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Screen Slant Lightlike Submanifolds of Indefinite Kenmotsu Manifolds

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Abstract. In this paper, we introduce the notion of a screen slant lightlike submanifold of an indefinite Kenmotsu manifold. We provide characterization theorem for existence of screen slant lightlike submanifold with examples. Also, we give an example of a minimal screen slant lightlike submanifold of R_2^9 and prove some characterization theorems.

1. Introduction

The study of the geometry of lightlike submanifolds of semi-Riemannian manifolds is interesting due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial and is remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented in a book by Duggal and Bejancu [4]. B. Y. Chen has introduced the notion of slant immersions by generalizing the concept of holomorphic and totally real immersions [2, 3] and it was A. Lotta [9] who introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. Slant submanifold of a Kenmotsu manifold was studied in [7]. To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two (radical and screen) distributions. The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Using these facts the notion of slant lightlike and screen slant lightlike submanifolds of an indefinite Hermitian manifold were introduced by B. Sahin {[10],[11]}.

The purpose of the present paper is to introduce the notion of screen slant lightlike submanifold of an indefinite Kenmotsu manifold.

In Section 2, we have collected the formulae and information which are useful in our subsequent sections. In Section 3, we introduce the concept of screen slant

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lightlike submanifold of an indefinite Kenmotsu manifold with examples. We prove a characterization theorem for the existence of screen slant lightlike submanifolds. Finally, in Section 4, we consider minimal screen slant lightlike submanifolds and give an example and prove two characterization theorems.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, V a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\text{(2.1)} \qquad \begin{cases} \phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \qquad \phi V = 0, \quad \eta(V) = 1 \\ \quad \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \overline{g}(X, V) = \eta(X) \end{cases}$$

for $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [8],

$$\text{(2.2)} \qquad (\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \eta(Y)\phi X, \quad \text{and} \quad \overline{\nabla}_X V = -X + \eta(X)V$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $\{\overline{M}^{m+n}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution of $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. Now, $Rad(TM) = TM \bigcap TM^{\perp}$, where

$$\text{(2.3)} \qquad TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M\}.$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$.

We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. For any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, and therefore, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}(cf.[4], page144)$. Let tr(TM) be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$\text{(2.4)} \qquad \begin{cases} tr(TM) = ltr(TM) \perp S(TM^{\perp}) \\ T\overline{M}|_M = S(TM) \perp [Rad(TM) \bigoplus ltr(TM)] \perp S(TM^{\perp}). \end{cases}$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is said to be

(i) *r-lightlike* if $r < min\{m, n\}$;

(ii) *Coisotropic* if $r = n < m, \quad S(TM^{\perp}) = \{0\}$;

(iii) *Isotropic* if $r = m < n, \quad S(TM) = \{0\}$;

(iv) *Totally lightlike* if $r = m = n, \quad S(TM) = \{0\} = S(TM^{\perp})$.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle tr(TM), respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \forall\, X, Y \in \Gamma(TM), \tag{2.5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \ \ \forall\, U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_U is the shape operator of M with respect to U. Moreover, according to the decomposition (2.4), h^l, h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued *lightlike second fundamental form* and *screen second fundamental form* of M, respectively. Then

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \ \ \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \ \ N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \ \ W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $D^l(X,W)$, $D^s(X,N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively and ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively. We call ∇^l, ∇^s the lightlike and screen transversal connections on M, and A_N, A_W are shape operators on M with respect to N and W, respectively. Using (2.5) and (2.7)∼(2.9), we obtain

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N), W) = \overline{g}(N, A_W X). \tag{2.11}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.13}$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and $Rad(TM)$.
From (2.12) and (2.13), we get

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X, \xi), \xi) = 0, \qquad A^*_\xi \xi = 0. \tag{2.16}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, from (2.7), we obtain

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y). \tag{2.17}$$

However, it is important to note that ∇^*, ∇^{*t} are metric connections on $S(TM)$ and $Rad(TM)$, respectively.

A general notion of a minimal lightlike submanifold in a semi-Riemannian manifold, as introduced by Bejan and Duggal [1], is as follows:

Definition 2.1. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if
(i) $h^s = 0$ on $Rad(TM)$;
(ii) trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

The following result is important for our subsequent use.

Proposition 2.1[4]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^\perp)$ and $ltr(TM)$.*

3. Screen slant lightlike submanifolds

In what follows we prove:

Lemma 3.1. *Let M be 2q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to M such that $2q < dim(M)$. Then the screen distribution $S(TM)$ is Riemannian.*

Proof. Let $\overline{M}$ be an $(m+n)$-dimensional indefinite Kenmotsu manifold with index $2q$ and M be an m-dimensional $2q$-lightlike submanifold of $\overline{M}$ such that $2q < m$. We can choose a local quasi orthonormal frame on $\overline{M}$ along M as follows:

$$\{\xi_i, N_i, X_\alpha, W_a\}, i \in \{1, ..., 2q\}, \alpha \in \{2q+1, ..., m\}, a \in \{2q+1, ..., n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $RadTM$ and $ltr(TM)$, respectively, and $\{X_\alpha\}$ is an orthonormal basis of $S(TM)$ and $\{W_a\}$ is an orthonormal basis of $S(TM^\perp)$. Now we can construct the orthonormal basis $\{U_1, U_2, ..., U_{4q}\}$ as follows:

$$\begin{array}{ll} U_1 = \frac{1}{\sqrt{2}}\{\xi_1 + N_1\}, & U_2 = \frac{1}{\sqrt{2}}\{\xi_1 - N_1\}, \\ U_3 = \frac{1}{\sqrt{2}}\{\xi_2 + N_2\}, & U_4 = \frac{1}{\sqrt{2}}\{\xi_2 - N_2\}, \\ \ldots & \ldots \\ \ldots & \ldots \\ U_{4q-1} = \frac{1}{\sqrt{2}}\{\xi_{2q} + N_{2q}\}, & U_{4q} = \frac{1}{\sqrt{2}}\{\xi_{2q} - N_{2q}\}. \end{array}$$

Hence, $\{\xi_i, N_i\}$ gives a non-degenerate space of constant index $2q$ which implies that $RadTM \oplus ltr(TM)$ is non-degenerate and of constant index $2q$ on $\overline{M}$. As $index(T\overline{M}) = index(RadTM \bigoplus ltr(TM)) + index(S(TM) \perp S(TM^{\perp}))$, we have $2q = 2q + index(S(TM) \perp S(TM^{\perp}))$, which implies that $index(S(TM) \perp S(TM^{\perp})) = 0$. Hence $S(TM)$ and $S(TM^{\perp})$ are Riemannian. □

As mentioned in the introduction, the purpose of this paper is to define screen slant lightlike submanifolds of indefinite Kenmotsu manifolds. To define this notion, one needs to consider angle between two vector fields. As we can see from Section 2, a lightlike submanifold has two distributions viz. radical and screen. The radical distribution is totally lightlike and, therefore, it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Thus one way to define slant lightlike submanifolds is to choose a Riemannian screen distribution on lightlike submanifolds, for which we use Lemma 3.1.

Similar to the definition of screen slant lightlike submanifold of indefinite Hermitian manifold [11], we state the following:

Definition 3.1. Let M be a 2q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to M such that $2q < dim(M)$. Then M is a screen slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $RadTM$ is invariant with respect to ϕ, i.e. $RadTM = \phi RadTM$

(ii) For all $x \in U \subset M$ and for each non zero vector field X tangent to $S(TM) = D \perp \{V\}$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $S(TM)$ is constant, where D is complementary distribution to V in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of $S(TM)$. A screen slant lightlike submanifold M is said to be proper if $\theta \neq 0, \frac{\pi}{2}$. In what follows, we suppose that $(M, g, S(TM))$ is $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with constant index $2q < dim(M)$. A real lightlike submanifold M of $\overline{M}$ is called invariant lightlike submanifold if [5]

$$\phi RadTM = RadTM \quad and \quad \phi S(TM) \subset S(TM).$$

A real lightlike submanifold M is called screen real submanifold if [5]

$$\phi RadTM = RadTM \quad and \quad \phi S(TM) \subset S(TM^{\perp}).$$

The following result is an easy consequence of Definition 3.1:

Proposition 3.1. *Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Then M is invariant(resp. Screen real) if and only if $\theta = 0$, (resp. $\theta = \frac{\pi}{2}$).*

Proof. If M is invariant, then $\phi RadTM = RadTM$ and $\phi S(TM) \subset S(TM)$, and consequently $\theta = 0$. Conversely, if M is screen slant lightlike submanifold with $\theta = 0$, then it is clear from Definition 3.1 that $\phi RadTM = RadTM$ and $\phi S(TM) \subset S(TM)$. Thus the proof follows. Similarly other assertion follows. □

Proposition 3.1 implies that invariant and screen real lightlike submanifolds are examples of screen slant lightlike submanifolds. Now we provide examples of proper screen slant lightlike submanifolds.

In what follows, $(R_q^{2m+1}, \phi_0, V, g)$ will denote the manifold R_q^{2m+1} with its usual Kenmotsu structure given by

$$\begin{cases} \eta = dz, \qquad V = \partial z, \\ \overline{g} = \eta \bigotimes \eta - e^{2z}(\sum_{i=1}^{q} dx^i \bigotimes dx^i + \sum_{i=q+1}^{m} dx^i \bigotimes dx^i + \sum_{i=1}^{m} dy^i \bigotimes dy^i), \\ \phi_0(X_1, X_2, ..., X_{m-1}, X_m, Y_1, Y_2, ..., Y_{m-1}, Y_m, Z) \\ \qquad = (-X_2, X_1, ..., -X_m, X_{m-1}, -Y_2, Y_1, ..., -Y_m, Y_{m-1}, 0), \end{cases}$$

where (x^i, y^i, z) are the cartesian coordinates.

Example 3.1. Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$.

Consider a submanifold M of R_2^9, defined by

$$X(u, v, \theta_1, \theta_2, t) = (u, v, \sin\theta_1, \cos\theta_1, -\theta_1 \sin\theta_2, -\theta_1 \cos\theta_2, u, v, t)$$

Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \partial y_3), \qquad Z_2 = e^{-z}(\partial x_2 + \partial y_4), \\ Z_3 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 - \sin\theta_2 \partial y_1 - \cos\theta_2 \partial y_2), \\ Z_4 = e^{-z}(-\theta_1 \cos\theta_2 \partial y_1 + \theta_1 \sin\theta_2 \partial y_2), \qquad Z_5 = V = \partial z. \end{cases}$$

Hence, $RadTM = span\{Z_1, Z_2\}$, which is invariant with respect to ϕ_0. Next, $S(TM) = D \perp \{V\} = \{Z_3, Z_4\} \perp \{V\}$ is slant distribution with slant angle $\frac{\pi}{4}$. By direct calculations, we get

$$S(TM^\perp) = \text{span} \begin{cases} W_1 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 + \sin\theta_2 \partial y_1 + \cos\theta_2 \partial y_2), \\ W_2 = e^{-z}(\sin\theta_1 \partial x_3 + \cos\theta_1 \partial x_4) \end{cases}$$

and $ltr(TM)$ =span$\{N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_3), N_2 = \frac{e^{-z}}{2}(-\partial x_2 + \partial y_4)\}$. It is easy to see that conditions (i) and (ii) of Definition 3.1 hold. Hence, M is a proper screen slant lightlike submanifold of R_2^9.

Example 3.2. Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$.

Consider a submanifold M of R^9_2, defined by

$$X(u,v,\theta_1,\theta_2,t) = (u, v, \sin\theta_1, \cos\theta_1, -\theta_1\sin\theta_2, \\ -\theta_1\cos\theta_2, u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha, t)$$

for $\alpha, \theta_1, \theta_2 \in (0, \frac{\pi}{2})$. Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \cos\alpha\partial y_3 + \sin\alpha\partial y_4), \quad Z_2 = e^{-z}(\partial x_2 - \sin\alpha\partial y_3 + \cos\alpha\partial y_4), \\ Z_3 = e^{-z}(\cos\theta_1\partial x_3 - \sin\theta_1\partial x_4 - \sin\theta_2\partial y_1 - \cos\theta_2\partial y_2), \\ Z_4 = e^{-z}(-\theta_1\cos\theta_2\partial y_1 + \theta_1\sin\theta_2\partial y_2), \quad Z_5 = V = \partial z. \end{cases}$$

Hence, $RadTM = span\{Z_1, Z_2\}$, which is invariant with respect to ϕ_0. Next, $S(TM) = D\perp\{V\} = \{Z_3, Z_4\}\perp\{V\}$ is slant distribution with slant angle $\frac{\pi}{4}$. By direct calculations, we get

$$S(TM^\perp) = \text{ span } \begin{cases} W_1 = e^{-z}(\cos\theta_1\partial x_3 - \sin\theta_1\partial x_4 + \sin\theta_2\partial y_1 + \cos\theta_2\partial y_2) \\ W_2 = e^{-z}(\sin\theta_1\partial x_3 + \cos\theta_1\partial x_4) \end{cases}$$

and

$$ltr(TM) = \text{ span } \begin{cases} N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \cos\alpha\partial y_3 + \sin\alpha\partial y_4) \\ N_2 = \frac{e^{-z}}{2}(-\partial x_2 - \sin\alpha\partial y_3 + \cos\alpha\partial y_4). \end{cases}$$

It is easy to see that conditions (i) and (ii) of Definition 3.1 hold. Hence, M is a proper screen slant lightlike submanifold of R^9_2.

Example 3.3. Let $\overline{M} = (R^{13}_2, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Consider a submanifold M of R^{13}_2, defined by

$$X(u,v,\theta_1,\theta_2,t) = (u\cosh\alpha, v\cosh\alpha, u, v, \theta_1, \theta_2, k\cos\theta_1, k\sin\theta_1, \\ k\cos\theta_2, k\sin\theta_2, u\sinh\alpha, v\sinh\alpha, t)$$

for $\alpha, k > 0$. Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\cosh\alpha\partial x_1 + \partial x_3 + \sinh\alpha\partial y_5), \\ Z_2 = e^{-z}(\cosh\alpha\partial x_2 + \partial x_4 + \sinh\alpha\partial y_6), \\ Z_3 = e^{-z}(\partial x_5 - k\sin\theta_1\partial y_1 + k\cos\theta_1\partial y_2), \\ Z_4 = e^{-z}(\partial x_6 - k\sin\theta_2\partial y_3 + k\cos\theta_2\partial y_4), \\ Z_5 = V = \partial z \end{cases}$$

Hence, $RadTM = \text{span}\{Z_1, Z_2\}$, which is invariant with respect to ϕ_0. Next, $S(TM) = D\perp\{V\} = \{Z_3, Z_4\}\perp\{V\}$ is slant distribution with slant angle $\theta = \cos^{-1}(\frac{1}{1+k^2})$. It is easy to see that conditions (i) and (ii) of Definition 3.1 hold. Hence, M is a proper screen slant lightlike submanifold of R^{13}_2.

Now, we give a result for non-existence of proper screen slant submanifold.

Proposition 3.2. *There exists no 3-dimensional proper screen slant lightlike submanifold M with structure vector field tangent to M in an indefinite Kenmotsu manifolds $\overline{M}$ with index 2.*

Proof. Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold with index 2. Then M is 2-lightlike or 1-lightlike. If M is 2-lightlike then $S(TM) = \{V\}$, and $\phi S(TM) \subset S(TM)$. Moreover, by Definition 3.1, $\phi RadTM = RadTM$. Hence, M is invariant. From Definition 3.1, M can not be 1-lightlike screen slant submanifold. □

We know that for any $X \in \Gamma(S(TM))$

$$\phi X = TX + \omega X \tag{3.1}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(tr(TM))$ are the tangential and transversal components of ϕX, respectively. Moreover, for a screen slant lightlike submanifold, we denote by Q, P and $\overline{P}$ the projections on the distributions $RadTM$, D and $S(TM) = D \perp \{V\}$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = QX + \overline{P}X \tag{3.2}$$

where $\overline{P}X = PX + \eta(X)V$. Using (2.1) in the above equation, we obtain

$$\phi X = TQX + \phi PX = TQX + TPX + \omega PX \tag{3.3}$$

for any $X \in \Gamma(TM)$. Thus, we conclude that

$$\phi QX = TQX, \;\; \omega QX = 0 \;\; \texttt{and} \quad TPX \in \Gamma(S(TM)). \tag{3.4}$$

On the other hand, the screen transversal vector bundle $S(TM^\perp)$ has tha following decomposition

$$S(TM^\perp) = \omega P(S(TM)) \perp \nu \tag{3.5}$$

Then for $W \in \Gamma(S(TM^\perp))$, we have

$$\phi W = BW + CW \tag{3.6}$$

where $BW \in \Gamma(S(TM))$ and $CW \in \Gamma(\nu)$.

We have:

Corollary 3.1. *Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. We have*

(i) *if $X \in \Gamma(S(TM))$, then $\omega X \in \Gamma(S(TM^\perp))$,*

(ii) *if $X \in \Gamma(RadTM)$, then $\omega X = 0$.*

Proof. (i) follows from the invariant properties of $ltr(TM)$ with respect to ϕ due to the fact that $RadTM$ is invariant and (ii) is obvious. □

We now prove a characterization for screen slant lightlike submanifolds.

Theorem 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M such that $2q < dim(M)$. Then M is screen slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) *$\phi ltr(TM) = ltr(TM)$ i.e. $ltr(TM)$ is invariant*

(b) *There exists a constant $\lambda \in [-1, 0]$ such that*

$$T^2\overline{P}X = \lambda(\overline{P}X - \eta(\overline{P}X)V) \tag{3.7}$$

$\forall\, X \in \Gamma(S(TM))$ linearly independent of structure vector field V. Moreover, in such a case, $\lambda = -\cos^2\theta\,|_{S(TM)}$,

where θ is the slant angle of M.

Proof. Let M be $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$. Then Lemma 3.1 implies that $S(TM)$ is a Riemannian vector bundle. If M is a screen slant lightlike submanifold of $\overline{M}$, then $RadTM$ is invariant distribution with respect to ϕ and from Corollary 3.1, we have that $\omega PX \in \Gamma(S(TM^{\perp}))$ for $X \in S(TM)$. Thus, using (3.1), we get

$$\overline{g}(\phi N, X) = -\overline{g}(N, \phi X) = -\overline{g}(N, TPX) - \overline{g}(N, \omega PX) = 0$$

for $X \in S(TM)$ and $N \in ltr(TM)$. Hence we conclude that ϕN does not belong to $S(TM)$. On the other hand, from (2.6), we find

$$\overline{g}(\phi N, W) = -\overline{g}(N, \phi W) = -\overline{g}(N, BW) - \overline{g}(N, CW) = 0$$

for $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$. Thus, ϕN does not belong to $S(TM^{\perp})$. Next, suppose that $\phi N \in \Gamma(RadTM)$. Then,

$\phi\phi N = -N + \eta(N)V = -N \in \Gamma(ltr(TM))$, as $RadTM$ is invariant , and we get a contradiction. Thus (a) is proved.

For $X \in \Gamma(S(TM)), PX \in S(TM) - \{V\}$, we have

$$\cos\theta(PX) = \frac{\overline{g}(\phi PX, TPX)}{|\phi PX||TPX|} = -\frac{\overline{g}(PX, \phi TPX)}{|\phi PX||TPX|} = -\frac{\overline{g}(PX, T^2PX)}{|PX||TPX|}. \tag{3.8}$$

On the other hand, we get

$$\cos\theta(PX) = \frac{|TPX|}{|\phi PX|}. \tag{3.9}$$

Thus, from (3.8) and (3.9), we find

$$\cos^2\theta(PX) = -\frac{\overline{g}(PX, T^2PX)}{|PX|^2}.$$

Since $\theta(PX)$ is constant on $S(TM)$, we conclude that

$$T^2PX = \lambda PX = \lambda(\overline{P}X - \eta(\overline{P}X)V), \lambda \in (-1, 0). \tag{3.10}$$

Moreover, in this case, $\lambda = -\cos^2\theta$. It is clear that equation (3.10) is valid for $\theta = 0$ and $\theta = \frac{\pi}{2}$. Hence, for $\overline{P}X \in S(TM)$, we find

$$T^2(\overline{P}X) = \lambda(\overline{P}X - \eta(\overline{P}X)V), \lambda \in [-1, 0]. \tag{3.11}$$

The converse can be obtained in a similar way. □

Using (2.1), (3.1) and Theorem 3.1, we have the following:

Corollary 3.2. *Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector tangent to M. Then we have*

$$g(T\overline{P}X, T\overline{P}Y) = \cos^2\theta|_{S(TM)}[g(\overline{P}X, \overline{P}Y) - \eta(\overline{P}X)\eta(\overline{P}Y)] \tag{3.12}$$

$$g(F\overline{P}X, F\overline{P}Y) = \sin^2\theta|_{S(TM)}[g(\overline{P}X, \overline{P}Y) - \eta(\overline{P}X)\eta(\overline{P}Y)] \tag{3.13}$$

for $X, Y \in \Gamma(TM)$.

4. Minimal screen slant lightlike submanifolds

In this section we study minimal screen slant lightlike submanifolds of indefinite Kenmotsu manifolds. We have the following:

Example 4.1. Let $\overline{M} = (R^9_2, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$.

Consider a submanifold M of R^9_2 defind by

$$\begin{aligned} x_1 &= u_1, & x_2 &= u_2,\\ x_3 &= u_1, & x_4 &= u_2,\\ y_1 &= \cos u_3\cosh u_4, & y_2 &= \cos u_3\sinh u_4,\\ y_3 &= \sin u_3\sinh u_4, & y_4 &= \sin u_3\cosh u_4,\\ & & z = t \end{aligned}$$

where $u_1 \in (0, \frac{\pi}{2})$.
Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \partial x_3), \quad Z_2 = e^{-z}(\partial x_2 + \partial x_4),\\ Z_3 = e^{-z}(-\sin u_3\cosh u_4\partial y_1 - \sin u_3\sinh u_4\partial y_2 + \cos u_3\sinh u_4\partial y_3 + \cos u_3\cosh u_4\partial y_4),\\ Z_4 = e^{-z}(\cos u_3\sinh u_4\partial y_1 + \cos u_3\cosh u_4\partial y_2 + \sin u_3\cosh u_4\partial y_3 + \sin u_3\sinh u_4\partial y_4),\\ Z_5 = V = \partial z \end{cases}$$

We define a (1,1) tensor ϕ_1 as follows:
$\phi_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = (-x_2, x_1, -x_4, x_3, -y_3\cos\alpha - y_2\sin\alpha, -y_4\cos\alpha + y_1\sin\alpha, y_1\cos\alpha + y_4\sin\alpha, y_2\cos\alpha - y_3\sin\alpha, 0)$ where $\alpha \in (0, \frac{\pi}{2})$.

For $X = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$, we have
$\phi_1^2 X = (-x_1, -x_2, -x_3, -x_4, -y_1\cos^2\alpha - y_4\sin\alpha\cos\alpha + y_4\sin\alpha\cos\alpha - y_1\sin^2\alpha, -y_2\cos^2\alpha + y_3\sin\alpha\cos\alpha - y_3\sin\alpha\cos\alpha - y_2\sin^2\alpha, -y_3\cos^2\alpha - y_2\cos\alpha\sin\alpha + y_2\cos\alpha\sin\alpha - y_3\sin^2\alpha, -y_4\cos^2\alpha + y_1\sin\alpha\cos\alpha - y_1\sin\alpha\cos\alpha - y_4\sin^2\alpha) = -X + \eta(X)V$
proving that ϕ_1 is an almost contact structure.
Hence, $RadTM = span\{Z_1, Z_2\}$, which is invariant with respect to ϕ_1. Next, $S(TM) = D\perp\{V\} = \{Z_3 + Z_4\}\perp\{V\}$ is Riemannian. Then M is screen slant lightlike with slant angle α with respect to ϕ_1. By direct calculation, we get

$$S(TM^\perp) = \text{span}\begin{cases} W_1 = e^{-z}(-\cosh u_4\partial y_1 + \sinh u_4\partial y_2 + \tan u_3\sinh u_4\partial y_3 \\ \qquad - \tan u_3\cosh u_4\partial y_4), \\ W_2 = e^{-z}(-\tan u_3\sinh u_4\partial y_1 + \tan u_3\cosh u_4\partial y_2 - \cosh u_4\partial y_3 \\ \qquad + \sinh u_4\partial y_4) \end{cases}$$

and $ltr(TM) = span\{N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial x_3),\ N_2 = \frac{e^{-z}}{2}(-\partial x_2 + \partial x_4)\}$. It is easy to see that condition (i) and (ii) of Definition 3.1 hold. Hence M is a proper screen slant lightlike submanifold of (R_2^9, ϕ_1).
By direct calculation and using Gauss formula, we get

$$\begin{cases} h^s(X, Z_1) = h^s(X, Z_2) = 0, \qquad h^l = 0, \forall X \in \Gamma(TM) \\ h^s(Z_3, Z_3) = \frac{e^{-z}(\cos u_3)}{(\cosh^2 u_4 + \sinh^2 u_4)}W_1, \quad h^s(Z_4, Z_4) = -\frac{e^{-z}(\cos u_3)}{(\cosh^2 u_4 + \sinh^2 u_4)}W_1 \\ h^s(Z_5, Z_5) = 0 \end{cases}$$

Thus M is a minimal proper screen slant lightlike submanifold of (R_2^9, ϕ_1).

In what follows, we prove two characterization results for minimal slant lightlike submanifolds.
We have the following:

Lemma 4.1. *Let M be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ such that $dim(D) = dim(S(TM^\perp))$. If $\{e_1, ..., e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{\csc\theta Fe_1, ..., \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^\perp)$.*

Proof. Since $\{e_1, ..., e_m\}$ is a local orthonormal basis of D and D is Riemannian, from Corollary 3.2, we find

$$\overline{g}\{\csc\theta Fe_i, \csc\theta Fe_j\} = \delta_{ij},$$

where $i, j = 1, 2, ..., m$. This proves the result. □

Theorem 4.1. *Let M be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Then M is minimal if and only if*

$$traceA_{W_j|S(TM)} = 0,\ traceA^*_{\xi_k|S(TM)} = 0,\ \text{ and }\ \overline{g}(D^l(X,W),Y) = 0,$$

for $X, Y \in \Gamma(RadTM)$ *and* $W \in \Gamma(S(TM^\perp))$*, where* $\{\xi_k\}_{k=1}^r$ *is a basis of* $Rad(TM)$ *and* $\{W_j\}_{j=1}^m$ *is a basis of* $S(TM^\perp)$*.*

Proof. Since $\overline{\nabla}_V V = 0$, from (2.7), we get $h^l(V,V) = h^s(V,V) = 0$. Now, take an orthonormal frame $\{e_1, ..., e_m\}$ of D. We know that $h^l = 0$ on $Rad(TM)$(cf.[1], Proposition 4.1). Thus M is minimal if and only if $\sum_{i=1}^m h(e_i,e_i) = 0$ and $h^s = 0$ on $RadTM$. Using (2.10) and (2.14), we obtain

$$\sum_{i=1}^m h(e_i,e_i) = \sum_{i=1}^m \frac{1}{r}\sum_{a=1}^r g(A^*_{\xi_a}e_i,e_i)N_a + \frac{1}{m}\sum_{j=1}^m g(A_{W_j}e_i,e_i)W_j. \tag{4.1}$$

On the other hand, from(2.10), we get $h^s = 0$ on $RadTM$ if

$$\overline{g}(D^l(X,W),Y) = 0,$$

for $X, Y \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.
Thus our assertion follows from (4.1) and (4.2). □

Theorem 4.2. *Let* M *be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *with structure vector field tangent to* M *such that* $dim(D) = dim(S(TM^\perp))$*. Then* M *is minimal if and only if*

$$traceA_{Fe_j|S(TM)} = 0, traceA^*_{\xi_k|S(TM)} = 0,\ \text{ and }\ \overline{g}(D^l(X,Fe_j),Y) = 0,$$

for $X, Y \in \Gamma(RadTM)$*, where* $\{\xi_k\}_{k=1}^r$ *is a basis of* $RadTM$ *and* $\{e_j\}_{j=1}^m$ *is a basis of* D*.*

Proof. Since $\overline{\nabla}_V V = 0$, from (1.7), we get $h^l(V,V) = h^s(V,V) = 0$. We know that $h^l = 0$ on $Rad(TM)$ (cf.[1], Proposition 3.1). Also, from Lemma 4.1, $\{\csc\theta Fe_1, ..., \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^\perp)$. Thus

$$h^s(X,X) = \sum_{i=1}^m \csc\theta g(A_{Fe_i}X,X)Fe_i$$

for $X \in \Gamma(D)$. Thus the proof follows from Theorem 4.1. □

Remarks. (a) It is known that a proper slant submanifold of a Kenmotsu manifold is odd dimensional [6], but this is not true in case of our definition of screen slant lightlike submanifold. For instance, see two examples given in this paper.
(b) We notice that the second fundamental forms and their shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (1.7)∼(1.11) that in case of lightlike submanifolds there are interrelations between these geometric objects and those of its screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $(S(TM), S(TM^\perp), ltr(TM))$. However, it is important to highlight that, as per Proposition 1.1 of this paper, our results are stable with respect to any change in

the above triplet.

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