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Generalization of Hardy-Hilbert's Inequality and Applications

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ABSTRACT. In this paper, by introducing some parameters we establish an extension of Hardy-Hilbert's integral inequality and the corresponding inequality for series. As an application, the reverses, some particular results and their equivalent forms are considered.

1. Introduction

First, let us recall the well-known Hilbert's integral inequality: If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, and $0 < \int_0^\infty g^2(x)dx < \infty$, then (see [2])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int_0^\infty f^2(x)dx . \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{1.1}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4 \left(\int_0^\infty f^2(x)dx . \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{1.2}$$

where the constant factors π and 4 are the best possible in (1.1) and (1.2), respectively. Inequality (1.1) is called Hilbert's integral inequality and (1.2) is called Hilbert's type which have been extended by Hardy (see [3]) as follows: if $p > 1$,

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$\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) \geq 0$ such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1.4}$$

where the constant factors $\frac{\pi}{\sin(\frac{\pi}{p})}$ and pq are the best possible in (1.3) and (1.4), respectively. Hardy-Hilbert's inequality and its applications are important in analysis (see [5]). In the recent years a lot of results with generalizations of these type of inequalities were obtained. Let's mention some of them which take our attention. We cite the results of Yang([6]-[9]), Li ([4]), Xi ([10]) and Azar ([1]).

We start with the result of Yang: In [6], Yang gave some generalizations and the reverse form of (1.3) as follows: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $f(x), g(x) \geq 0$ such that $0 < \int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \left(\int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(x)dx \right)^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor $\frac{\pi}{\lambda \sin(\frac{\pi}{r})}$ is the best possible.

The corresponding inequalities for series (1.3) and (1.4) are the following forms

$$\begin{aligned} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} &< \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{n=0}^\infty a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^\infty b_n^q \right)^{\frac{1}{q}}, \\ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} &< pq \left(\sum_{n=0}^\infty a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^\infty b_n^q \right)^{\frac{1}{q}}, \end{aligned} \tag{1.6}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 < \sum_{n=1}^\infty a_n^p < \infty$, $0 < \sum_{n=1}^\infty b_n^q < \infty$, and the constant factors $\frac{\pi}{\sin(\frac{\pi}{p})}$ and pq are the best possible. By introducing a parameter $0 < \lambda < 2$, some extensions of (1.6)($p = q = 2$) where given Yang [7], [8] as follows:

$$(1.7)\quad \begin{aligned} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} &< \frac{\pi}{\lambda}\left(\sum_{n=0}^{\infty} n^{1-\lambda}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} n^{1-\lambda}b_n^2\right)^{\frac{1}{2}},\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &< \frac{4}{\lambda}\left(\sum_{n=0}^{\infty} n^{1-\lambda}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} n^{1-\lambda}b_n^2\right)^{\frac{1}{2}}.\end{aligned}$$

Another result of the same type inequality is given by Li,

$$(1.8)\quad \int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dxdy<D(A,B)\left(\int_0^{\infty} f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^{\infty} g^2(x)dx\right)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}}<D(A,B)\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}},$$

where the constant factor $D(A,B)$ (see [4], Lemma 2.1) is the best possible in both inequalities. For more information related to this subject see, for example, [9], [10].

By introducing a parameter $\lambda > 0$, in [1], Azar gave a result of this similar type inequality (1.8):

$$(1.9)\quad \int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dxdy$$
$$< C_\lambda(A,B)\left(\int_0^{\infty} x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx\right)^{\frac{1}{q}},$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}$$
$$< C_\lambda(A,B)\left(\sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^2\right)^{\frac{1}{2}},$$

where the constant factor $C_\lambda(A,B)$ (see [1], Theorem 2.3) is the best possible in both inequalities.

In this paper, we generalize the inequality (1.9) and we obtain the reverse form for each of them. Some particular results and the equivalent form are also considered.

2. Main results

First, we introduce some Lemmas:

Lemma 1. *Suppose that* $\lambda > 0,\ A_1 \geq 0,\ A_2 \geq 0,\ A_3 \geq 0,\ A_4 > 0$. *Define the weight coefficients* $\varpi_\lambda(A_1, A_2, A_3, A_4, x)$ *and* $\varpi_\lambda(A_1, A_2, A_3, A_4, y)$ *by*

$$\varpi_\lambda(A_1, A_2, A_3, A_4, x) := \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A_1 x^\lambda + A_2 y^\lambda + A_3 \min\{x^\lambda, y^\lambda\} + A_4 \max\{x^\lambda, y^\lambda\}} dy, \tag{2.1}$$

$$\varpi_\lambda(A_1, A_2, A_3, A_4, y) := \int_0^\infty \frac{x^{-1+\frac{\lambda}{2}} y^{\frac{\lambda}{2}}}{A_1 x^\lambda + A_2 y^\lambda + A_3 \min\{x^\lambda, y^\lambda\} + A_4 \max\{x^\lambda, y^\lambda\}} dx, \tag{2.2}$$

then $\varpi_\lambda(A_1, A_2, A_3, A_4, x) = \varpi_\lambda(A_1, A_2, A_3, A_4, y) = C_\lambda(A_1, A_2, A_3, A_4)$ *is a constant defined by*

$$C_\lambda(A_1, A_2, A_3, A_4) = \begin{cases} \dfrac{2}{\lambda\sqrt{(A_1+A_4)(A_2+A_3)}} \arctan\sqrt{\dfrac{A_2+A_3}{A_1+A_4}} \\ \quad + \dfrac{2}{\lambda\sqrt{(A_1+A_3)(A_2+A_4)}} \arctan\sqrt{\dfrac{A_1+A_3}{A_2+A_4}} \\ \qquad ,\textit{for } A_i > 0,\ i = 1,2,3,4 \\ \dfrac{4}{\lambda A_4} \qquad ,\textit{for } A_i = 0,\ A_4 > 0,\ i = 1,2,3. \end{cases} \tag{2.3}$$

Proof. By putting $t = \left(\frac{y}{x}\right)^\lambda$, we have

$$\begin{aligned} \varpi_\lambda(A_1, A_2, A_3, A_4, x) &= \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A_1 x^\lambda + A_2 y^\lambda + A_3 \min\{x^\lambda, y^\lambda\} + A_4 \max\{x^\lambda, y^\lambda\}} dy, \\ &= \frac{1}{\lambda} \int_0^\infty \frac{t^{-\frac{1}{2}}}{A_1 + A_2 t + A_3 \min\{1, t\} + A_4 \max\{1, t\}} dt := I \end{aligned} \tag{2.4}$$

(i) For $A_1,\ A_2,\ A_3,\ A_4 > 0$, we obtain

$$(2.5)\quad I = \frac{1}{\lambda}\left\{\int_0^1 \frac{t^{-\frac{1}{2}}}{A_1 + A_2 t + A_3 t + A_4} dt + \int_1^\infty \frac{t^{-\frac{1}{2}}}{A_1 + A_2 t + A_3 + A_4 t} dt\right\}$$

$$= \frac{1}{\lambda}\left\{\frac{2}{\sqrt{(A_1+A_4)(A_2+A_3)}} \int_0^{\sqrt{\frac{A_2+A_3}{A_1+A_4}}} \frac{dt}{t^2+1} + \frac{2}{\sqrt{(A_1+A_3)(A_2+A_4)}} \int_0^{\sqrt{\frac{A_1+A_3}{A_2+A_4}}} \frac{dt}{t^2+1}\right\}$$

$$= \frac{2}{\lambda\sqrt{(A_1+A_4)(A_2+A_3)}} \arctan\sqrt{\frac{A_2+A_3}{A_1+A_4}} + \frac{2}{\lambda\sqrt{(A_1+A_3)(A_2+A_4)}} \arctan\sqrt{\frac{A_1+A_3}{A_2+A_4}};$$

(ii) For $A_1 = A_2 = A_3 = 0,\ A_4 > 0$, we find

$$(2.6)\qquad I = \frac{1}{\lambda}\left\{\int_0^1 \frac{t^{-\frac{1}{2}}}{A_4} dt + \int_1^\infty \frac{t^{-\frac{1}{2}}}{A_4 t} dt\right\} = \frac{4}{\lambda A_4}.$$

Hence, $\varpi_\lambda(A_1, A_2, A_3, A_4, x) = C_\lambda(A_1, A_2, A_3, A_4)$. By the symmetry, we still have $\varpi_\lambda(A_1, A_2, A_3, A_4, y) = C_\lambda(A_1, A_2, A_3, A_4)$.

Lemma 2. *For* $p > 1$ (*or* $0 < p < 1$), $\frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 0,\ A_1,\ A_2,\ A_3 \geq 0,\ A_4 > 0$ *and* $0 < \varepsilon < \frac{p\lambda}{2}$, *setting*

$$(2.7)\qquad J(\varepsilon) = \int_1^\infty \int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1 x^\lambda + A_2 y^\lambda + A_3 \min\{x^\lambda, y^\lambda\} + A_4 \max\{x^\lambda, y^\lambda\}} dxdy,$$

then for $\varepsilon \longrightarrow 0^+$,

$$(2.8)\quad \frac{1}{\varepsilon}[C_\lambda(A_1, A_2, A_3, A_4) + o(1)] - O(1) < J(\varepsilon) < \frac{1}{\varepsilon}[C_\lambda(A_1, A_2, A_3, A_4) + \tilde{o}(1)].$$

Proof. By putting $t=\left(\frac{x}{y}\right)^{\lambda}$, we have

$$J(\varepsilon)=\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dxdy \tag{2.9}$$

$$=\frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon}\int_{y^{-\lambda}}^\infty \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_1t+A_2+A_3\min\{t,1\}+A_4\max\{t,1\}}dtdy$$

$$=\frac{1}{\lambda\varepsilon}\int_0^\infty \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_1t+A_2+A_3\min\{t,1\}+A_4\max\{t,1\}}dt$$

$$-\frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon}\int_0^{y^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_1t+A_2+A_3\min\{t,1\}+A_4\max\{t,1\}}dtdy$$

$$=\frac{1}{\varepsilon}\left[C_\lambda(A_1,A_2,A_3,A_4)+o(1)\right]-\frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon}\int_0^{y^{-\lambda}}\frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_1t+A_2+A_3t+A_4}dtdy$$

$$\geq\frac{1}{\varepsilon}\left[C_\lambda(A_1,A_2,A_3,A_4)+o(1)\right]-\frac{1}{\lambda A_4}\int_1^\infty y^{-1}\left(\int_0^{y^{-\lambda}} t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}dt\right)dy$$

$$=\frac{1}{\varepsilon}\left[C_\lambda(A_1,A_2,A_3,A_4)+o(1)\right]-O(1).$$

On the other hand,

$$J(\varepsilon)=\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dxdy$$

$$<\int_1^\infty\left[\int_0^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dx\right]y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}dy$$

$$=\frac{1}{\varepsilon}\left[C_\lambda(A_1,A_2,A_3,A_4)+\tilde{o}(1)\right].$$

Hence, (2.7) is valid. The lemma is proved. □

Theorem 1. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda>0,\ A_1,\ A_2,\ A_3\ge 0,\ A_4>0,\ f(x),\ g(x)\ge 0$ *such that* $0<\int\limits_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx<\infty$ *and* $0<\int\limits_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx<\infty$, *then*

$$S:=\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,g(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dxdy$$
$$<C_\lambda(A_1,A_2,A_3,A_4)\left(\int\limits_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int\limits_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx\right)^{\frac{1}{q}}, \tag{2.10}$$

where the constant factor $C_\lambda(A_1,A_2,A_3,A_4)$ *defined by* (2.3) *is the best possible.*

Proof. By the Hölder inequality, taking into account (2.1), we have

$$S=\int\limits_0^\infty\int\limits_0^\infty\left[\frac{1}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}\right]^{\frac{1}{p}}\frac{x^{\frac{1-\frac{\lambda}{2}}{q}}}{y^{\frac{1-\frac{\lambda}{2}}{p}}}f(x) \tag{2.11}$$

$$\times\left[\frac{1}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}\right]^{\frac{1}{q}}\left[\frac{y^{\frac{1-\frac{\lambda}{2}}{p}}}{x^{\frac{1-\frac{\lambda}{2}}{q}}}g(y)\right]dxdy$$
$$\le\left\{\int\limits_0^\infty\int\limits_0^\infty\frac{x^{\left(1-\frac{\lambda}{2}\right)(p-1)}y^{\frac{\lambda}{2}-1}}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}f^p(x)\,dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int\limits_0^\infty\int\limits_0^\infty\frac{y^{\left(1-\frac{\lambda}{2}\right)(q-1)}x^{\frac{\lambda}{2}-1}}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}g^q(y)\,dy\right\}^{\frac{1}{q}}$$
$$=\left\{\int\limits_0^\infty\varpi_\lambda(A_1,A_2,A_3,A_4,x)x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)\,dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int\limits_0^\infty\varpi_\lambda(A_1,A_2,A_3,A_4,y)y^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(y)\,dy\right\}^{\frac{1}{q}}$$
$$\le C_\lambda(A_1,A_2,A_3,A_4)\left\{\int\limits_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int\limits_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(y)\,dy\right\}^{\frac{1}{q}}.$$

If (2.11) takes the form of equality, then there exist constants M and N which are not all zero such that

$$M\frac{x^{\left(1-\frac{\lambda}{2}\right)(p-1)}y^{\frac{\lambda}{2}-1}}{A_1x^{\lambda}+A_2y^{\lambda}+A_3\min\{x^{\lambda},y^{\lambda}\}+A_4\max\{x^{\lambda},y^{\lambda}\}}f^p(x) = N\frac{y^{\left(1-\frac{\lambda}{2}\right)(q-1)}x^{\frac{\lambda}{2}-1}}{A_1x^{\lambda}+A_2y^{\lambda}+A_3\min\{x^{\lambda},y^{\lambda}\}+A_4\max\{x^{\lambda},y^{\lambda}\}}g^q(y), \tag{2.12}$$

$$Mx^{p\left(1-\frac{\lambda}{2}\right)}f^p(x) = Ny^{q\left(1-\frac{\lambda}{2}\right)}g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

Hence, there exists a constant c such that

$$Mx^{p\left(1-\frac{\lambda}{2}\right)}f^p(x) = Ny^{q\left(1-\frac{\lambda}{2}\right)}g^q(y) = c \text{ a.e. in } (0,\infty)\times(0,\infty). \tag{2.13}$$

We claim that $M=0$. In fact, if $M\neq 0$, then

$$x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x) = \frac{c}{Mx} \quad \text{a.e. in } (0,\infty) \tag{2.14}$$

which contradicts the fact that $0<\int\limits_0^{\infty}x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx<\infty$. Hence, by (2.11) we get (2.10).

If the constant factor $C_{\lambda}(A_1,A_2,A_3,A_4)$ is not the best possible, then there exists a positive constant K(with $K<C_{\lambda}(A_1,A_2,A_3,A_4)$), thus (2.10) is still valid if we replace $C_{\lambda}(A_1,A_2,A_3,A_4)$ by K. For $0<\varepsilon<\frac{p\lambda}{2}$, setting $\widetilde{f}$ and $\widetilde{g}$ as $\widetilde{f}(x)=\widetilde{g}(x)=0$ for $x\in(0,1)$, $\widetilde{f}(x)=x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}$; $\widetilde{g}(x)=x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$ for $x\in[1,\infty)$, then we have

$$K\left(\int\limits_0^{\infty}x^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{f}^p(x)\,dx\right)^{\frac{1}{p}}\left(\int\limits_0^{\infty}x^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{g}^q(x)dx\right)^{\frac{1}{q}} = K\left(\int\limits_0^{\infty}x^{-1-\varepsilon}dx\right)^{\frac{1}{p}}\left(\int\limits_0^{\infty}x^{-1-\varepsilon}dx\right)^{\frac{1}{q}} = \frac{K}{\varepsilon}. \tag{2.15}$$

By using (2.7), we find

$$\begin{aligned} &\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{\widetilde{f}(x)\,\widetilde{g}(x)}{A_1x^{\lambda}+A_2y^{\lambda}+A_3\min\{x^{\lambda},y^{\lambda}\}+A_4\max\{x^{\lambda},y^{\lambda}\}}dxdy \\ &= \int\limits_1^{\infty}\left[\int\limits_1^{\infty}\frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}}{A_1x^{\lambda}+A_2y^{\lambda}+A_3\min\{x^{\lambda},y^{\lambda}\}+A_4\max\{x^{\lambda},y^{\lambda}\}}dx\right]y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}dy \\ &> \frac{1}{\varepsilon}\left[C_{\lambda}(A_1,A_2,A_3,A_4)+o(1)\right]-O(1). \end{aligned} \tag{2.16}$$

Therefore, we get

$$\frac{1}{\varepsilon}\left[C_\lambda(A_1,A_2,A_3,A_4)+o(1)\right]-O(1)<\frac{K}{\varepsilon}, \tag{2.17}$$

or

$$\frac{1}{\lambda}\left[C_\lambda(A_1,A_2,A_3,A_4)+o(1)\right]-\varepsilon O(1)<K. \tag{2.18}$$

For $\varepsilon \to 0^+$, it follows that $C_\lambda(A_1,A_2,A_3,A_4) \le K$ which contradicts the fact that $K < C_\lambda(A_1,A_2,A_3,A_4)$. Hence, the constant factor $C_\lambda(A_1,A_2,A_3,A_4)$ in (2.10) is the best possible. The theorem is proved. □

Remark 1. (i) In Theorem 1, if we take $A_1 = A_2 = 0$ and $\lambda = A_3 = A_4 = 1$, then $C_1(0,0,1,1) = \pi$ and inequality (2.10) reduces to Hardy-Hilbert's inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^\infty x^{\frac{p}{2}-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{\frac{q}{2}-1}g^q(x)dx\right)^{\frac{1}{q}}.$$

(ii) In Theorem 1, if we take $A_1 = A_2 = A_3 = 0$, $\lambda = A_4 = 1$, then $C_1(0,0,0,1) = 4$ and (2.10) reduces to

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\left(\int_0^\infty x^{\frac{p}{2}-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{\frac{q}{2}-1}g^q(x)dx\right)^{\frac{1}{q}}.$$

(iii) In Theorem 1, if we take $A_1 = A_2 = 0$ and $\lambda > 0$, $A_3 = A$, $A_4 = B$, then

$$C_\lambda(0,0,A,B) = \begin{cases} \dfrac{4}{\lambda\sqrt{AB}}\arctan\sqrt{\dfrac{A}{B}} & \text{, for } A,B>0, \\[2ex] \dfrac{4}{\lambda B} & \text{, for } A=0,\ B>0, \end{cases}$$

and inequality (2.10) reduces to

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dxdy$$
$$< C_\lambda(A,B)\left(\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx\right)^{\frac{1}{q}}.$$

Theorem 2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, A_1, A_2, $A_3 \geq 0$, $A_4 > 0$, $f(x)$, $g(x) \geq 0$ *such that* $0 < \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx < \infty$ *and* $0 < \int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx < \infty$, *then*

$$(2.19)\qquad \int_0^\infty\int_0^\infty \frac{f(x)\,g(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dxdy$$
$$> C_\lambda(A_1, A_2, A_3, A_4)\left(\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx\right)^{\frac{1}{q}},$$

where the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ *defined by* (2.3) *is the best possible.*

Proof. By reverse Hölder's inequality, and the same way, we have (2.19). If the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ in (2.19) is not the best possible, then there exists a positive constant H(with $H > C_\lambda(A_1, A_2, A_3, A_4)$) such that (2.19) is still valid if we replace $C_\lambda(A_1, A_2, A_3, A_4)$ by H. For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{f}$ and $\widetilde{g}$ as in Theorem 1, then we have

$$(2.20)\qquad H\left(\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{f}^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{g}^q(x)dx\right)^{\frac{1}{q}}$$
$$= H\left(\int_0^\infty x^{-1-\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{-1-\varepsilon}dx\right)^{\frac{1}{q}} = \frac{H}{\varepsilon}.$$

By using (2.7), we find

$$(2.21)\qquad \int_0^\infty\int_0^\infty \frac{\widetilde{f}(x)\,\widetilde{g}(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dxdy$$
$$= \int_1^\infty\left[\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\}+A_4\max\{x^\lambda, y^\lambda\}}dx\right] y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}dy$$
$$< \frac{1}{\varepsilon}\left[C_\lambda(A_1, A_2, A_3, A_4) + \tilde{o}(1)\right].$$

Therefore, we get

$$(2.22)\qquad \frac{1}{\varepsilon}\left[C_\lambda(A_1, A_2, A_3, A_4) + \tilde{o}(1)\right] > \frac{H}{\varepsilon},$$

or

$$(2.23)\qquad C_\lambda(A_1, A_2, A_3, A_4) + \tilde{o}(1) \geq H.$$

For $\varepsilon \to 0^+$, it follows that $C_\lambda(A_1, A_2, A_3, A_4) \geq H$ which contradicts the fact that $H > C_\lambda(A_1, A_2, A_3, A_4)$. Hence, the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ in (2.19) is the best possible. The theorem is proved. □

Remark 2. (i) In Theorem 2, if we take $A_1 = A_2 = 0$ and $\lambda = A_3 = A_4 = 1$, then $C_1(0, 0, 1, 1) = \pi$ and inequality (2.19) reduces to Hardy-Hilbert's inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy > \pi \left(\int_0^\infty x^{\frac{p}{2}-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{\frac{q}{2}-1} g^q(x)dx \right)^{\frac{1}{q}}.$$

(ii) In Theorem 2, if we take $A_1 = A_2 = A_3 = 0$, $\lambda = A_4 = 1$, then $C_1(0, 0, 0, 1) = 4$ and (2.19) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dxdy > 4 \left(\int_0^\infty x^{\frac{p}{2}-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{\frac{q}{2}-1} g^q(x)dx \right)^{\frac{1}{q}}.$$

(iii) In Theorem 2, if we take $A_1 = A_2 = 0$ and $\lambda > 0,\ A_3 = A,\ A_4 = B$, then

$$C_\lambda(0, 0, A, B) = \begin{cases} \dfrac{4}{\lambda\sqrt{AB}} \arctan \sqrt{\dfrac{A}{B}} & \text{, for } A, B > 0, \\[2ex] \dfrac{4}{\lambda B} & \text{, for } A = 0,\ B > 0, \end{cases}$$

and inequality (2.19) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dxdy$$
$$> C_\lambda (A, B) \left(\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx \right)^{\frac{1}{q}}.$$

Theorem 3. *Under the assumption of Theorem 1,*

$$\int_0^\infty y^{\frac{\lambda p}{2}-1} \left[\int_0^\infty \frac{f(x)}{A_1 x^\lambda + A_2 y^\lambda + A_3 \min\{x^\lambda, y^\lambda\} + A_4 \max\{x^\lambda, y^\lambda\}} dx \right]^p dy \tag{2.24}$$
$$< [C_\lambda(A_1, A_2, A_3, A_4)]^p \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx,$$

where the constant factor $[C_\lambda(A_1,A_2,A_3,A_4)]^p$ *is the best possible. Inequalities* (2.10) *and* (2.24) *are equivalent.*

Proof. If we take

$$g(y)=y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dx\right\}^{p-1}, \tag{2.25}$$

then by (2.10) we have

$$\begin{aligned}
&\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(y)\,dy \\
&=\int_0^\infty y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dx\right\}^p dy \\
&=\int_0^\infty \left\{\int_0^\infty \frac{f(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dx\right\} \\
&\quad\times\left\{y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dx\right\}^{p-1}\right\}dy \\
&=\int_0^\infty\int_0^\infty \frac{f(x)\,g(x)}{A_1x^\lambda+A_2y^\lambda+A_3\min\{x^\lambda,y^\lambda\}+A_4\max\{x^\lambda,y^\lambda\}}dxdy \\
&\le C_\lambda(A_1,A_2,A_3,A_4)\left(\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.26}$$

Hence, we obtain

$$\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(y)\,dy \le [C_\lambda(A_1,A_2,A_3,A_4)]^p\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx. \tag{2.27}$$

Thus, by (2.10), both (2.26) and (2.27) keep the form of strict inequalities, then we have (2.24).

By Hölder's inequality, we obtain

$$(2.28)\quad \int_0^\infty \int_0^\infty \frac{f(x)\,g(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dxdy$$

$$= \int_0^\infty \left\{ y^{\frac{\lambda}{2}-\frac{1}{p}} \int_0^\infty \frac{f(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dx \right\}$$

$$\times \left\{ y^{\frac{1}{p}-\frac{\lambda}{2}} g(y) \right\} dy$$

$$\leq \left\{ \int_0^\infty y^{\frac{\lambda p}{2}-\frac{1}{p}} \left\{ \int_0^\infty \frac{f(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dx \right\}^p \right\}^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)\, dy \right)^{\frac{1}{q}}.$$

Therefore, by (2.24) we have (2.10), and inequalities (2.24) and (2.10) are equivalent. If the constant factor in (2.24) is not the best possible, then by (2.28) we can get a contradiction that the constant factor in (2.10) is not the best possible. The theorem is proved. □

Theorem 4. *Under the assumption of Theorem 2,*

$$(2.29)\quad \int_0^\infty y^{\frac{\lambda p}{2}-1} \left[\int_0^\infty \frac{f(x)}{A_1x^\lambda + A_2y^\lambda + A_3\min\{x^\lambda, y^\lambda\} + A_4\max\{x^\lambda, y^\lambda\}}dx \right]^p dy$$

$$> [C_\lambda(A_1, A_2, A_3, A_4)]^p \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx,$$

where the constant factor $[C_\lambda(A_1, A_2, A_3, A_4)]^p$ *is the best possible. Inequalities* (2.19) *and* (2.29) *are equivalent.*

The proof of Theorem 4 is similar to that of Theorem 3, so we omit it.

3. Discrete analogous

Lemma 3. *Suppose that* $0 < \lambda \leq 2$, A_1, A_2, $A_3 \geq 0$, $A_4 > 0$. *Then the weight coefficients* $\varpi_\lambda(A_1, A_2, A_3, A_4, m)$ *and* $\varpi_\lambda(A_1, A_2, A_3, A_4, n)$*, defined, respectively,* *by*

$$\text{(3.1)}\quad \varpi_\lambda(A_1, A_2, A_3, A_4, m) := \sum_{n=1}^{\infty} \frac{m^{\frac{\lambda}{2}} n^{-1+\frac{\lambda}{2}}}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \; (m \in N),$$

$$\text{(3.2)}\quad \varpi_\lambda(A_1, A_2, A_3, A_4, n) := \sum_{m=1}^{\infty} \frac{n^{\frac{\lambda}{2}} m^{-1+\frac{\lambda}{2}}}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \; (n \in N),$$

satisfy the following inequalities:

$$\text{(3.3)}\quad C_\lambda(A_1, A_2, A_3, A_4)\,[1 - \theta_\lambda(A_1, A_2, A_3, A_4, m)] < \varpi_\lambda(A_1, A_2, A_3, A_4, m) < C_\lambda(A_1, A_2, A_3, A_4),$$

$$\text{(3.4)}\quad C_\lambda(A_1, A_2, A_3, A_4)\,[1 - \theta_\lambda(A_1, A_2, A_3, A_4, n)] < \varpi_\lambda(A_1, A_2, A_3, A_4, n) < C_\lambda(A_1, A_2, A_3, A_4),$$

where

$$\theta_\lambda(A_1, A_2, A_3, A_4, r) := \left(\frac{1}{C_1(A_1, A_2, A_3, A_4)}\right) \int_0^{r^{-\lambda}} \frac{t^{-\frac{1}{2}}}{A_1 + A_4 + (A_2 + A_3)\, t} dt = O\left(\frac{1}{r^{\frac{\lambda}{2}}}\right) \in (0,1)\,(r \in N)\,(r \to \infty),$$

and $C_\lambda(A_1, A_2, A_3, A_4)$ *is defined by* (2.3).

Proof. For $0 < \lambda \leq 2$, A_1, A_2, $A_3 \geq 0$, $A_4 > 0$, by Lemma 1 we get

$$\text{(3.5)}\quad \varpi_\lambda(A_1, A_2, A_3, A_4, m) < \int_0^{\infty} \frac{m^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A_1 m^\lambda + A_2 y^\lambda + A_3 \min\{m^\lambda, y^\lambda\} + A_4 \max\{m^\lambda, y^\lambda\}} dy \;\; (m \in N), = \varpi_\lambda(A_1, A_2, A_3, A_4, m) = C_\lambda(A_1, A_2, A_3, A_4).$$

On the other hand, we have

$$\begin{aligned}(3.6)\qquad &\varpi_\lambda(A_1,A_2,A_3,A_4,m)\\ &>\int_0^\infty \frac{m^{\frac{\lambda}{2}}y^{-1+\frac{\lambda}{2}}}{A_1m^\lambda+A_2y^\lambda+A_3\min\{m^\lambda,y^\lambda\}+A_4\max\{m^\lambda,y^\lambda\}}dy\\ &=\frac{1}{\lambda}\int_{m^{-\lambda}}^\infty \frac{t^{\frac{-1}{2}}}{A_1+A_2t+A_3\min\{1,t\}+A_4\max\{1,t\}}dt\\ &=I-\frac{1}{\lambda}\int_0^{m^{-\lambda}}\frac{t^{\frac{-1}{2}}}{A_1+A_4+(A_2+A_3)\,t}dt\\ &=I\,(1-\theta_\lambda(A_1,A_2,A_3,A_4,m))\,,\end{aligned}$$

where $I=\frac{1}{\lambda}C_1(A_1,A_2,A_3,A_4)$ and

$$\begin{aligned}(3.7)\qquad &0<\theta_\lambda(A_1,A_2,A_3,A_4,m)\\ &=\frac{1}{C_1(A_1,A_2,A_3,A_4)}\int_0^{m^{-\lambda}}\frac{t^{\frac{-1}{2}}}{A_1+A_4+(A_2+A_3)\,t}dt<1.\end{aligned}$$

Since

$$(3.8)\qquad \int_0^{m^{-\lambda}}\frac{t^{\frac{-1}{2}}}{A_1+A_4+(A_2+A_3)\,t}dt\le\int_0^{m^{-\lambda}}\frac{t^{\frac{-1}{2}}}{A_1+A_4}dt=\frac{2}{(A_1+A_4)\,m^{\frac{\lambda}{2}}},$$

then $\theta_\lambda(A_1,A_2,A_3,A_4,m)=O\left(\frac{1}{m^{\frac{\lambda}{2}}}\right)$. Therefore, (3.3) is valid. By the symmetry, (3.4) is still valid. The lemma is proved. □

Lemma 4. *If* $p>0\,(p\neq1)\,,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\lambda\le2,\ A_1,\ A_2,\ A_3\ge0,\ A_4>0,$ *and* $0<\varepsilon<\frac{p\lambda}{2}$, *setting*

$$(3.9)\quad L(\varepsilon)\ =\ \sum_{n=1}^\infty\sum_{m=1}^\infty\frac{m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1m^\lambda+A_2n^\lambda+A_3\min\{m^\lambda,n^\lambda\}+A_4\max\{m^\lambda,n^\lambda\}},$$

then for $\varepsilon\to0^+$,

$$\begin{aligned}(3.10)\quad &[C_\lambda(A_1,A_2,A_3,A_4)-o\,(1)]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}<L(\varepsilon)\\ &\qquad<[C_\lambda(A_1,A_2,A_3,A_4)+\widetilde{o}\,(1)]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}.\end{aligned}$$

Proof. By putting $t=\left(\frac{x}{n}\right)^{\lambda}$ and by (3.4), we have

$$
\begin{aligned}
L(\varepsilon) &< \sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1x^{\lambda}+A_2n^{\lambda}+A_3\min\{x^{\lambda},n^{\lambda}\}+A_4\max\{x^{\lambda},n^{\lambda}\}}dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\frac{1}{\lambda}\int_{0}^{\infty}\frac{t^{-\frac{1}{2}-1-\frac{\varepsilon}{p}}}{A_1t+A_2+A_3\min\{t,1\}+A_4\max\{t,1\}}dt\right] \\
&= [C_{\lambda}(A_1,A_2,A_3,A_4)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\ \ (\varepsilon\to 0^{+}).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(3.11)\quad L(\varepsilon) &> \sum_{n=1}^{\infty}\int_{1}^{\infty}\frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A_1x^{\lambda}+A_2n^{\lambda}+A_3\min\{x^{\lambda},n^{\lambda}\}+A_4\max\{x^{\lambda},n^{\lambda}\}}dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\frac{1}{\lambda}\int_{n^{-\lambda}}^{\infty}\frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_1t+A_2+A_3\min\{t,1\}+A_4\max\{t,1\}}dt\right] \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A_1,A_2,A_3,A_4)+\widetilde{o}(1)-\frac{1}{\lambda}\int_{0}^{n^{-\lambda}}\frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{(A_1+A_3)t+(A_2+A_4)}dt\right] \\
&> \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[C_{\lambda}(A_1,A_2,A_3,A_4)+\widetilde{o}(1)]-\frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n}\int_{0}^{n^{-\lambda}}\frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A_2+A_3}dt \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[C_{\lambda}(A_1,A_2,A_3,A_4)+\widetilde{o}(1)]-\frac{1}{\frac{\lambda(A_2+A_3)}{2}-\frac{\varepsilon(A_2+A_3)}{p}}\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\Bigg[C_{\lambda}(A_1,A_2,A_3,A_4)+\widetilde{o}(1)-\frac{1}{\frac{\lambda(A_2+A_3)}{2}-\frac{\varepsilon(A_2+A_3)}{p}} \\
&\qquad\qquad\times\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\Bigg] \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[C_{\lambda}(A_1,A_2,A_3,A_4)-o(1)]\quad\left(\varepsilon\to 0^{+}\right).
\end{aligned}
$$

Thus, inequality (3.10) holds. The lemma is proved. □

Theorem 5. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 2$, $A_1, A_2, A_3 \geq 0$, $A_4 > 0$, $a_n, b_n \geq 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q < \infty$, *then*

$$(3.12)\qquad D := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}}$$
$$< C_\lambda(A_1, A_2, A_3, A_4) \left\{\sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q\right\}^{\frac{1}{q}},$$

where the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ *defined by* (2.3) *is the best possible.*

Proof. By Hölder's inequality, we get

$$(3.13)\qquad D = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left\{\frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}}\right\}^{\frac{1}{p}} \left[\frac{m^{\frac{\left(1-\frac{\lambda}{2}\right)}{q}}}{n^{\frac{\left(1-\frac{\lambda}{2}\right)}{p}}} a_m\right]$$
$$\times \left\{\frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}}\right\}^{\frac{1}{q}} \left[\frac{n^{\frac{\left(1-\frac{\lambda}{2}\right)}{p}}}{m^{\frac{\left(1-\frac{\lambda}{2}\right)}{q}}} b_n\right]$$
$$\leq \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \frac{m^{\left(1-\frac{\lambda}{2}\right)(p-1)}}{n^{1-\frac{\lambda}{2}}} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \frac{n^{\left(1-\frac{\lambda}{2}\right)(q-1)}}{m^{1-\frac{\lambda}{2}}} b_n^q\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{m=1}^{\infty} \varpi_\lambda(A_1, A_2, A_3, A_4, m) m^{p\left(1-\frac{\lambda}{2}\right)-1} a_m^p\right\}^{\frac{1}{p}}$$
$$\left\{\sum_{n=1}^{\infty} \varpi_\lambda(A_1, A_2, A_3, A_4, n) n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q\right\}^{\frac{1}{q}}.$$

Then, by (3.3) and (3.4) we obtain (3.12).

It remains to show that the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ is the best possible, to do that we set for $0 < \varepsilon < \frac{p\lambda}{2}$, $\widetilde{a}_m = m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}$; $\widetilde{b}_n = n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$, by (3.9) we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{A_1m^{\lambda}+A_2n^{\lambda}+A_3\min\{m^{\lambda},n^{\lambda}\}+A_4\max\{m^{\lambda},n^{\lambda}\}}=L(\varepsilon). \tag{3.14}$$

If there exists a constant $0<K\leq C_{\lambda}(A_1,A_2,A_3,A_4)$ such that (3.12) is valid if we replace $C_{\lambda}(A_1,A_2,A_3,A_4)$ by K, then in particular by (3.10) we find

$$\begin{aligned}(3.15)\quad & [C_{\lambda}(A_1,A_2,A_3,A_4)-o(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \\ & < K\left\{\sum_{n=1}^{\infty}n^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}}=K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},\end{aligned}$$

it follows that $C_{\lambda}(A_1,A_2,A_3,A_4)-o(1)<K$ and then $C_{\lambda}(A_1,A_2,A_3,A_4)\leq K$ $(\varepsilon\to 0^+)$. Therefore, $K=C_{\lambda}(A_1,A_2,A_3,A_4)$ the best constant factor in (3.12). The theorem is proved. □

Remark 3. (i) In Theorem 5, if we take $A_1=A_2=0$ and $\lambda=A_3=A_4=1$, then $C_1(0,0,1,1)=\pi$, and inequality (3.12) reduces to Hardy-Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{m+n}<\pi\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

(ii) In Theorem 5, if we take $A_1=A_2=A_3=0$, $\lambda=A_4=1$, then $C_1(0,0,0,1)=4$ and (3.12) reduces to Hardy-Hilbert's type inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m,n\}}<4\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

(iii) In Theorem 5, if we take $A_1=A_2=0$ and $\lambda>0,\ A_3=A,\ A_4=B$, then

$$C_{\lambda}(0,0,A,B)=\begin{cases}\dfrac{4}{\lambda\sqrt{AB}}\arctan\sqrt{\dfrac{A}{B}} & ,\text{for } A,B>0,\\[2ex] \dfrac{4}{\lambda B} & ,\text{for } A=0,\ B>0,\end{cases}$$

and inequality (3.12) reduces to

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{A\min\{m^{\lambda},n^{\lambda}\}+B\max\{m^{\lambda},n^{\lambda}\}}\\ &\quad<C_{\lambda}(A,B)\left(\sum_{n=1}^{\infty}n^{p\left(1-\frac{\lambda}{2}\right)-1}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^2\right)^{\frac{1}{2}}.\end{aligned}$$

Theorem 6. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \le 2$, A_1, A_2, $A_3 \ge 0$, $A_4 > 0$, a_n, $b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q < \infty$, *then*

$$\text{(3.16)} \qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}}$$
$$> C_\lambda(A_1, A_2, A_3, A_4) \left\{ \sum_{n=1}^{\infty} [1 - \theta_\lambda(A_1, A_2, A_3, A_4, n)]\, n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p \right\}^{\frac{1}{p}}$$
$$\times \left\{ \sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q \right\}^{\frac{1}{q}},$$

where the constant factor $C_\lambda(A_1, A_2, A_3, A_4)$ *defined in* (2.3) *is the best possible.*

Proof. By reverse Hölder's inequality, we get

$$\text{(3.17)} \quad D = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}}$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left\{ \frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \right\}^{\frac{1}{p}} \times \left[\frac{m^{\frac{\left(1-\frac{\lambda}{2}\right)}{q}}}{n^{\frac{\left(1-\frac{\lambda}{2}\right)}{p}}} a_m \right]$$
$$\left\{ \frac{1}{A_1 m^\lambda + A_2 n^\lambda + A_3 \min\{m^\lambda, n^\lambda\} + A_4 \max\{m^\lambda, n^\lambda\}} \right\}^{\frac{1}{q}} \times \left[\frac{n^{\frac{\left(1-\frac{\lambda}{2}\right)}{p}}}{m^{\frac{\left(1-\frac{\lambda}{2}\right)}{q}}} b_n \right]$$
$$\ge \left\{ \sum_{m=1}^{\infty} \varpi_\lambda(A_1, A_2, A_3, A_4, m) m^{p\left(1-\frac{\lambda}{2}\right)-1} a_m^p \right\}^{\frac{1}{p}}$$
$$\times \left\{ \sum_{n=1}^{\infty} \varpi_\lambda(A_1, A_2, A_3, A_4, n) n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q \right\}^{\frac{1}{q}}.$$

Then by (3.3) and (3.4), in view of $q < 0$, we have (3.16). For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{a}_m = m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}\ (m, n \in N)$. If there exists a constant $K \ge C_\lambda(A_1, A_2, A_3, A_4)$ such that (3.16) is still valid if we replace $C_\lambda(A_1, A_2, A_3, A_4)$ by

K, then in particular by (3.9) and (3.10) we find

$$(3.18)\qquad \left[C_\lambda(A_1,A_2,A_3,A_4)+\widetilde{o}(1)\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} > L(\varepsilon)$$

$$> K\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(A_1,A_2,A_3,A_4,n)\right] n^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}}$$

$$= K\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{\frac{1}{q}}$$

$$= K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}},$$

it follows

$$(3.19)\qquad C_\lambda(A_1,A_2,A_3,A_4)+\widetilde{o}(1) > K\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}}.$$

Hence, if $\varepsilon\to 0^+$, we get $C_\lambda(A_1,A_2,A_3,A_4) > K$. Thus, $K = C_\lambda(A_1,A_2,A_3,A_4)$ is the best constant factor in (3.16). □

Remark 4. (i) In Theorem 6, if we take $A_1 = A_2 = 0$ and $\lambda = A_3 = A_4 = 1$, then $C_1(0,0,1,1) = \pi$, and inequality (3.16) reduces to Hardy-Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} > \pi\left\{\sum_{n=1}^{\infty}\left[1-\frac{2}{\pi}\arctan\frac{1}{n^{\frac{1}{2}}}\right] n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

(ii) In Theorem 6, if we take $A_1 = A_2 = A_3 = 0$, $\lambda = A_4 = 1$, then $C_1(0,0,0,1) = 4$ and (3.12) reduces to Hardy-Hilbert's type inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} > 4\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{2n^{\frac{1}{2}}}\right] n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

(iii) In Theorem 6**,** if we take $A_1 = A_2 = 0$ and $\lambda > 0$, $A_3 = A$, $A_4 = B$, then

$$C_\lambda(0,0,A,B) = \begin{cases} \dfrac{4}{\lambda\sqrt{AB}}\arctan\sqrt{\dfrac{A}{B}} & ,\text{for } A,B>0,\\[2ex] \dfrac{4}{\lambda B} & ,\text{for } A=0,\ B>0,\end{cases}$$

and inequality (3.16) reduces to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}$$
$$> C_\lambda(A,B)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(A,B,n)]\,n^{p\left(1-\frac{\lambda}{2}\right)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q\right\}^{\frac{1}{q}}.$$

Theorem 7. *Under the assumption of Theorem 5,*

$$\sum_{n=1}^{\infty}n^{\frac{\lambda p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A_1m^\lambda+A_2n^\lambda+A_3\min\{m^\lambda,n^\lambda\}+A_4\max\{m^\lambda,n^\lambda\}}\right]^p \tag{3.20}$$
$$< [C_\lambda(A_1,A_2,A_3,A_4)]^p\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p,$$

where the constant factor $[C_\lambda(A_1,A_2,A_3,A_4)]^p$ *is the best possible. Inequalities* (3.12) *and* (3.20) *are equivalent.*

Proof. Setting

$$b_n = n^{\frac{\lambda p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A_1m^\lambda+A_2n^\lambda+A_3\min\{m^\lambda,n^\lambda\}+A_4\max\{m^\lambda,n^\lambda\}}\right]^{p-1}, \tag{3.21}$$

we get

$$\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A_1m^\lambda+A_2n^\lambda+A_3\min\{m^\lambda,n^\lambda\}+A_4\max\{m^\lambda,n^\lambda\}}. \tag{3.22}$$

By (3.12) and using the same method of Theorem 3, we obtain (3.20). We may show that the constant factor in (3.20) is the best possible and inequality (3.12) is equivalent to (3.20). □

Theorem 8. *Under the assumption of Theorem 6,*

$$\sum_{n=1}^{\infty}n^{\frac{\lambda p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A_1m^\lambda+A_2n^\lambda+A_3\min\{m^\lambda,n^\lambda\}+A_4\max\{m^\lambda,n^\lambda\}}\right]^p \tag{3.23}$$
$$> [C_\lambda(A_1,A_2,A_3,A_4)]^p\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p,$$

where the constant factor $[C_\lambda(A_1,A_2,A_3,A_4)]^p$ *is the best possible. Inequalities* (3.16) *and* (3.23) *are equivalent.*

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