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A Study of Generalized Weyl Differintegral Operator Associated with a General Class of Polynomials and the Multivariable H-function

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Abstract. In the present paper, we obtain a new formula for the generalized Weyl differintegral operator in a compact form avoiding the occurrence of infinite series and thus making it useful in applications. Our findings provide interesting generalizations and unifications of the results given by several authors and lying scattered in the literature.

1. Introduction

Generalized differintegral operators

We shall define the generalized Weyl differintegral operator of a function $f(x)$ [10, p. 529,eq.(2.2)] (see also [5-8, 16]) as follows :

Let α, β and γ be complex numbers. The generalized Weyl fractional integral (Re(α) > 0) and derivative (Re(α) < 0) of a function $f(x)$ defined on $(0,\infty)$ is given by

$$(1.1)\quad J_{x,\infty}^{\alpha,\beta,\gamma} f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_x^\infty (t-x)^{\alpha-1} F\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{x}{t}\right) t^{-\alpha-\beta} f(t)\,dt, & (Re(\alpha) > 0), \\ (-1)^q \frac{d^q}{dx^q} J_{x,\infty}^{\alpha+q,\beta-q,\gamma} f(x), & (Re(\alpha) \le 0, 0 < Re(\alpha)+q \le 1, q=1,2,3,\cdots). \end{cases}$$

where F stands for the well known Gauss hypergeometric function.

The operator J includes both the Weyl and the Erdélyi-Kober fractional operators as follows:
The Weyl operator:

$$(1.2)\quad W_{x,\infty}^{\alpha} f(x) = \begin{cases} J_{x,\infty}^{\alpha,-\alpha,\gamma} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^\infty (t-x)^{\alpha-1} f(t)\,dt, & (Re(\alpha) > 0), \\ (-1)^q \frac{d^q}{dx^q} W_{x,\infty}^{\alpha+q} f(x), & (Re(\alpha) \le 0, 0 < Re(\alpha)+q \le 1, q=1,2,3,\cdots). \end{cases}$$

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The Erdélyi-Kober operator:

$$(1.3)\quad K_{x,\infty}^{\alpha,\gamma} f(x) = J_{x,\infty}^{\alpha,0,\gamma} f(x) = \frac{x^{\gamma}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\gamma} f(t)\, dt, \ (Re(\alpha) > 0).$$

The established results in the present paper can be reduced for both the Weyl and the Erdélyi-Kober operators.

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [11, p. 1, eq.(1)]:

$$(1.4)\qquad S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \ n = 0, 1, 2, \cdots,$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n, k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields as number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [17, pp. 158-161].

The H-function of r complex variables $z_1, \cdots, z_r$ was introduced by Srivastava and Panda [15]. We shall define and represent it in the following form [14, p. 251, eq.(C.1)]:

$$(1.5)\quad H[z_1, \cdots, z_r]$$

$$= H_{P,Q:P',Q';\cdots;P^{(r)},Q^{(r)}}^{0,N:M',N';\cdots;M^{(r)},N^{(r)}} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} \left(a_j; \alpha'_j, \cdots, \alpha_j^{(r)}\right)_{1,P} : (c'_j, \gamma'_j)_{1,P'}; \cdots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,P^{(r)}} \\ \\ \left(b_j; \beta'_j, \cdots, \beta_j^{(r)}\right)_{1,Q} : (d'_j, \delta'_j)_{1,Q'}; \cdots; \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,Q^{(r)}} \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(\xi_1) \cdots \phi_r(\xi_r)\, \psi(\xi_1, \cdots, \xi_r)\, z_1^{\xi_1} \cdots z_r^{\xi_r} d\xi_1 \cdots d\xi_r$$

where $\omega = \sqrt{-1}$,

$$(1.6)\quad \phi_i(\xi_i) = \frac{\prod_{j=1}^{M^{(i)}} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \xi_i\right) \prod_{j=1}^{N^{(i)}} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i\right)}{\prod_{j=M^{(i)}+1}^{Q^{(i)}} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i\right) \prod_{j=N^{(i)}+1}^{P^{(i)}} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} \xi_i\right)}, \ \forall\, i \in \{1, \cdots, r\}$$

and

$$(1.7)\qquad \psi(\xi_1, \cdots, \xi_r) = \frac{\prod_{j=1}^{N} \Gamma\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right)}{\prod_{j=N+1}^{P} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right) \prod_{j=1}^{Q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i\right)}.$$

The nature of contours $L_1,\cdots,\ L_r$ in (1.5), the various special cases and other details of the above function can be found in the book referred to above. It may be remarked here that all the Greek letters occurring in the left-hand side of (1.5) are assumed to be positive real numbers for standardization purposes; the definition of this function will, however, be meaningful even if some of these quantities are zero. Again, it is assumed that the various multivariable H-functions occurring in the paper always satisfy their appropriate conditions of convergence [14, pp. 252-253, eqs. (C.4) – (C.6)].

2. Main result

We establish here the following formula for the generalized Weyl differintegral operator given by equation (1.1), involving the product of a general class of polynomials and the multivariable H-function

$$(2.1)\qquad J_{x,\infty}^{\alpha,\beta,\gamma}\Big\{x^{\rho}\left(x^{-t_1}+\alpha_1\right)^{\sigma} S_n^m\left[a\,x^{\lambda}\left(x^{-t_1}+\alpha_1\right)^{\eta}\right]$$

$$H\left[z_1x^{-u_1}\left(x^{-t_1}+\alpha_1\right)^{-v_1},\cdots,z_rx^{-u_r}\left(x^{-t_1}+\alpha_1\right)^{-v_r}\right]\Big\}$$

$$=\alpha_1^{\sigma}x^{\rho-\beta}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}\,A_{n,k}\,a^k\alpha_1^{\eta k}x^{\lambda k}H^{0,N+3:M',N';\cdots;M^{(r)},N^{(r)};1,0}_{P+3,Q+3:P',Q';\cdots;P^{(r)},Q^{(r)};0,1}$$

$$\left[\begin{matrix} z_1\alpha_1^{-v_1}x^{-u_1}\\ \vdots \\ z_r\alpha_1^{-v_r}x^{-u_r}\\ \alpha_1^{-1}x^{-t_1}\end{matrix}\left|\begin{matrix}(1-\beta+\rho+\lambda k;u_1,\cdots,u_r,t_1),\\ \\ \\ (1+\rho+\lambda k;u_1,\cdots,u_r,t_1),\end{matrix}\right.\right.$$

$$(1-\gamma+\rho+\lambda k;u_1,\cdots,u_r,t_1),(1+\sigma+\eta k;v_1,\cdots,v_r,1),$$
$$(1-\alpha-\beta-\gamma+\rho+\lambda k;u_1,\cdots,u_r,t_1),(1+\sigma+\eta k;v_1,\cdots,v_r,0),$$

$$\left.\begin{matrix}\left(a_j;\alpha_j',\cdots,\alpha_j^{(r)},0\right)_{1,P}:\left(c_j',\gamma_j'\right)_{1,P'};\cdots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}};\text{---}\\ \\ \left(b_j;\beta_j',\cdots,\beta_j^{(r)},0\right)_{1,Q}:\left(d_j',\delta_j'\right)_{1,Q'};\cdots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}};(0,1)\end{matrix}\right]$$

provided that

(i) Re $(\alpha)>0$; the quantities $t_1,\lambda,\eta,u_1,v_1,\cdots,u_r,v_r$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning)

(ii) $\mathrm{Re}(\gamma-\beta)>0$

(iii) $\mathrm{Re}(\beta-\rho)+\sum_{i=1}^{r}u_i\min_{1\le j\le M^{(i)}}\left[Re\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]>0$.

Proof of (2.1). To prove the formula (2.1), we first express the general class of polynomials occurring on its left-hand side in the series form given by (1.4), replace the multivariable H-function occurring therein by its well known Mellin-Barnes contour integral given by (1.5), interchange the order of summation, $(\xi_1,\ldots,\xi_r)$ – integrals and taking the operator $J_{x,\infty}^{\alpha,\beta,\gamma}$ inside (which is permissible under the conditions stated with (2.1)) and make a little simplification. Next, we express the term $(x^{-t_1}+\alpha_1)^{\sigma+\eta k-v_1\xi_1-\cdots-v_r\xi_r}$ so obtained in terms of Mellin-Barnes contour integral [14, p.18, eq.(2.6.4); p.10, eq.(2.1.1)]. Now, interchange the order of ξ_{r+1}– and $(\xi_1,\ldots,\xi_r)$ – integrals (which is also permissible under the conditions stated with (2.1)), and evaluate the t-integral thus obtained by using the known formula [9, p. 16, Lemma 2]

$$J_{x,\infty}^{\alpha,\beta,\gamma}x^{\mu}=\frac{\Gamma(\beta-\mu)\ \Gamma(\gamma-\mu)}{\Gamma(-\mu)\ \Gamma(\alpha+\beta+\gamma-\mu)}x^{\mu-\beta}, \tag{2.2}$$

where Re $(\alpha)>0$, Re$(\mu)<$min[Re(β), Re(γ)] or Re$(\alpha)\leq 0$, $0<$Re$(\alpha)+q\leq 1$ and Re $(\mu)<$ min[Re$(\beta)-q$, Re(γ)] for some positive integer q as occurring in (1.1).

On reinterpreting the multiple Mellin-Barnes contour integral so obtained in terms of the H-function of $r+1$ variables, we easily arrive at the desired formula (2.1) after a little simplification.

3. Special cases and applications

The formula (2.1) established here is unified in nature and acts as the key formula. Thus the general class of polynomials involved in it reduce to a large number of polynomials listed by Srivastava and Singh [17, pp. 158-161], and so from the formula (2.1) we can further obtain various formulae involving a number of simpler polynomials. Again the multivariable H-function occurring in this formula can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of E,F,G and H-functions of one, two or more variables. For example, if $N=P=Q=M=0$, the multivariable H-function occurring in the left-hand side of our main result would reduce immediately to the product of r-different H-functions of Fox [2], thus the table listing various special cases of the H-function [4, pp. 145-159] can be used to derive from our main formula a number of other formulae involving any of these special functions.

We record below two special cases of (2.1) that are believed to be new

(i) On reducing the general class of polynomials S_n^m occurring in the left-hand side of (2.1) to the Hermite polynomials [17, p. 158, eq.(1.4)] and the multivariable H-function occurring therein to the product of r-different H-functions of Fox [2], we get the following formula after a little simplification

$$J_{x,\infty}^{\alpha,\beta,\gamma}\left\{x^{\rho+\frac{n}{2}}\left(x^{-t_1}+\alpha_1\right)^{\sigma}H_n\left(\frac{1}{2\sqrt{x}}\right)\prod_{i=1}^{r}H_{P^{(i)},Q^{(i)}}^{M^{(i)},N^{(i)}}\left[z_i x^{-u_i}\left|\begin{matrix}\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,P^{(i)}}\\ \left(d_j^{(i)},\delta_j^{(i)}\right)_{1,Q^{(i)}}\end{matrix}\right.\right]\right\} \tag{3.1}$$

$$= \frac{\alpha_1^{\sigma} x^{\rho-\beta}}{\Gamma(-\sigma)} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!} (-1)^k x^k H^{0,2:M',N';...;M^{(r)},N^{(r)};1,1}_{2,2:P',Q';...;P^{(r)},Q^{(r)};1,1}$$

$$\left[\begin{array}{l|l} z_1 x^{-u_1} & (1-\beta+\rho+k;u_1,...,u_r,t_1), \\ \vdots & \\ z_r x^{-u_r} & \\ \alpha_1^{-1} x^{-t_1} & (1+\rho+k;u_1,...,u_r,t_1), \end{array}\right.$$

$$\left.\begin{array}{l} (1-\gamma+\rho+k;u_1,...,u_r,t_1) : \left(c'_j,\gamma'_j\right)_{1,P'};...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}};(1+\sigma,1) \\ (1-\alpha-\beta-\gamma+\rho+k;u_1,...,u_r,t_1) : \left(d'_j,\delta'_j\right)_{1,Q'};...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}};(0,1) \end{array}\right].$$

The conditions of validity of (3.1) can be easily obtained from those of (2.1).
(ii) On reducing the general class of polynomials S_n^m occurring in the left-hand side of (2.1) to the Jacobi polynomials [17, p. 159, eq.(1.6)] and the multivariable H-function occurring therein to the product of r-different modified Bessel functions of the third kind[14, p. 18, eq.(2.6.6)], we arrive at the following result after a little simplification

$$(3.2)\quad J_{x,\infty}^{\alpha,\beta,\gamma}\left\{x^{\rho-\frac{r}{2}}\left(x^{-t_1}+\alpha_1\right)^{\sigma} P_n^{(\delta,\tau)}[1-2x]\prod_{i=1}^{r} K_{v_i}\left[z_i x^{-1}\right]\right\}$$

$$=2^{-\frac{3r}{2}}\prod_{i=1}^{r} z_i^{-\frac{1}{2}}\frac{\alpha_1^{\sigma} x^{\rho-\beta}}{\Gamma(-\sigma)}\sum_{k=0}^{n}\frac{(-n)_k}{k!}\binom{n+\delta}{n}\frac{(\delta+\tau+n+1)_k}{(\delta+1)_k} x^k H^{0,2:2,0;...2,0;1,1}_{2,2:2,0;...2,0;1,1}\left[\begin{array}{l} z_1 x^{-1}/2 \\ \vdots \\ z_r x^{-1}/2 \\ \alpha_1^{-1} x^{-t_1} \end{array}\right|$$

$$\left.\begin{array}{l} (1-\beta+\rho+k;1,...,1,t_1),(1-\gamma+\rho+k;1,...,1,t_1) : \_\_;...;\_\_;(1+\sigma,1) \\ (1+\rho+k;1,..,1,t_1),(1-\alpha-\beta-\gamma+\rho+k;1,..,1,t_1) : \left(\frac{1}{4}\pm\frac{\nu_1}{2},\frac{1}{2}\right);..;\left(\frac{1}{4}\pm\frac{\nu_r}{2},\frac{1}{2}\right);(0,1) \end{array}\right].$$

The conditions of validity of (3.2) can be easily obtained from those of (2.1).

On reducing the general class of polynomials S_n^m occurring in the left-hand side of (2.1) to the Konhauser biorthogonal polynomials [13, p. 225, eq.(3.23); 3, p. 304, eq.(5)], we can obtain a very useful formula involving this polynomial. On simplification the Konhauser biorthogonal polynomials further reduces to the Laguerre polynomials. A comprehensive investigation on biorthogonal polynomials can be seen in the well known paper by Srivastava [12].

Further, we observe that if in the left-hand side of our main formula (2.1) we take $\sigma = 0$ and $n = 0$ (the polynomial S_0^m will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality), and also take $v_1 = v_2 = \cdots = v_r = 0$, our main result will reduce to a known formula given by Saigo and Raina [10, p.532, eq.(4.2)]. The results given by Saigo and Raina [9,p.19, Th. 2] and Banerjee and Choudhary [1, p 272, eq.(5)] will also become as simple special cases of our main formula (2.1).

Several other interesting and useful special cases of our main result (2.1) involving the product of a large variety of polynomials (which are special cases of S_n^m) and numerous simple special functions involving one or more variables (which are particular cases of the multivariable H-function) can also be obtained but we do not record them here for lack of space.

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