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A New Class of Hermite-Konhauser Polynomials together with Differential Equations

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Abstract. It is shown that an appropriate combination of methods, relevant to operational calculus and to special functions, can be a very useful tool to establish and treat a new class of Hermite and Konhauser polynomials. We explore the formal properties of the operational identities to derive a number of properties of the new class of Hermite and Konhauser polynomials and discuss the links with various known polynomials.

1. Introduction

Various types of generalized polynomials, for example, Bessel polynomials, Laguerre polynomials, Laguerre-Bessel polynomials, Laguerre-Hermite polynomials, Bessel-Hermite polynomials and Laguerre-Konhauser polynomials have been proposed during the last years (see [2-10]). In [6] Dattoli et. al. introduced two Laguerre polynomials of two variables of the forms

$$ {}_1L_{n,\alpha}(y,z) = n!\sum_{k=0}^{n}\frac{z^{n-k}y^{\alpha+k}}{k!(n-k)!\Gamma(\alpha+k+1)}, \tag{1.1}$$

$$ L_n^{(m)}(y,z) = (m+n)!\sum_{k=0}^{n}\frac{(-1)^k z^{n-k}y^k}{k!(n-k)!(m+k)!}. \tag{1.2}$$

Clearly

$$ \frac{\Gamma(\alpha+n+1)(-x)^{\alpha}}{n!}\,{}_1L_{n,\alpha}(-x,1) = L_n^{(\alpha)}(x), \tag{1.3}$$

$$ L_n^{(m)}(x,y) = y^n L_n^{(m)}(x/y), \tag{1.4}$$

where $L_n^{(\alpha)}(x)$ is the associated Laguerre polynomials [15,p.200(1)].
Also, these so-called modified Laguerre polynomials $L_{a,b,c,n}(x)$ were introduced by

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Goyal [13] in the form

$$L_{a,b,c,n}(x) = \frac{b^n (c)_n}{n!} {}_1F_1\left[-n; c; \frac{ax}{b}\right], \tag{1.5}$$

where $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$, Γ : Gamma function, which is a slight variant of the associated Laguerre polynomials $L_n^{(\alpha)}$. In [10] the two variable Hermite -Kampé de Fériet polynomials are specified by the series

$$H_n(x,z) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{z^k x^{n-2k}}{k!(n-2k)!}. \tag{1.6}$$

and by the operational rule(see [5]):

$$e^{z\frac{\partial^2}{\partial x^2}} \{x^n\} = H_n(x,z). \tag{1.7}$$

From (1.6) it follows that

$$H_n(2x,-1) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!} = H_n(x), \tag{1.8}$$

where $H_n(x)$ being ordinary Hermite polynomials. A further interesting set of polynomials is provided by the Laguerre-Hermite polynomials ${}_LH_n^*(x,y)$ which defined by the series [11, p.233(41)]:

$${}_LH_n^*(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r L_{n-2r}(x,y)}{2^r r!(n-2r)!}, \tag{1.9}$$

where $L_n(x,y)$ are Laguerre polynomials of two variables [8]:

$$L_n(x,y) = n! \sum_{k=0}^{n} \frac{(-1)^k y^{n-k} x^k}{(k!)^2 (n-k)!}. \tag{1.10}$$

For the purpose of this our present study, we recall here the following explicit expression for the Konhauser polynomials $Z_n^\alpha(x;k)$ [14]:

$$Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{x^{ks}}{\Gamma(ks+\alpha+1)}, \ \alpha > -1, k = 1,2,..., \tag{1.11}$$

which for $k = 1$, these polynomials reduces to the Laguerre polynomials $L_n^{(\alpha)}(x)$ and their special case when $k = 2$, were encountered earlier by Spencer and Fano[16] in certain calculation involving penetration of gamma rays through matter. In this paper we exploit operational techniques combined with the monomiality principle to introduce and discuss a new class of Hermite-Konhauser polynomials, which provide a further generalization of a number of known polynomials including all polynomials mentioned above.

2. Hermite-Konhauser polynomials

Let us consider the generating relation

$$f(x,y;z|t) = exp\left[zt\left(1-\frac{1}{z}\hat{D}_y^{-k}\right)-xt^2\right]\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}, \tag{2.1}$$

where $k=1,2,...;\alpha>-1$ and $\hat{D}_y$ denotes the derivative operator and $\hat{D}_y^{-1}$ its inverse (see [8]). Expressing the exponential function in series and applying the result (see e.g. [1]) :

$$\hat{D}_x^n x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}x^{\alpha-n}, \quad \alpha\geq 0, n\in Z,$$

we can conclude that

$$f(x,y;z|t) = \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^{s+r}x^r z^{n-s}y^{ks+\alpha}}{s!r!(n-s)!\Gamma(ks+\alpha+1)}\right\}t^{n+2r}.$$

Finally, the change of index $n=n-2r$ leads to

$$exp\left[zt\left(1-\frac{1}{z}\hat{D}_y^{-k}\right)-xt^2\right]\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\} = \sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z)\frac{t^n}{n!}, \tag{2.2}$$

where ${}_kH_n^{(\alpha)}(x,y;z)$ is the Hermite- Konhauser polynomials defined by

$${}_kH_n^{(\alpha)}(x,y;z) = n!\sum_{s=0}^{n}\sum_{r=0}^{[\frac{n-s}{2}]}\frac{(-1)^{s+r}x^r y^{ks+\alpha}z^{n-s-2r}}{s!r!(n-s-2r)!\Gamma(ks+\alpha+1)}. \tag{2.3}$$

In view (1.8) and (1.11), it is easily seen that

$$\sqrt{x^n}\,{}_1H_n^{(0)}(x,0;z) = H_n(z/2\sqrt{x}). \tag{2.4}$$

$$\frac{y^{-\alpha}\Gamma(kn+\alpha+1)}{z^n n!}{}_kH_n^{(\alpha)}(0,y;z) = Z_n^\alpha(y/\sqrt[k]{z};k), \tag{2.5}$$

It may of interest to point out that the series representation (2.3), in particular, yields the following relationships:

$$\frac{y^{-\alpha}\Gamma(kn+\alpha+1)}{n!}{}_kH_n^{(\alpha)}(0,y;1) = Z_n^\alpha(y;k), \tag{2.6}$$

$$(2.7)\qquad (-1)^{\alpha}{}_1H_n^{(\alpha)}(0,-y;z) = {}_1L_{n,\alpha}(y,z),$$

$$(2.8)\qquad (ay)^{1-\alpha}\frac{\Gamma(\alpha+n)}{n!}{}_1H_n^{(\alpha-1)}(0,ay;z) = L_{a,z,\alpha,n}(y),$$

$$(2.9)\qquad y^{-m}\frac{(m+n)!}{n!}{}_1H_n^{(m)}(0,y;z) = L_n^{(m)}(y,z),$$

$$(2.10)\qquad \frac{y^{-\alpha}\Gamma(n+\alpha+1)}{n!}{}_1H_n^{(\alpha)}(0,y;1) = L_n^{\alpha}(y),$$

$$(2.11)\qquad {}_1H_n^{(0)}(-x,0;z) = H_n(z,x),$$

$$(2.12)\qquad y^{-\alpha}{}_1H_n^{(0)}(\frac{1}{2},\sqrt[k]{y};z) = {}_LH_n^{*}(y,z),$$

$$(2.13)\qquad {}_1H_n^{(0)}(1,0;2z) = H_n(z).$$

Moreover, the psedu Laguerre polynomials $L_n(x,y;k,j)$ and the psedu Hermite polynomials $\Delta_n(x,y;k,j)$ introduced recently by Dattoli et al.(see [3]) are a special cases of our polynomials as given below:

$$(2.14)\qquad {}_rH_n^{(j)}(0,y;z) = n!\sum_{s=0}^{n}\frac{(-1)^s z^{n-s}y^{ks+j}}{s!(n-s)!(ks+j)!} = L_n(x,y;k,j),$$

For the purpose of this work, we introduce the following obvious straightforward extension of (2.14)

$$(2.15)\qquad L_n(y,z;k,\alpha) = n!\sum_{s=0}^{n}\frac{(-1)^s z^{n-s}y^{ks+\alpha}}{s!(n-s)!\Gamma(ks+\alpha+1)}.$$

Clearly,

$$(2.16)\qquad L_n(y,z;k,\alpha) = {}_kH_n^{(\alpha)}(0,y;z).$$

The relevant generating function for the polynomials $L_n(x,y;k,\alpha)$ can be obtained by the method suggested in [3], thus getting

$$(2.17)\qquad \sum_{n=0}^{\infty}L_n(y,z;k,\alpha)\frac{t^n}{n!} = y^{\alpha}e^{zt}C_{\alpha}(y^kt;k),$$

where $C_\alpha(x;k)$ being the α order Tricomi function defined by [7]

$$C_\alpha(x;k) = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{s!\Gamma(ks+\alpha+1)}.$$

We must emphasize that the polynomials in (2.14) and (2.15) are a generalized forms of Konhauser polynomials defined by (1.11) and indeed we have

$$Z_n^\alpha(y,k) = \frac{y^{-\alpha}\Gamma(kn+\alpha+1)}{n!} L_n(y,1;k,\alpha). \tag{2.18}$$

Taking into account the nature of the series representation (2.3), we can write the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ in the more elegant forms:

$${}_kH_n^{(\alpha)}(x,y;z) = n! \sum_{r=0}^{[\frac{n-s}{2}]} \frac{(-1)^r z^{n-2r} x^r y^\alpha Z_{n-2r}^\alpha(y/\sqrt[k]{z};k)}{r!\Gamma(kn-2kr+\alpha+1)}, \tag{2.19}$$

$${}_kH_n^{(\alpha)}(x,y;z) = n! \sum_{r=0}^{n} \frac{(-1)^r \sqrt{x^n} y^{kr+\alpha} H_{n-r}(z/2\sqrt{x})}{r!\sqrt{x^r}(n-r)!\Gamma(kr+\alpha+1)}, \tag{2.20}$$

$${}_kH_n^{(\alpha)}(x,y;z) = n! \sum_{r=0}^{n} \frac{(-1)^r y^{kr+\alpha} H_{n-r}(z,-x)}{r!(n-r)!\Gamma(kr+\alpha+1)}, \tag{2.21}$$

$${}_kH_n^{(\alpha)}(x,y;z) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r x^r L_{n-2r}(y,z;k,\alpha)}{r!(n-2r)!}. \tag{2.22}$$

From the series representation (2.3) and the fact that

$$\hat{D}_x^{-k} = \frac{x^k}{k!},$$

it follows that
(2.23)

$${}_kH_n^{(\alpha)}\left(\frac{(1-x^2)}{z}\hat{D}_z^{-1}, y; 2x\right) = n! \sum_{s=0}^{n} \sum_{r=0}^{[\frac{n-s}{2}]} \frac{(-1)^{s+r}(1-x^2)^r y^{ks+\alpha}(2x)^{n-s-2r}}{s!r!(n-s-2r)!\Gamma(ks+\alpha+1)},$$

from which we find the following link between the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ and the Legendre polynomials $P_n(x)$ [15, pp.167]:

$${}_kH_n^{(\alpha)}\left(\frac{(1-x^2)}{z}\hat{D}_z^{-1}, y; 2x\right) = n! \sum_{s=0}^{n} \frac{(-1)^s 2^{n-s} y^{ks+\alpha} P_{n-s}(x)}{r!(n-s)!\Gamma(ks+\alpha+1)}. \tag{2.24}$$

where

$$P_n(x) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(x^2-1)^k x^{n-2k}}{2^{2k}(k!)^2(n-2k)!}.$$

Obviously,

$$(2.25) \qquad {}_kH_n^{(\alpha)}\left(\frac{(1-x^2)}{z}\hat{D}_z^{-1}, 0; 2x\right) = 2^n P_n(x).$$

3. Operational methods and monomiality principle

First of all, since

$$\frac{\hat{D}_z^{2r} z^{n-s}}{(n-s)!} = \frac{z^{n-s-2r}}{(n-s-2r)!} = \frac{\hat{D}_z^{s} z^{n-2r}}{(n-2r)!} \quad \text{and} \quad \frac{\hat{D}_y^{-ks} y^{\alpha}}{\Gamma(\alpha+1)} = \frac{y^{ks+\alpha}}{\Gamma(\alpha+1)}$$

we infer, from the series representation (2.3) the identities

$$(3.1) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-x\frac{\partial^2}{\partial z^2}} \left\{ \frac{n! y^{\alpha} z^n}{\Gamma(kn+\alpha+1)} Z_n^{\alpha}(y/\sqrt[k]{z}; k) \right\},$$

$$(3.2) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-\frac{\partial}{\partial z}\hat{D}_y^{-k}} \left\{ \frac{x^{n/2} y^{\alpha}}{\Gamma(\alpha+1)} H_n(z/2\sqrt{x}) \right\},$$

$$(3.3) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-\frac{\partial}{\partial z}\hat{D}_y^{-k}} \left\{ \frac{y^{\alpha}}{\Gamma(\alpha+1)} H_n(z,-x) \right\},$$

$$(3.4) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-x\frac{\partial^2}{\partial z^2}} \left\{ L_n(y,z;k,\alpha) \right\}.$$

The operational re presentations in (3.1) to (3.4) yield an idea of how further properties for the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ can be established. In the forthcoming sections we will show how to exploit the exponential operators in (3.1) to (3.4) in wider context involving the derivation of generating functions and expansions for the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ from the corresponding known results of the classical Hermite polynomials, Konhauser polynomials, Hermite-Kampé de Fériet polynomials and psedu Laguerre polynomials.
Secondly, according to the identity

$$(3.5) \qquad \frac{\hat{D}_z^{r} z^{n-s-r}}{(n-s-r)!} = \frac{z^{n-s-2r}}{(n-s-2r)!},$$

we find from (2.3) that

$$(3.6)\qquad {}_kH_n^{(\alpha)}(x,y;z)=\sum_{s=0}^{n}\sum_{r}^{\left[\frac{n-s}{2}\right]}\frac{(-n)_{s+r}x^r}{s!r!}\hat{D}_z^r\hat{D}_y^{-ks}\left\{\frac{z^{n-s-r}y^\alpha}{\Gamma(\alpha+1)}\right\},$$

which further can be handled to get the symbolic relation:

$$(3.7)\qquad {}_kH_n^{(\alpha)}(x,y;z)=\left(1-x\hat{D}_zz^{-1}-z^{-1}\hat{D}_y^{-k}\right)^n\left\{\frac{z^ny^\alpha}{\Gamma(\alpha+1)}\right\},$$

or equivalently, in the more compact form

$$(3.8)\qquad {}_kH_n^{(\alpha)}(x,y;z)=\left(z-x\hat{D}_z-\hat{D}_y^{-k}\right)^n\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Similarly, the series representation (2.3) can be exploited to derive the following operational representations

$$(3.9)\qquad {}_kH_n^{(\alpha)}(x,y;z)=\left[1-x\hat{D}_zz^{-1}\left(1-\frac{1}{z}\hat{D}_y^{-k}\right)^{-1}\right]^n\left(1-\frac{1}{z}\hat{D}_y^{-k}\right)^n\left\{\frac{z^ny^\alpha}{\Gamma(\alpha+1)}\right\},$$

and

$$(3.10)\qquad {}_kH_n^{(\alpha)}(x,y;z)=\left[1-\frac{1}{z}\hat{D}_y^{-k}\left(1-x\hat{D}_zz^{-1}\right)^{-1}\right]^n\left(1-x\hat{D}_zz^{-1}\right)^n\left\{\frac{z^ny^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Next, in view of the relationships (2.4), (2.5), (2.11) and (2.14). Equations (3.9) and (3.10) can be further handled to get the following Rodrigues-type relations

$$(3.11)\qquad \frac{1}{\sqrt{x^n}}H_n(z/2\sqrt{x})=\left(1-x\hat{D}_zz^{-1}\right)^n\{z^n\},$$

$$(3.12)\qquad L_n(y,z;k,\alpha)=\left(1-\frac{1}{z}\hat{D}_y^{-k}\right)^n\left\{\frac{z^ny^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Further, according to the definition of Kampé de Fériet's double hypergeometric series $F_{E:G;F}^{A:B;C}$ (see [18, Eq. 1.3(28)]) and identity (3.5), we can easily derive the following explicit representation for the Hermite-Konhauser polynomials ${}_kH_n^{(\alpha)}(x,y;z)$:

$$(3.13)\qquad {}_kH_n^{(\alpha)}(x,y;z)=F_{0:0;k}^{1:0;0}\left[\begin{array}{l}-n:--;-----;\\ \\ --:--;\Delta(k;\alpha+1);\end{array}\; x\hat{D}_zz^{-1},z^{-1}\left(\tfrac{y}{k}\right)^k\right]\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\},$$

where $\Delta(k;\lambda)$ denotes the array of k parameters $\frac{\lambda}{k}, \frac{\lambda+1}{k}, ..., \frac{\lambda+k-1}{k}, k \geq 1$. Again, we can rewrite the series representation (2.3) in the form

$$(3.14) \qquad {}_kH_n^{(\alpha)}(x,y;z) = \sum_{s=0}^{n} \sum_{r}^{\left[\frac{n-s}{2}\right]} \frac{(-1)^r(-n)_{s+2r}x^r z^{n-s-2r}}{s!r!} \hat{D}_y^{-ks} \left\{ \frac{y^\alpha}{\Gamma(\alpha+1)} \right\},$$

which by exploiting the same procedure leading to (3.7) yields the following direct connection between the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ and the classical Hermite $H_n(x)$:

$$(3.15) \qquad {}_kH_n^{(\alpha)}(x,y;z) = H_n\left(\frac{z}{2\sqrt{x}}\left(1 - \frac{1}{z}D_y^{-k}\right)\right)\left\{\frac{\sqrt{x^n}y^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Furthermore, from the identity in (3.2) and the identity in (3.4) in conjunction with (3.11) and (3.12), we get the following new relations

$$(3.16) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-\frac{\partial}{\partial z}\hat{D}_y^{-k}}\left(1 - x\hat{D}_z z^{-1}\right)^n \left\{\frac{z^n y^\alpha}{\Gamma(\alpha+1)}\right\},$$

$$(3.17) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-\frac{\partial^2}{\partial z^2}}\left(1 - \frac{1}{z}\hat{D}_y^{-k}\right)^n \left\{\frac{z^n y^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Alternatively, by combining the identity in (3.1) and (3.4), we find that

$$(3.18) \qquad {}_kH_n^{(\alpha)}(x,y;z) = e^{-\frac{\partial}{\partial z}\hat{D}_y^{-k} - x\frac{\partial^2}{\partial z^2}}\left\{\frac{z^n y^\alpha}{\Gamma(\alpha+1)}\right\}.$$

At this point let us stress that, the schema suggested in this section can be applied to find other operational relations connecting the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ with other polynomials presented in section 1. For instance, for the polynomials ${}_1L_{n,\alpha}(y,z)$ defined by (1.1) we find the following operational formulas

$$(3.19) \qquad {}_kH_n^{(\alpha)}(x,y;z) = (-1)^\alpha e^{-\frac{\partial^2}{\partial z^2}}\hat{D}_y^{k\alpha}\left\{{}_1L_{n,\alpha}(-\hat{D}_y^{-k}y,z)\right\},$$

$$(3.20) \qquad {}_1L_{n,\alpha}(y,z) = \left(1 - \frac{1}{z}\hat{D}_y^{-1}\right)^n \left\{\frac{z^n y^\alpha}{\Gamma(\alpha+1)}\right\}.$$

The two series and three variables Hermite-Konhauser polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ are quasi-monomials under the action of the multiplicative operator

$$(3.21) \qquad \hat{M} = z - 2x\hat{D}_z - \hat{D}_y^{-k},$$

and the derivatives operators

$$(3.22) \qquad \hat{P}_1 = -\frac{1}{k}y^{\alpha-k+1}\hat{D}_y y^{k-\alpha}\hat{D}_y^k,$$

$$\hat{P}_2 = \frac{\partial}{\partial z}, \tag{3.23}$$

$$\hat{P}_3 = -\hat{D}_z^{-1}\frac{\partial}{\partial x}. \tag{3.24}$$

According to the quasi-monomiality properties, we have

$$\hat{M}_k H_n^{(\alpha)}(x,y;z) = {}_kH_{n+1}^{(\alpha)}(x,y;z), \tag{3.25a}$$

$$\hat{P}_{1k} H_n^{(\alpha)}(x,y;z) = n_k H_{n-1}^{(\alpha)}(x,y;z), \tag{3.25b}$$

$$\hat{P}_{2k} H_n^{(\alpha)}(x,y;z) = n_k H_{n-1}^{(\alpha)}(x,y;z), \tag{3.25c}$$

$$\hat{P}_{3k} H_n^{(\alpha)}(x,y;z) = n_k H_{n-1}^{(\alpha)}(x,y;z), \tag{3.25d}$$

$$\frac{1}{3}\left(\hat{P}_1 + \hat{P}_2 + \hat{P}_3\right){}_kH_n^{(\alpha)}(x,y;z) = n_k H_{n-1}^{(\alpha)}(x,y;z). \tag{3.25e}$$

Therefore, the identities

$$\hat{P}_1\hat{M}_k H_n^{(\alpha)}(x,y;z) = (n+1)_k H_n^{(\alpha)}(x,y;z), \tag{3.26}$$

$$\hat{M}\hat{P}_{2k} H_n^{(\alpha)}(x,y;z) = n_k H_n^{(\alpha)}(x,y;z), \tag{3.27}$$

$$\hat{M}\hat{P}_{3k} H_n^{(\alpha)}(x,y;z) = n_k H_n^{(\alpha)}(x,y;z). \tag{3.28}$$

in differential forms give us

$$\left[y\left(2x\frac{\partial}{\partial z} - z\right)\hat{D}_y^{k+1} + (k-\alpha)\left(2x\frac{\partial}{\partial z} - z\right)\hat{D}_y^k + y\hat{D}_y - (\alpha + kn)\right]$$

$${}_kH_n^{(\alpha)}(x,y;z) = 0 \tag{3.29}$$

$$\left[\left(2x\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z} + n\right)\hat{D}_y^k + \frac{\partial}{\partial z}\right]{}_kH_n^{(\alpha)}(x,y;z) = 0 \tag{3.30}$$

$$\left[\left(z\frac{\partial^2}{\partial x \partial z} + 2\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial x}\frac{\partial^2}{\partial z^2} + n\frac{\partial^2}{\partial z^2}\right)\hat{D}_y^k - \frac{\partial^2}{\partial x \partial z}\right]{}_kH_n^{(\alpha)}(x,y;z) = 0. \tag{3.31}$$

It can also be easily checked that ${}_kH_n^{(\alpha)}(x,y;z)$ are the natural solution of

$$\frac{\partial^2}{\partial z^2}{}_kH_n^{(\alpha)}(x,y;z)+\frac{\partial}{\partial x}{}_kH_n^{(\alpha)}(x,y;z)=0. \tag{3.32}$$

It is important to note that equations (3.22) and (3.32) allow us to establish another multiplicative operator $\hat{M}^*$ of the form

$$\hat{M}^*=z+2x\frac{\partial}{\partial x}\hat{D}_z^{-1}-\hat{D}_y^{-k}, \tag{3.33}$$

together with the property

$$\hat{M}^*{}_kH_n^{(\alpha)}(x,y;z)={}_kH_{n+1}^{(\alpha)}(x,y;z).$$

Moreover, the derivative operators in (3.22) to (3.24) can be further handled to get the new differential relations

$$\frac{\partial}{\partial z}{}_kH_n^{(\alpha)}(x,y;z)=-\frac{1}{k}y^{\alpha-k+1}\frac{\partial}{\partial y}y^{k-\alpha}\frac{\partial^k}{\partial y^k}{}_kH_n^{(\alpha)}(x,y;z), \tag{3.34}$$

$$\frac{\partial}{\partial z}{}_kH_n^{(\alpha)}(x,y;z)=-\hat{D}_z^{-1}\frac{\partial}{\partial x}{}_kH_n^{(\alpha)}(x,y;z), \tag{3.35}$$

$$-\hat{D}_z^{-1}\frac{\partial}{\partial x}{}_kH_n^{(\alpha)}(x,y;z)=\frac{1}{k}y^{\alpha-k+1}\frac{\partial}{\partial y}y^{k-\alpha}\frac{\partial^k}{\partial y^k}{}_kH_n^{(\alpha)}(x,y;z). \tag{3.36}$$

Next, regarding the Lie bracket $[\ \ ,\ \ ]$ defined by $[A,B]=AB-BA$, we led to

$$[\hat{P}_1,\hat{M}^*]{}_kH_n^{(\alpha)}(x,y;z)=[\hat{P}_1,\hat{M}]{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x,y;z), \tag{3.37a}$$

$$[\hat{P}_2,\hat{M}^*]{}_kH_n^{(\alpha)}(x,y;z)=[\hat{P}_2,\hat{M}]{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x,y;z), \tag{3.37b}$$

$$[\hat{P}_3,\hat{M}^*]{}_kH_n^{(\alpha)}(x,y;z)=[\hat{P}_3,\hat{M}]{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x,y;z). \tag{3.37c}$$

From the lowering operators $\hat{P}_1,\hat{P}_2$ and $\hat{P}_3$, we can define operators playing the role of the inverse operators $\hat{P}_1^{-1},\hat{P}_2^{-1}$ and $\hat{P}_3^{-1}$ (see [4, Equation (15)]). Thus, we get

$$\hat{P}_1^{-1}=\hat{D}_z^{-1}, \tag{3.38}$$

$$\hat{P}_2^{-1}=-ky^{\alpha}\hat{D}_y^{-1}y^{-(\alpha+1)}\hat{D}_y^{-k}, \tag{3.39}$$

$$\hat{P}_3^{-1}=-\hat{D}_x^{-1}\hat{D}_z, \tag{3.40}$$

and they satisfy

$$\hat{P}_1^{-1}{}_kH_n^{(\alpha)}(x,y;z)=\hat{P}_2^{-1}{}_kH_n^{(\alpha)}(x,y;z)=\hat{P}_3^{-1}{}_kH_n^{(\alpha)}(x,y;z)=\frac{{}_kH_{n+1}^{(\alpha)}(x,y;z)}{(n+1)}. \tag{3.41}$$

Clearly, we have

$$\begin{aligned}\hat{P}_1\hat{P}_1^{-1}\left\{{}_kH_n^{(\alpha)}(x,y;z)\right\} &= \hat{P}_2\hat{P}_2^{-1}\left\{{}_kH_n^{(\alpha)}(x,y;z)\right\}\\ &= \hat{P}_3\hat{P}_3^{-1}\left\{{}_kH_n^{(\alpha)}(x,y;z)\right\}\\ &= {}_kH_n^{(\alpha)}(x,y;z).\end{aligned} \tag{3.42}$$

Also, from definition (2.3), we find that

$$\frac{\partial}{\partial x}{}_kH_n^{(\alpha)}(x,y;z) = -n(n-1){}_kH_{n-2}^{(\alpha)}(x,y;z), \tag{3.43a}$$

$$\frac{\partial}{\partial y}{}_kH_n^{(\alpha)}(x,y;z) = {}_kH_n^{(\alpha-1)}(x,y;z), \tag{3.43b}$$

$$\frac{\partial}{\partial z}{}_kH_n^{(\alpha)}(x,y;z) = n{}_kH_{n-1}^{(\alpha)}(x,y;z). \tag{3.43c}$$

In general, we have

$$\hat{D}_x^m{}_kH_n^{(\alpha)}(x,y;z) = (-1)^m n(n-1)\cdots(n-2m+1){}_kH_{n-2m}^{(\alpha)}(x,y;z), \tag{3.44a}$$

$$\hat{D}_y^m{}_kH_n^{(\alpha)}(x,y;z) = {}_kH_n^{(\alpha-m)}(x,y;z), \tag{3.44b}$$

$$\hat{D}_z^m{}_kH_n^{(\alpha)}(x,y;z) = n(n-1)\cdots(n-m+1){}_kH_{n-m}^{(\alpha)}(x,y;z). \tag{3.44c}$$

Similarly, we can show that

$$\hat{D}_x^{-m}{}_kH_n^{(\alpha)}(x,y;z) = \frac{1}{(n+2m)(n+2m-1)\cdots(n+1)}{}_kH_{n+2m}^{(\alpha)}(x,y;z), \tag{3.45a}$$

$$\hat{D}_y^{-m}{}_kH_n^{(\alpha)}(x,y;z) = {}_kH_n^{(\alpha+m)}(x,y;z), \tag{3.45b}$$

$$\hat{D}_z^{-m}{}_kH_n^{(\alpha)}(x,y;z) = \frac{1}{(n+m)(n+m-1)\cdots(n+1)}{}_kH_{n+m}^{(\alpha)}(x,y;z). \tag{3.45c}$$

4. Generating functions via operational identities

In this section we show how readily new generating functions for the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$ can be derived from the operational re presentations of the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$. First, in the identity (3.2) multiply throughout by $\frac{t^n}{n!}$, sum and then employ the well-known generating function [15]

$$exp\left[2zt-t^2\right]=\sum_{n=0}^{\infty}H_n(z)\frac{t^n}{n!}, \tag{4.1}$$

to get

$$e^{-\frac{\partial}{\partial z}\hat{D}_y^{-k}}e^{(zt-xt^2)}\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z)\frac{t^n}{n!}. \tag{4.2}$$

In the same manner, from the operational identity in (3.4),(see (2.17)) and (3.8) one can derive the following generating functions

$$y^\alpha e^{-x\frac{\partial^2}{\partial z^2}}e^{zt}C_\alpha(y^kt;k)=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z)\frac{t^n}{n!}, \tag{4.3}$$

and

$$e^{\left(zt-xt\frac{\partial}{\partial z}-t\hat{D}_y^{-k}\right)}\left\{\frac{z^ny^\alpha}{\Gamma(\alpha+1)}\right\}=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z)\frac{t^n}{n!}, \tag{4.4}$$

respectively. Again, by starting from equation (3.8) multiplying throughout by t^n and exploiting the previous outlined method, we can show that

$$\begin{aligned}\frac{y^\alpha}{\Gamma(\alpha+1)(1-zt+xt\hat{D}_z)}{}_1F_k\left[1;\Delta(k;\alpha+1);\frac{-(y/k)^kt}{(1-zt+xt\hat{D}_z)}\right]\\=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z)t^n.\end{aligned} \tag{4.5}$$

The previously outlined procedure offers a useful tool for the derivation of other families of generating functions for the polynomials ${}_kH_n^{(\alpha)}(x,y;z)$. For instance, let us consider the generating relation

$$f(x,y,w,u;z,v|t)=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z){}_pH_n^{(\beta)}(w,u;v)\frac{t^n}{n!}, \tag{4.6}$$

which according to Equations (3.18) and (4.2) yields the following bilinear generating function

$$\begin{aligned}exp\left[-\left(\frac{\partial}{\partial z}\hat{D}_y^{-k}+x\frac{\partial^2}{\partial z^2}\right)\right]\exp\left[-\left(\frac{\partial}{\partial v}\hat{D}_u^{-p}-vt+t^2w\right)\right]\left\{\frac{z^ny^\alpha u^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}\\=\sum_{n=0}^{\infty}{}_kH_n^{(\alpha)}(x,y;z){}_pH_n^{(\beta)}(w,u;v)\frac{t^n}{n!}.\end{aligned} \tag{4.7}$$

In [2] the following Laguerre-Konhauser polynomials have been introduced

$$(4.8)\qquad {}_kL_n^{(\alpha,\beta)}(x,y)=n!\sum_s^n\sum_r^{n-s}\frac{(-1)^{s+r}x^{\alpha+r}y^{\beta+ks}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(ks+\beta+1)}$$

together with the operational identity

$$(4.9)\qquad \left(1-\hat{D}_x^{-1}-\hat{D}_y^{-k}\right)^n\left\{\frac{x^\alpha y^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}={}_kL_n^{(\alpha,\beta)}(x,y).$$

Let us consider the generating relation

$$(4.10)\qquad f(x,y,w,u;z,v|t)=\sum_{n=0}^\infty {}_kH_n^{(\alpha)}(x,y;z)_pL_n^{(\beta,\gamma)}(w,u)\frac{t^n}{n!}.$$

Now, directly from (3.8) and (4.9) by employing the previously outlined method leading to the bilinear generating function, we obtain from (4.10) the following bilateral generating function

$$(4.11)\quad exp\left[t\left(z-x\hat{D}_z-\hat{D}_y^{-k}\right)\left(1-\hat{D}_w^{-1}-\hat{D}_u^{-p}\right)\right]\left\{\frac{y^\alpha w^\beta u^\gamma}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}\right\}$$
$$=\sum_{n=0}^\infty {}_kH_n^{(\alpha)}(x,y;z)_pL_n^{(\alpha,\beta)}(w,u)\frac{t^n}{n!}$$

5. Expansions

Let $\hat{N}=z-x\hat{D}_z-\hat{D}_y^{-k}$, then from (3.8) we can state that

$$(5.1)\qquad [z-\hat{N}]^n\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}=\sum_{s=0}^n(-1)^s\binom{n}{s}{}_kH_s^{(\alpha)}(x,y;z)z^{n-s}.$$

Alternatively, from (3.21) by applying the same method leading to (5.1), we find that

$$(5.2)\qquad [\hat{M}]^n\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}=\sum_{s=0}^n(-1)^s\binom{n}{s}{}_kH_{n-s}^{(\alpha)}(x,y;z)x^s\frac{\partial^s}{\partial z^s}.$$

Next, consider the expression:

$$(5.3)\qquad f(x,y;z)=\left[\hat{M}+x\frac{\partial}{\partial z}\right]^n\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\},$$

which in view of (3.8) and (5.2), yields the expansion

$$\left[\hat{M}+x\frac{\partial}{\partial z}\right]^n\left\{\frac{y^\alpha}{\Gamma(\alpha+1)}\right\}={}_kH_n^{(\alpha)}(x,y;z) \tag{5.4}$$

$$=\sum_{s=0}^{n}\sum_{r=0}^{s}(-1)^r\binom{n}{s}\binom{s}{r}{}_kH_{s-r}^{(\alpha)}(x,y;z)\left(x\hat{D}_z\right)^{n-s+r}$$

Further, consider the known summation formula [17,p.248(3.28)]:

$$\frac{Z_n^\alpha(u;k)}{\Gamma(kn+\alpha+1)}=\left(\frac{u}{w}\right)^{kn}\sum_{s=0}^{n}\frac{Z_{n-s}^\alpha(w;k)}{s!\Gamma(kn-ks+\alpha+1)}\left[\left(\frac{u}{w}\right)^k-1\right]^s. \tag{5.5}$$

Upon replacing u and w in (5.5) by $\frac{u}{\sqrt[k]{z}}$ and $\frac{w}{\sqrt[k]{z}}$ respectively, multiplying both sides by $(n!u^\alpha w^\alpha z^n)$ and then applying the operational identity in (3.1), we obtain the following explicit expansion

$$ {}_kH_n^{(\alpha)}(x,y;z)=\frac{u^\alpha n!}{\Gamma(\alpha+1)}\left(\frac{u}{w}\right)^{kn}\sum_{s=0}^{n}\frac{1}{s!} \tag{5.6}$$

$$F^{1:0;1}_{0:0;k+1}\left[\begin{array}{c} -n:-;-n+s; \\ \\ -:-;-n,\Delta(k;\alpha+1); \end{array} x\hat{D}_z z^{-1}, \left(\frac{w}{k}\right)^k z^{-1}\right]\left(\left(\frac{u}{w}\right)^k-1\right)^s.$$

Moreover, the definition (2.3), in view of Taylor's formulas

$$\sum_{m=0}^{\infty}\frac{z^m}{m!}\hat{D}_x^m f(x)=f(x+z)$$

and

$$\sum_{m=0}^{\infty}\frac{[(z-1)x]^m}{m!}\hat{D}_x^m f(x)=f(xz),$$

yields the following interesting addition and multiplication formulas:

$$e^{w\hat{D}_x}\,{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x+w,y;z), \tag{5.7a}$$

$$e^{w\hat{D}_y}\,{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x,y+w;z), \tag{5.7b}$$

$$e^{w\hat{D}_z}\,{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(x,y;z+w), \tag{5.7c}$$

$$e^{(w-1)x\hat{D}_x}\,{}_kH_n^{(\alpha)}(x,y;z)={}_kH_n^{(\alpha)}(xw,y;z), \tag{5.8a}$$

$$e^{(w-1)y\hat{D}_y}\,{}_kH_n^{(\alpha)}(x,y;z) = {}_kH_n^{(\alpha)}(x,yw;z), \tag{5.8b}$$

$$e^{(w-1)z\hat{D}_z}\,{}_kH_n^{(\alpha)}(x,y;z) = {}_kH_n^{(\alpha)}(x,y;zw). \tag{5.8c}$$

Similarly, the use of the inverse operator $\hat{D}^{-1}$ allows to conclude

$$\sum_{m=0}^{\infty} \frac{w^m}{m!} \hat{D}_y^{-m} \left\{ {}_kH_n^{(\alpha-2m)}(x,y;z) \right\} = {}_kH_n^{(\alpha)}(x,y+w;z), \tag{5.9a}$$

$$\sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2m} w^m}{m!} \hat{D}_z^{-m} \left\{ {}_kH_{n-2m}^{(\alpha)}(x,y;z) \right\} = {}_kH_n^{(\alpha)}(x,y;z+w), \tag{5.9b}$$

$$\sum_{m=0}^{\infty} \frac{[(1-w)y]^m}{m!} \hat{D}_y^{-m} \left\{ {}_kH_n^{(\alpha-2m)}(x,y;z) \right\} = {}_kH_n^{(\alpha)}(x,yw;z), \tag{5.10a}$$

$$\sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2m}[(1-w)y]^m}{m!} \hat{D}_z^{-m} \left\{ {}_kH_{n-2m}^{(\alpha)}(x,y;z) \right\} = {}_kH_n^{(\alpha)}(x,y;zw). \tag{5.10b}$$

For $w = -x (w = -y, w = -z)$, the results (5.7a), (5.7b), (5.7c), (5.9a) and (5.9b) would reduce immediately to curious results since $x(y,z)$, does not appear on the right-hand side of equations (5.7a), (5.7b), (5.7c), (5.9a) and (5.9b). Finally, let us consider the expression

$$\Omega(x,y,z) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \hat{D}_z^m \hat{D}_y^{-km} \left\{ H_{n-m}(z,-x) \times L_{n-m}(y,z;k,\alpha) \frac{x^m}{z^n} \right\}. \tag{5.11}$$

In view of the identity (see (1.6))

$$H_n(z,x) = \left(1 + x\hat{D}_z z^{-1}\right)^n \{z^n\}$$

and upon using (3.12) one obtains by routine calculations

$$\Omega(x,y,z) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \sum_{s=0}^{n-m} \sum_{r=0}^{n-m} \frac{(-n+m)_s(-n+m)_r}{s!r!}$$

$$\hat{D}_z^{m+r} \hat{D}_y^{-k(m+s)} \left\{ \frac{z^{n-s-r-2m} x^{m+r} y^\alpha}{\Gamma(\alpha+1)} \right\}.$$

On letting $s \mapsto s-m$ and $r \mapsto r-m$, employing the Gaussian theorem

$${}_2F_1[a,b;c;1] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, Re(c-a-b) > 0, Re(c) > 0,$$

and simplify, we led to the desired result

$$(5.12)\quad {}_kH_n^{(\alpha)}(x,y;z)=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\hat{D}_z^m\hat{D}_y^{-km}\left\{H_{n-m}(z,-x)\times L_{n-m}(y,z;k,\alpha)\frac{x^m}{z^n}\right\}.$$

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