KYUNGPOOK Math. J. 50(2010), 165-175

The Fourth and Eighth Order Mock Theta Functions

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Abstract. In the paper we consider deemed three mock theta functions introduced by Hikami. We have given their alternative expressions in double summation analogous to Hecke type expansion. In proving we also give a new Bailey pair relative to a^2. I presume they will be helpful in getting fundamental transformations.

1. Introduction

In his last letter to Hardy [6, pp. 354-355] Ramanujan gave a list of seventeen functions which he called "mock theta functions". The results in this letter formed the basis of the study of these functions. The mock theta functions are not modular, but have a nice asymptotic behaviour when q is a root of unity. Ramanujan included in his letter four separate classes of mock theta functions, one class of third order, two of fifth order and one of seventh order. The systematic understanding and even the "order" are still missing.

Many deep results were obtained about the third order mock theta functions. This was possible because Watson was able to find representations of them from which he was able to study their behaviour under fundamental transformation of modular group [7]. For example,

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \tag{1.1}$$

is one of the third order functions. Watson showed that

$$f(q) \prod_{n=1}^{\infty} (1-q^n) = 1 + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{\frac{n(3n+1)}{2}}}{1+q^n}, \tag{1.2}$$

and then using the Poisson summation formula to the right hand side of (1.2) obtained the modular transformations of $f(q)$.

In [5] Hikami introduced the functions $D_6(q)$, $I_{12}(q)$ and $I_{13}(q)$ defined as

$$D_6(q) = \sum_{n=0}^{\infty} q^n \frac{(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}}, \tag{1.3}$$

Received July 30, 2009; accepted September 9, 2009.
2000 Mathematics Subject Classification: 33D15.
Key words and phrases: Mock theta functions, Basic hypergeometric series, Bailey pair.

(1.4) $$I_{12}(q) = \sum_{n=0}^{\infty} q^{2n} \frac{(-q;q^2)_n}{(q^{n+1};q)_{n+1}},$$

(1.5) $$I_{13}(q) = \sum_{n=0}^{\infty} q^{n} \frac{(-q;q^2)_n}{(q^{n+1};q)_{n+1}},$$

and conjectured that they are of fourth order, eighth order and eighth order respectively. Hikami also considered their behaviour outside the unit circle and gave their WRT invariants [5].

The object of this paper is to give counterparts of (1.2) for these fourth order and eighth order mock theta functions. This will be helpful in finding the transformation theory for these functions. These will be obtained by the method of Bailey pairs. We also find a new Bailey pair with respect to $a = q^2$.

We will be using the standard notation of q-Calculus :

$$(x)_n = (x;q)_n = \prod_{k=1}^{n}(1 - xq^{k-1}).$$

If $\mid q \mid < 1$, we let

$$(a, b, c, ...; q)_\infty = (a;q)_\infty, (b;q)_\infty, (c;q)_\infty, ...,$$

$$\left[{n \atop m}\right]_q = \frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}.$$

We have for $n > 0$

(1.6) $$\sum_{j=-n}^{n} q^{-j^2} = q^{-n^2} \sum_{j=0}^{2n} q^{j(2n-j)},$$

(1.7) $$\sum_{j=-(n-1)}^{n-1} q^{-j^2} = q^{-n^2} \sum_{j=-2n+1}^{-1} q^{j(-2n-j)},$$

and

(1.8) $$2\sum_{j=0}^{n} q^{-j^2-j} = q^{-n^2-n} \sum_{j=0}^{2n+1} q^{j(2n+1-j)},$$

(1.9) $$2\sum_{j=0}^{n-1} q^{-j^2-j} = q^{-n^2-n} \sum_{j=-2n}^{-1} q^{j(-2n-1-j)}.$$

2. Bailey lemma

Two sequences $\{\alpha_n\}$and $\{\beta_n\}$, $n \geq 0$, form a Bailey pair relative to a number a if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{2.1}$$

for all $n \geq 0$. Clearly, the $\alpha's$ are uniquely determined by the $\beta's$. In fact $\{\alpha_n\}$and $\{\beta_n\}$ form a Bailey pair if and only if

$$\alpha_n = (1-aq^{2n}) \sum_{j=0}^{n} \frac{(aq;q)_{n+j-1}}{(q;q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \beta_j \tag{2.2}$$

for all $n \geq 0$. See [2,(2.1), p. 70].

Corollary 2.1[2, Cor. 2.1, p. 70]**.** *If $\{\alpha_n\}$and $\{\beta_n\}$ form a Bailey pair relative to a, then*

$$\sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n} \tag{2.3}$$
$$= \frac{(aq)_\infty \left(\frac{aq}{\rho_1\rho_2}\right)_\infty}{\left(\frac{aq}{\rho_1}\right)_\infty \left(\frac{aq}{\rho_2}\right)_\infty} \sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n,$$

provided that both sums converge absolutely.

3. Bailey pair

We now give a Bailey pair $\{\alpha_n\}$, $\{\beta_n\}$ relative to $a = q^2$. For this we shall require the following result of Andrews [1, Lemma 11, p. 130]. Let a sequence of polynomials (in q) U_n be defined by

$$U_n = \sum_{j=1}^{n} \begin{bmatrix} n+j-1 \\ n-j \end{bmatrix}_q (q;q)_{j-1}(-1)^{n-j} q^{\binom{n-j}{2}}. \tag{3.1}$$

(There is a slight misprint which has been corrected.)

For the U_n defined in (3.1), we have

$$U_{2n} = -2q^{3n^2-2n} \sum_{j=0}^{n-1} q^{-j^2-j}, \tag{3.2}$$

$$U_{2n+1} = q^{3n^2+n} \sum_{j=-n}^{n} q^{-j^2}. \tag{3.3}$$

Setting $a = q^2$ in (2.1), we get

$$\begin{aligned} \alpha_n &= (1-q^{2n+2}) \sum_{j=0}^{n} \frac{(q^3;q)_{n+j-1}}{(q;q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \beta_j \\ &= \frac{(1-q^{2n+2})}{(1-q)(1-q^2)} \sum_{j=0}^{n} \frac{(q;q)_{n+j+1}}{(q;q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \beta_j. \end{aligned} \tag{3.4}$$

In (3.4) set

$$\beta_j = \frac{1}{(q^{j+1};q)_{j+1}} = \frac{(q;q)_j}{(q;q)_j (q^{j+1};q)_{j+1}} = \frac{(q;q)_j}{(q;q)_{2j+1}}. \tag{3.5}$$

So

$$\begin{aligned} \alpha_n &= \frac{(1-q^{2n+2})}{(1-q)(1-q^2)} \sum_{j=0}^{n} \frac{(q;q)_{n+j+1}}{(q;q)_{n-j}(q;q)_{2j+1}} (-1)^{n-j} (q;q)_j q^{\binom{n-j}{2}} \\ &= \frac{(1-q^{2n+2})}{(1-q)(1-q^2)} \sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ n-j \end{bmatrix}_q (-1)^{n-j} (q;q)_j q^{\binom{n-j}{2}}. \end{aligned} \tag{3.6}$$

Now by (3.1)

$$\begin{aligned} U_{n+1} &= \sum_{j=1}^{n+1} \begin{bmatrix} n+j \\ n-j+1 \end{bmatrix}_q (q;q)_{j-1} (-1)^{n+1-j} q^{\binom{n+1-j}{2}} \\ &= \sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ n-j \end{bmatrix}_q (q;q)_j (-1)^{n-j} q^{\binom{n-j}{2}}. \end{aligned} \tag{3.7}$$

From (3.6)

$$\alpha_n = \frac{(1-q^{2n+2})}{(1-q)(1-q^2)} U_{n+1},$$

so

$$\begin{aligned} \alpha_{2n} &= \frac{(1-q^{4n+2})}{(1-q)(1-q^2)} U_{2n+1} \\ &= \frac{(1-q^{4n+2})}{(1-q)(1-q^2)} q^{3n^2+n} \sum_{j=-n}^{n} q^{-j^2}, \quad \text{by (3.3)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \alpha_{2n+1} &= \frac{(1-q^{4n+4})}{(1-q)(1-q^2)} U_{2n+2} \\ &= -\frac{(1-q^{4n+4})}{(1-q)(1-q^2)} 2q^{3n^2+4n+1} \sum_{j=0}^{n} q^{-j^2-j}, \quad \text{by (3.2)} \end{aligned} \tag{3.9}$$

Hence we finally have

$$\beta_n = \frac{1}{(q^{n+1};q)_{n+1}} \tag{3.10}$$

$$\alpha_{2n} = \frac{(1-q^{4n+2})}{(1-q)(1-q^2)} q^{2n^2+n} \sum_{j=0}^{2n} q^{j(2n-j)}, \quad \text{by (1.6)} \tag{3.11}$$

and

$$\alpha_{2n+1} = -\frac{(1-q^{4n+4})}{(1-q)(1-q^2)} q^{2n^2+3n+1} \sum_{j=0}^{2n+1} q^{j(2n+1-j)}, \quad \text{by(1.8).} \tag{3.12}$$

4. Another representation for $D_6(q)$, $I_{12}(q)$ and $I_{13}(q)$

We shall prove the following theorems giving another representation for $D_6(q)$, $I_{12}(q)$ and $I_{13}(q)$.

Theorem 1.

(a)
$$\frac{(q;q)_\infty^2}{(-q^2;q^2)_\infty} D_6(q)$$
$$= -\sum_{n=0}^{\infty} q^{3n^2+3n} \sum_{j=-n}^{n} q^{-j^2} + 2\sum_{n=0}^{\infty} \frac{q^{3n^2+3n}}{1+q^{4n+2}} \sum_{j=-n}^{n} q^{-j^2}$$
$$+ 2\sum_{n=0}^{\infty} q^{3n^2+6n+2} \sum_{j=0}^{n} q^{-j^2-j} - 4\sum_{n=0}^{\infty} \frac{q^{3n^2+6n+2}}{1+q^{4n+4}} \sum_{j=0}^{n} q^{-j^2-j}.$$

(b)
$$\frac{(q;q)_\infty^2}{(-q;q^2)_\infty} I_{13}(q)$$
$$= \left(\sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{1+q^{4n+1}} - \sum_{n=-\infty}^{-1} \frac{q^{2n^2+3n}}{1+q^{4n+1}} \right) \sum_{j=0}^{2n} q^{j(2n-j)} - 2\sum_{n=-\infty}^{-1} \frac{q^{2n^2+n-1}}{1+q^{4n+1}}$$
$$+ \left(\sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{1+q^{4n+1}} - \sum_{n=-\infty}^{-1} \frac{q^{2n^2+3n}}{1+q^{4n+1}} \right) \sum_{j=-2n}^{-1} q^{j(-2n-1-j)} - 2\sum_{n=-\infty}^{-1} \frac{q^{2n^2+3n}}{1+q^{4n+1}}.$$

(c)
$$\frac{(q;q)_\infty^2}{(-q;q^2)_\infty} I_{12}(q)$$
$$= \sum_{n=0}^{\infty} \frac{q^{2n^2+5n}}{1+q^{4n+1}} \sum_{j=0}^{2n} q^{j(2n-j)} - \sum_{n=-\infty}^{-1} \frac{q^{2n^2+5n}}{1+q^{4n+1}} \sum_{j=-1}^{2n+1} q^{j(2n-j)}$$
$$+ \left(\sum_{n=1}^{\infty} \frac{q^{2n^2+n-1}}{1+q^{4n+1}} - \sum_{n=1}^{\infty} \frac{q^{2n^2+n-2}}{1+q^{4n-1}} \right) \sum_{j=-2n}^{-1} q^{j(-2n-1-j)}.$$

Proof of Theorem 1(a)

We start with $D_6(q)$.

$$D_6(q) = \sum_{n=0}^{\infty} q^n \frac{(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} q^n \frac{(iq;q)_n(-iq;q)_n}{(q^{n+1};q)_{n+1}}. \tag{4.1}$$

Setting $\rho_1 = iq$, $\rho_2 = -iq$ and $a = q^2$ in (2.2), we have

$$\sum_{n=0}^{\infty} \frac{(iq;q)_n(-iq;q)_n q^n}{(iq^2;q)_n(-iq^2;q)_n} \alpha_n = \frac{(q;q)_\infty(q^3;q)_\infty}{(iq^2;q)_\infty(-iq^2;q)_\infty} \sum_{n=0}^{\infty} (iq;q)_n(-iq;q)_n q^n \beta_n$$

and this simplifies to

$$\frac{(q;q)_\infty(q^3;q)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-q^2;q^2)_n q^n \beta_n = \sum_{n=0}^{\infty} \frac{q^n}{1+q^{2n+2}} \alpha_n. \tag{4.2}$$

Taking the Bailey pair $\{\alpha_n\}, \{\beta_n\}$as given in (3.8), (3.9) and (3.10), we have

$$\begin{aligned} \frac{(q;q)_\infty(q^3;q)_\infty}{(-q^2;q^2)_\infty} D_6(q) &= \frac{1}{(1-q)(1-q^2)} \sum_{n=0}^{\infty} \frac{1-q^{4n+2}}{1+q^{4n+2}} q^{3n^2+3n} \sum_{j=-n}^{n} q^{-j^2} \\ &\quad - \frac{2}{(1-q)(1-q^2)} \sum_{n=0}^{\infty} \frac{1-q^{4n+4}}{1+q^{4n+4}} q^{3n^2+6n+2} \sum_{j=0}^{n} q^{-j^2-j} \end{aligned} \tag{4.3}$$

or

$$\begin{aligned} \frac{(q;q)_\infty^2}{(-q^2;q^2)_\infty} D_6(q) &= -\sum_{n=0}^{\infty} q^{3n^2+3n} \sum_{j=-n}^{n} q^{-j^2} + 2\sum_{xn=0}^{\infty} \frac{q^{3n^2+3n}}{1+q^{4n+2}} \sum_{j=-n}^{n} q^{-j^2} \\ &\quad +2\sum_{n=0}^{\infty} q^{3n^2+6n+2} \sum_{j=0}^{n} q^{-j^2-j} - 4\sum_{n=0}^{\infty} \frac{q^{3n^2+6n+2}}{1+q^{4n+4}} \sum_{j=0}^{n} q^{-j^2-j}, \end{aligned} \tag{4.4}$$

which proves Theorem 1(a).

Proof of Theorem 1(b)

We start with $I_{13}(q)$.
$I_{13}(q)$ is defined as :

$$I_{13}(q) = \sum_{n=0}^{\infty} q^n \frac{(-q;q^2)_n}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} q^n \frac{(iq^{\frac{1}{2}};q)_n(-iq^{\frac{1}{2}};q)_n}{(q^{n+1};q)_{n+1}}. \tag{4.5}$$

Setting $\rho_1 = iq^{\frac{1}{2}}$, $\rho_2 = -iq^{\frac{1}{2}}$ and $a = q$ in (2.2), we get

$$\sum_{n=0}^{\infty} \frac{(iq^{\frac{1}{2}};q)_n(-iq^{\frac{1}{2}};q)_n q^n}{(iq^{\frac{3}{2}};q)_n(-iq^{\frac{3}{2}};q)_n}\alpha_n = \frac{(q;q)_\infty(q^2;q)_\infty}{(iq^{\frac{3}{2}};q)_\infty(-iq^{\frac{3}{2}};q)_\infty}\sum_{n=0}^{\infty}(iq^{\frac{1}{2}};q)_n(-iq^{\frac{1}{2}};q)_n q^n\beta_n$$

or

$$\sum_{n=0}^{\infty} \frac{q^n}{1+q^{2n+1}}\alpha_n = \frac{(q;q)_\infty^2}{(1-q)(-q;q^2)_\infty}\sum_{n=0}^{\infty}(-q;q^2)_n q^n\beta_n. \tag{4.6}$$

In (4.6) we take the Bailey pair $\{\alpha_n\},\{\beta_n\}$ see [1, (7.20)–(7.21), p. 131], namely,

$$\beta_n = \frac{1}{(q^{n+1};q)_{n+1}}, \tag{4.7}$$

$$\alpha_{2n} = \frac{1}{(1-q)}\left(q^{3n^2+n}\sum_{j=-n}^{n} q^{-j^2} + 2q^{3n^2+2n}\sum_{j=0}^{n-1} q^{-j^2-j}\right), \tag{4.8}$$

and

$$\alpha_{2n+1} = -\frac{1}{(1-q)}\left(2q^{3n^2+4n+1}\sum_{j=0}^{n} q^{-j^2-j} + q^{3n^2+5n+2}\sum_{j=-n}^{n} q^{-j^2}\right), \tag{4.9}$$

to get

$$\begin{aligned} \frac{(q;q)_\infty^2}{(-q;q^2)_\infty}I_{13}(q) &= (1-q)\left(\sum_{n=0}^{\infty}\frac{q^{2n}}{1+q^{4n+1}}\alpha_{2n} + \sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{4n+3}}\alpha_{2n+1}\right) \\ &= \sum_{n=0}^{\infty}\frac{q^{3n^2+3n}}{1+q^{4n+1}}\sum_{j=-n}^{n} q^{-j^2} + 2\sum_{n=0}^{\infty}\frac{q^{3n^2+4n}}{1+q^{4n+1}}\sum_{j=0}^{n-1} q^{-j^2-j} \\ &\quad - 2\sum_{n=0}^{\infty}\frac{q^{3n^2+6n+2}}{1+q^{4n+3}}\sum_{j=0}^{n} q^{-j^2-j} - \sum_{n=0}^{\infty}\frac{q^{3n^2+7n+3}}{1+q^{4n+3}}\sum_{j=-n}^{n} q^{-j^2}. \end{aligned} \tag{4.10}$$

Now we couple the like summations:

$$\begin{aligned} &\frac{(q;q)_\infty^2}{(-q;q^2)_\infty}I_{13}(q) \\ &= \left(\sum_{n=0}^{\infty}\frac{q^{3n^2+3n}}{1+q^{4n+1}}\sum_{j=-n}^{n} q^{-j^2} - \sum_{n=0}^{\infty}\frac{q^{3n^2+7n+3}}{1+q^{4n+3}}\sum_{j=-n}^{n} q^{-j^2}\right) \\ &\quad + 2\left(\sum_{n=0}^{\infty}\frac{q^{3n^2+4n}}{1+q^{4n+1}}\sum_{j=0}^{n-1} q^{-j^2-j} - \sum_{n=0}^{\infty}\frac{q^{3n^2+6n+2}}{1+q^{4n+3}}\sum_{j=0}^{n} q^{-j^2-j}\right) \end{aligned} \tag{4.11}$$

We take the second summation in the first bracket and first write $n-1$ for n to get

$$\text{(4.12)}\quad \sum_{n=0}^{\infty}\frac{q^{3n^2+7n+3}}{1+q^{4n+3}}\sum_{j=-n}^{n}q^{-j^2}=\sum_{n=1}^{\infty}\frac{q^{3n^2+n-1}}{1+q^{4n-1}}\sum_{j=-(n-1)}^{n-1}q^{-j^2}$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n^2+n-1}}{1+q^{4n-1}}\sum_{j=-2n+1}^{-1}q^{j(-2n-j)},\quad \text{by (1.7)}.$$

Writing $-n$ for n on the right hand side of (4.12)

$$\sum_{n=0}^{\infty}\frac{q^{3n^2+7n+3}}{1+q^{4n+3}}\sum_{j=-n}^{n}q^{-j^2}=\sum_{n=-\infty}^{-1}\frac{q^{2n^2-n-1}}{1+q^{-4n-1}}\sum_{j=-1}^{2n+1}q^{j(2n-j)}$$
$$=\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\left(\sum_{j=0}^{2n}q^{j(2n-j)}+2q^{-2n-1}\right)$$

or

$$\text{(4.13)}\quad \sum_{n=0}^{\infty}\frac{q^{3n^2+7n+3}}{1+q^{4n+3}}\sum_{j=-n}^{n}q^{-j^2}$$
$$=\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\sum_{j=0}^{2n}q^{j(2n-j)}+2\sum_{n=-\infty}^{-1}\frac{q^{2n^2+n-1}}{1+q^{4n+1}}.$$

Hence the two summations in the first bracket on the right hand side of (4.11) equals

$$\text{(4.14)}\quad \left(\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{4n+1}}-\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\right)\sum_{j=0}^{2n}q^{j(2n-j)}-2\sum_{n=-\infty}^{-1}\frac{q^{2n^2+n-1}}{1+q^{4n+1}}.$$

We now simplify the second summation in the second bracket on the right hand side of (4.11),

$$\text{(4.15)}\quad 2\sum_{n=0}^{\infty}\frac{q^{3n^2+6n+2}}{1+q^{4n+3}}\sum_{j=0}^{n}q^{-j^2-j}=\sum_{n=0}^{\infty}\frac{q^{2n^2+5n+2}}{1+q^{4n+3}}\sum_{j=0}^{2n+1}q^{j(2n+1-j)},\quad \text{by (1.8)}.$$

First writing $n-1$ for n and then $-n$ for n on the right hand side of the above identity,

$$\text{(4.16)}\quad 2\sum_{n=0}^{\infty}\frac{q^{3n^2+6n+2}}{1+q^{4n+3}}\sum_{j=0}^{n}q^{-j^2-j}=\sum_{n=1}^{\infty}\frac{q^{2n^2+n-1}}{1+q^{4n-1}}\sum_{j=0}^{2n-1}q^{j(2n-1-j)}$$
$$=\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\sum_{j=-2n-1}^{0}q^{j(-2n-1-j)}$$
$$=\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\left(\sum_{j=-2n}^{-1}q^{j(-2n-1-j)}+2\right).$$

So the two summations in the second bracket on the right hand side of (4.11) equals

$$\left(\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{4n+1}}-\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\right)\sum_{j=-2n}^{-1}q^{j(-2n-1-j)}-2\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}. \tag{4.17}$$

Using (4.14) and (4.17) in (4.11), we have

$$\begin{aligned}&\frac{(q;q)_\infty^2}{(-q;q^2)_\infty}I_{13}(q)\\&=\left(\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{4n+1}}-\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\right)\sum_{j=0}^{2n}q^{j(2n-j)}-2\sum_{n=-\infty}^{-1}\frac{q^{2n^2+n-1}}{1+q^{4n+1}}\\&+\left(\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}}{1+q^{4n+1}}-\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}}\right)\sum_{j=-2n}^{-1}q^{j(-2n-1-j)}-2\sum_{n=-\infty}^{-1}\frac{q^{2n^2+3n}}{1+q^{4n+1}},\end{aligned} \tag{4.18}$$

which proves Theorem 1(b).

Proof of Theorem 1(c)

Now we consider $I_{12}(q)$.

$$I_{12}(q)=\sum_{n=0}^{\infty}q^{2n}\frac{(-q;q^2)_n}{(q^{n+1};q)_{n+1}}=\sum_{n=0}^{\infty}q^{2n}\frac{(iq^{\frac12};q)_n(-iq^{\frac12};q)_n}{(q^{n+1};q)_{n+1}}. \tag{4.19}$$

Setting $\rho_1=iq^{\frac12}$, $\rho_2=-iq^{\frac12}$ and $a=q^2$ in (2.2), we have

$$\sum_{n=0}^{\infty}\frac{(iq^{\frac12};q)_n(-iq^{\frac12};q)_nq^{2n}}{(iq^{\frac52};q)_n(-iq^{\frac52};q)_n}\alpha_n=\frac{(q^2;q)_\infty(q^3;q)_\infty}{(iq^{\frac52};q)_\infty(-iq^{\frac52};q)_\infty}\sum_{n=0}^{\infty}(iq^{\frac12};q)_n(-iq^{\frac12};q)_nq^{2n}\beta_n,$$

which simplifies to

$$\sum_{n=0}^{\infty}\frac{q^{2n}}{(1+q^{2n+1})(1+q^{2n+3})}\alpha_n=\frac{(q^2;q)_\infty(q^3;q)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}(-q;q^2)_nq^{2n}\beta_n. \tag{4.20}$$

Taking the Bailey pair $\{\alpha_n\},\{\beta_n\}$ given in (3.8), (3.9) and (3.10) and setting in the above identity, we get

$$\begin{aligned}\frac{(q^2;q)_\infty^2}{(-q;q^2)_\infty}I_{12}(q)&=\frac{1}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+2}}{(1+q^{4n+1})(1+q^{4n+3})}q^{3n^2+5n}\sum_{j=-n}^{n}q^{-j^2}\\&-\frac{2}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+4}}{(1+q^{4n+3})(1+q^{4n+5})}q^{3n^2+8n+3}\sum_{j=0}^{n}q^{-j^2-j}.\end{aligned} \tag{4.21}$$

We will split the first summation on the right hand side of (4.21) into two parts using

$$\frac{1}{(1-q)}\frac{(1-q^{4n+2})}{(1+q^{4n+1})(1+q^{4n+3})}=\frac{1}{(1-q)^2}\left(\frac{1}{1+q^{4n+1}}-\frac{q}{1+q^{4n+3}}\right).$$

So

$$\begin{aligned}(4.22)\quad &\frac{1}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+2}}{(1+q^{4n+1})(1+q^{4n+3})}q^{3n^2+5n}\sum_{j=-n}^{n}q^{-j^2}\\ &=\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\left(\frac{1}{1+q^{4n+1}}-\frac{q}{1+q^{4n+3}}\right)q^{3n^2+5n}\sum_{j=-n}^{n}q^{-j^2}\\ &=\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{3n^2+5n}}{1+q^{4n+1}}\sum_{j=-n}^{n}q^{-j^2}-\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{3n^2+5n+1}}{1+q^{4n+3}}\sum_{j=-n}^{n}q^{-j^2}.\end{aligned}$$

Writing $n-1$ for n in the second summation on the right hand side in the above, we have

$$\begin{aligned}(4.23)\quad &\frac{1}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+2}}{(1+q^{4n+1})(1+q^{4n+3})}q^{3n^2+5n}\sum_{j=-n}^{n}q^{-j^2}\\ &=\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{3n^2+5n}}{1+q^{4n+1}}\sum_{j=-n}^{n}q^{-j^2}-\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{3n^2-n-1}}{1+q^{4n-1}}\sum_{j=-(n-1)}^{n-1}q^{-j^2}.\end{aligned}$$

By (1.6) and (1.7)

$$\begin{aligned}(4.24)\quad &\frac{1}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+2}}{(1+q^{4n+1})(1+q^{4n+3})}q^{3n^2+5n}\sum_{j=-n}^{n}q^{-j^2}\\ &=\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{2n^2+5n}}{1+q^{4n+1}}\sum_{j=0}^{2n}q^{j(2n-j)}-\frac{1}{(1-q)^2}\sum_{n=1}^{\infty}\frac{q^{2n^2-n-1}}{1+q^{4n-1}}\sum_{j=-2n+1}^{-1}q^{j(-2n-j)}\\ &=\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}\frac{q^{2n^2+5n}}{1+q^{4n+1}}\sum_{j=0}^{2n}q^{j(2n-j)}-\frac{1}{(1-q)^2}\sum_{n=-\infty}^{-1}\frac{q^{2n^2+5n}}{1+q^{4n+1}}\sum_{j=-1}^{2n+1}q^{j(2n-j)}.\end{aligned}$$

Here we have put $-n$ for n in the second summation on the right hand side. We will now split the second summation of the right hand side in (4.21) into two parts using

$$\frac{1}{(1-q)}\frac{(1-q^{4n+4})}{(1+q^{4n+3})(1+q^{4n+5})}=\frac{1}{(1-q)^2}\left(\frac{1}{1+q^{4n+3}}-\frac{q}{1+q^{4n+5}}\right),$$

to get

$$(4.25)\qquad \frac{2}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+4}}{(1+q^{4n+3})(1+q^{4n+5})}q^{3n^2+8n+3}\sum_{j=0}^{n}q^{-j^2-j}$$
$$=\frac{2}{(1-q)^2}\sum_{n=0}^{\infty}\left(\frac{1}{1+q^{4n+3}}-\frac{q}{1+q^{4n+5}}\right)q^{3n^2+8n+3}\sum_{j=0}^{n}q^{-j^2-j}.$$

Writing $n-1$ for n in the summations on the right hand side in the above, we have

$$(4.26)\qquad \frac{2}{(1-q)}\sum_{n=0}^{\infty}\frac{1-q^{4n+4}}{(1+q^{4n+3})(1+q^{4n+5})}q^{3n^2+8n+3}\sum_{j=0}^{n}q^{-j^2-j}$$
$$=\frac{2}{(1-q)^2}\sum_{n=1}^{\infty}\frac{q^{3n^2+2n-2}}{1+q^{4n-1}}\sum_{j=0}^{n-1}q^{-j^2-j}-\frac{2}{(1-q)^2}\sum_{n=1}^{\infty}\frac{q^{3n^2+2n-1}}{1+q^{4n+1}}\sum_{j=0}^{n-1}q^{-j^2-j}$$

Hence putting (4.24) and (4.26) in (4.21), we have

$$(4.27)\qquad \frac{(q;q)_\infty^2}{(-q;q^2)_\infty}I_{12}(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2+5n}}{1+q^{4n+1}}\sum_{j=0}^{2n}q^{j(2n-j)}-\sum_{n=-\infty}^{-1}\frac{q^{2n^2+5n}}{1+q^{4n+1}}\sum_{j=-1}^{2n+1}q^{j(2n-j)}$$
$$+\left(\sum_{n=1}^{\infty}\frac{q^{2n^2+n-1}}{1+q^{4n+1}}-\sum_{n=1}^{\infty}\frac{q^{2n^2+n-2}}{1+q^{4n-1}}\right)\sum_{j=-2n}^{-1}q^{j(-2n-1-j)},$$

which proves Theorem 1(c).

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