [6] proved that $\|P\| = \|\overline{P}\|$. It is also worth to remark that $\overline{A}$ is not symmetric in general. Next, for a vector-valued k-homogeneous polynomial $P \in \mathcal{P}({}^kE : F)$, the Aron-Berner extension $\overline{P} \in \mathcal{P}({}^kE^{**} : F^{**})$ is defined as follows: Given $z \in E^{**}$ and $w \in F^*$,

$$\overline{P}(z)(w) = \overline{w \circ P}(z).$$

For $x \in E$, we define $\delta_x : E^* \to \mathbb{C}$ by $\delta_x(x^*) = x^*(x)$ for each $x^* \in E^*$. Then $\delta_x \in E^{**}$. Let (x_α) be a net in E and $x_0^{**} \in E^{**}$. We say that (x_α) converges polynomial-star to x_0^{**} if for every $P \in \mathcal{P}({}^kE)(k \in \mathbb{N})$, we have $P(x_\alpha)$ converges to $\overline{P}(x_0^{**})$, where $\overline{P}$ is the Aron-Berner extension of P.

Proposition 2.7. *For $n, k \in \mathbb{N}$ and E a Banach space, we have $c_{num}(n, k : E^{**}) \leq c_{num}(n, k : E)$.*

Proof. Let $\epsilon > 0$ and $P_1, \ldots, P_n \in \mathcal{P}(^kE : E)$ with $v(P_j) = 1$ for all $j = 1, \ldots, n$. Let $\overline{P}_1, \ldots, \overline{P}_n \in \mathcal{P}(^kE^{**} : E^{**})$ be the Aron-Berner extensions of $P_1, \ldots, P_n$, respectively. By Corollary 2.14 of [5], $v(\overline{P}_j) = v(P_j) = 1$ for all $j = 1, 2, \cdots, n$. By the definition of $c_{\text{num}}(n, k : E^{**})$, there is some $(x_0^{**}, x_0^{***}) \in \Pi(E^{**})$ such that

$$|x_0^{***}(\overline{P}_j(x_0^{**}))| \geq c_{\text{num}}(n, k : E^{**}) - \epsilon$$

for all $j = 1, 2, \cdots, n$. From the result of Davie-Gamelin [6] that B_E (B_{E^*}, resp) is polynomial-star dense in $B_{E^{**}}$ ($B_{E^{***}}$, resp), there are nets (x_α) in B_E and (x_β^*) in B_{E^*} such that (x_α) converges polynomial-star to x_0^{**} and (x_β^*) converges polynomial-star to x_0^{***}. Since P_j's are uniformly continuous on B_E, there is some $0 < \delta < \frac{\epsilon}{3\max\{\|P_j\| \,:\, j=1,\ldots,n\}}$ such that $w_1, w_2 \in B_E$ with $\|w_1 - w_2\| < \delta$ implies that $\|P_j(w_1) - P_j(w_2)\| < \frac{\epsilon}{3}$ for all $j = 1, 2, \cdots, n$. Note that

$$\lim_\beta x_0^{**}(x_\beta^*) = 1, \quad \lim_\beta \lim_\alpha |x_\beta^*(P_j(x_\alpha))| = |x_0^{***}(\overline{P}_j(x_0^{**}))|.$$

Thus there are α_0 and β_0 such that

$$|x_{\beta_0}^*(P_j(x_{\alpha_0})) - x_0^{***}(\overline{P}_j(x_0^{**}))| < \frac{\epsilon}{3}, \quad |1 - x_{\beta_0}^*(x_{\alpha_0})| < \frac{\delta^2}{4}.$$

By the Bishop-Phelps-Bollobás Theorem ([4], p7, Theorem 1), there is $(z_0, z_0^*) \in \Pi(E)$ such that

$$\|z_0^* - x_{\beta_0}^*\| < \delta, \ \|z_0 - x_{\alpha_0}\| < \delta.$$

It follows that for all $j = 1, 2, \cdots, n$,

$$\begin{aligned}
& |z_0^*(P_j(z_0)) - x_0^{***}(\overline{P}_j(x_0^{**}))| \\
\leq\ & |z_0^*(P_j(z_0)) - x_{\beta_0}^*(P_j(z_0))| + |x_{\beta_0}^*(P_j(z_0)) - x_{\beta_0}^*(P_j(x_{\alpha_0}))| \\
+\ & |x_{\beta_0}^*(P_j(x_{\alpha_0})) - x_0^{***}(\overline{P}_j(x_0^{**}))| \\
\leq\ & \|z_0^* - x_{\beta_0}^*\| \, \|P_j(z_0)\| + \|P_j(z_0) - P_j(x_{\alpha_0})\| + |x_{\beta_0}^*(P_j(x_{\alpha_0})) - x_0^{***}(\overline{P}_j(x_0^{**}))| \\
<\ & \epsilon,
\end{aligned}$$

which shows the proposition. □

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