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The Signless Laplacian Spectral Radius for Bicyclic Graphs with k Pendant Vertices

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Abstract. In this paper, we study the signless Laplacian spectral radius of bicyclic graphs with given number of pendant vertices and characterize the extremal graphs.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. The order of a graph is the cardinality of its vertex set. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of graph G, where $D(G) = diag(d_u, u \in V)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. It is known that $Q(G)$ is a positive semi-definite matrix, we call this matrix the *Q-matrix* and its largest eigenvalue is denoted by $\mu(G)$ or μ for simplicity. For the background on the Laplacian eigenvalues of a graph, the reader is referred to [20] and the references therein.

It is well known that the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix*, and $\lambda(G) \le \mu(G)$ (see, for example, [14]), the equality holds if and only if G is bipartite.

A bicyclic graph is a connected graph with vertex number equal to edge number minus one. A *pendant path* in a connected graph is a path attached to a connected graph. For $S \subset V$, $G[S]$ denotes the subgraph induced by S. For $u \in V$, d_u is the degree of u, $N(u)$ is the neighbor set of u.

Denote by C_n and P_n the cycle and the path on n vertices, respectively. We will use $\mathcal{B}_n(k)$ to denote the set of bicyclic graphs on n vertices with k pendant vertices. Let C_p and C_q be two vertex disjoint cycles. Suppose that v_1 is a vertex of C_p and v_t is a vertex of C_q. Joining v_1 and v_t by a path $v_1v_2\cdots v_t$ of length $t-1$, where $t \ge 1$ and $t = 1$ means identifying v_1 with v_t, the resulting graph, denoted by $B(p, t, q)$. The set of bicyclic graphs obtained from $B(p, t, q)$ by attaching trees is denoted by $\mathcal{B}_n^+(k)$. Let P_{t+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $t, p, q \ge 1$ and at most one of them is 1. Identifying the three initial vertices and

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terminal vertices of them, respectively, the resulting graph is denoted by $P(t,p,q)$. The set of bicyclic graphs obtained from $P(t,p,q)$ by attaching trees is denoted by $\mathcal{B}_n^{++}(k)$. Obviously, $\mathcal{B}_n(k) = \mathcal{B}_n^{+}(k) \cup \mathcal{B}_n^{++}(k)$. For other notations in graph theory, we follow [1].

The Laplacian spectral radius of unicyclic graphs is well studied. In [17], the upper and lower bounds for Laplacian spectral radius of unicyclic graphs were studied. In [13], the author characterized the maximum Laplacian spectral radius of unicyclic graphs with fixed girth. In [18], the Laplacian spectral radius of bicyclic graphs were studied. In [16], the spectral radius of bicyclic graphs with given number of pendant vertices were studied. The study of the signless Laplacian spectral radius attracts researchers attention just recently. In [7], Fan et.al. studied the signless Laplacian spectral radius of bicyclic graph with fixed order. In [6], the authors discussed the smallest eigenvalue of $Q(G)$ as a parameter reflecting the nonbipartiteness of the graph G. Some other use of the signless Laplacian can be found in [12], [3]. For a survey paper of this new direction, see [5].

In this paper, we study the Laplacian spectral radius of bicyclic graphs with given pendant vertices. We also characterize the extremal graphs.

2. Some lemmas

Lemma 2.1([19]). *Let G be a connected graph and u, v be two vertices of G. Suppose $v_1, v_2, \cdots, v_s \in N(v) \setminus (N(u) \cup \{u\})$ $(1 \le s \le d_v)$, and G^* is the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \le i \le s)$. Let $X = (x_1, x_2, \cdots, x_n)^t$ be the principal eigenvector of $Q(G)$, where x_i corresponds to v_i $(1 \le i \le n)$. If $x_u \ge x_v$, then $\mu(G) < \mu(G^*)$.*

We generalize Lemma 2.1 next.

Lemma 2.2. *Let G be a connected graph of order n and S, T be its two disjoint nonempty vertex subset. Suppose $S = \{v_1, v_2, \cdots, v_s\}$ and the neighbors of v_i in T are $v_{i1}, v_{i2}, \cdots, v_{il_i}$ $(l_i \ge 1,\ i = 1, 2, \cdots, s)$. Let $X = (x_{v_1}, x_{v_2}, \cdots, x_{v_n})^t$ be the Perron vector of $Q(G)$, where x_{v_k} corresponds to the vertex v_k $(1 \le k \le n)$. Suppose $x_{v_1} = \max\{x_{v_i} : i = 1, 2, \cdots, s\}$. Let H be the graph obtained from G by deleting edges v_iv_{ij} and adding the edges v_1v_{ij} $(i = 2, 3, \cdots, s; j = 1, 2, \cdots, l_i)$. Then we have $\mu(G) < \mu(H)$.*

Proof. The proof is similar to that in [11], we present it here for completeness. Obviously,

$$\begin{aligned} X^t(Q(H) - Q(G))X &= X^t(D(H) + A(H) - D(G) - A(G))X \\ &= \sum_{i=2}^{s}\sum_{j=1}^{l_i}\Big((x_{v_1} + x_{v_{ij}})^2 - (x_{v_i} + x_{v_{ij}})^2\Big) \end{aligned}$$

$$= \sum_{i=2}^{s}\sum_{j=1}^{l_i}\Big((x_{v_1}^2 - x_{v_i}^2) + 2x_{v_{ij}}(x_{v_1} - x_{v_i})\Big)$$
$$\geq 0.$$

Thus,

$$\mu(H) = \max_{||Y||=1} Y^t Q(H) Y \geq X^t Q(H) X \geq X^t Q(G) X = \mu(G).$$

If $\mu(H) = \mu(G)$, then the inequalities above should be equalities. So

$$\mu(H) = X^t Q(H) X = X^t Q(G) X = \mu(G).$$

Thus, $\mu(H)X = Q(H)X$ and $Q(G)X = \mu(G)X$. Thus,

$$\mu(H)x_{v_1} = d_H(v_1) + \sum_{w \in N_H(v_1)} x_w.$$

$$\mu(G)x_{v_1} = d_G(v_1) + \sum_{w \in N_G(v_1)} x_w.$$

Since $d_H(v_1) \geq d_G(v_1)$, $\sum_{w\in N_H(v_1)} x_w > \sum_{w \in N_G(v_1)} x_w$, so we have $\mu(H)x_{v_1} > \mu(G)x_{v_1}$. Since $x_{v_1} > 0$, hence $\mu(H) > \mu(G)$, a contradiction. □

Now, we consider the graph G_{uv} obtained from the connected graph G by subdividing the edge uv, that is, by replacing uv with edges uw, vw, where w is an additional vertex. We call the following two types of paths *internal paths* : (a) A sequence of vertices $v_0, v_1, \cdots, v_{k+1}$ $(k \geq 2)$, where $v_0, v_1, \cdots, v_k$ are distinct, $v_{k+1} = v_0$ of degree at least 3, $d_{v_i} = 2$ for $i = 1, \cdots, k$, and v_{i-1} and v_i $(i = 1, \cdots, k+1)$ are adjacent. (b) A sequence of distinct vertices $v_0, v_1, \cdots, v_{k+1} (k \geq 0)$ such that v_{i-1} and v_i $(i = 1, \cdots, k+1)$ are adjacent, $d_{v_0} \geq 3$, $d_{v_{k+1}} \geq 3$ and $d_{v_i} = 2$ whenever $1 \leq i \leq k$.

Lemma 2.3([8], [2]). *Let G be a connected graph and uv be some edge on the internal path of G as we defined above. If we subdivide uv, that is, substitute it by uw, wv, with the new vertex w, and denote the new graph by G_{uv}, then $\mu(G_{uv}) < \mu(G)$.*

Lemma 2.4. *Let G be a connected graph. Suppose v_1 and v_2 are vertices each of degree at least three and v_1v_2 is an edge of G. Let G' be the connected graph obtained form G by contracting v_1v_2 (i.e., deleting the edge and identifying v_1 and v_2). Then $\mu(G) < \mu(G')$.*

Proof. The proof of the result is similar to Theorem 4.11 in [14] and we omit it. □

Lemma 2.5([2]). *Suppose G is a nontrivial simple connected graph. Let u be a* *vertex of G. For nonnegative integers k, l, let $G(k, l)$ denote the graph obtained from G by adding pendant paths of length k and l at u. If $k \geq l \geq 1$, then*

$$\mu(G(k, l)) > \mu(G(k+1, l-1)).$$

Lemma 2.6. *Let G be a connected graph and P be a pendant path in G. Suppose e is an edge in P and G' is the graph obtained from G by subdividing e, then we have $\mu(G) < \mu(G')$.*

Proof. Since G is a proper subgraph of G', we have $\mu(G) < \mu(G')$. □

3. Main results

Suppose the vertices of the graphs $B(4,1,4)$, $B(4,1,3)$, $B(3,1,3)$ are labeled as in Fig. 1.

Let B_1 be the graph on n vertices obtained from $B(4,1,4)$ by attaching k paths of almost equal lengths (i.e., the lengths of these paths differ in size by at most one) at u; B_2 be the graph obtained from $B(4,1,4)$ by attaching k paths of almost equal lengths at u_4; B_3 be the graph obtained from $B(4,1,4)$ by attaching k paths of almost equal lengths at u_6.

Let C_1 be the graph on n vertices obtained from $B(4,1,3)$ by attaching k paths of almost equal lengths at u; C_2 be the graph on n vertices obtained from $B(4,1,3)$ by attaching k paths of almost equal lengths at u_2; C_3 be the graph obtained from $B(4,1,3)$ by attaching k paths of almost equal lengths at u_5; C_4 be the graph obtained from $B(4,1,3)$ by attaching k paths of almost equal lengths at u_4.

Let D_1 be the graph on n vertices obtained from $B(3,1,3)$ by attaching k paths of almost equal lengths at u; D_2 be the graph on n vertices obtained from $B(3,1,3)$ by attaching k paths of almost equal lengths at u_2.

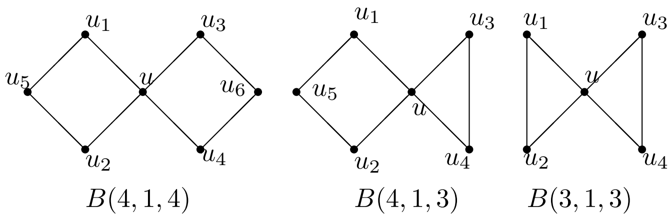


Fig.1

Theorem 3.1. *Let G be a bicyclic graph in $\mathcal{B}_n^+(k)$. Then $\mu(G) \leq \mu(D_1)$. The equality holds if and only if $G \cong D_1$.*

Proof. Let G be a bicyclic graph in $\mathcal{B}_n^+(k)$. Comparing the eigencomponents of the vertices on $B(p, l, q)$, by Lemma 2.2, identifying the roots of the trees attached to $B(p, l, q)$, the signless Laplacian spectral radius increases. Next, by Lemmas 2.3, 2.4, contracting the internal path and Lemma 2.5 to make all the pendant paths having almost equal lengths, the signless Laplacian spectral radius again increases. At last, subdividing the pendant paths several times if necessary to keep the order of graphs unchanged, by Lemma 2.6, $\mu(G)$ increases.

So we conclude the following three cases hold.

(1). If $p \geq q \geq 4$ and $l \geq 1$, then $\mu(G) \leq max\{\mu(B_1), \mu(B_2), \mu(B_3)\}$.

(2). If $p = 4, q = 3$ and $l \geq 1$, then $\mu(G) \leq max\{\mu(C_1), \mu(C_2), \mu(C_3), \mu(C_3)\}$.

(3). If $p = q = 3$, $l \geq 1$, then $\mu(G) \leq max\{\mu(D_1), \mu(D_2)\}$.

For case (1), we claim that $max\{\mu(B_1), \mu(B_2), \mu(B_3)\} = \mu(B_1)$.

In fact, for B_2, consider the eigencomponents corresponding to u and u_4, say, x_u and x_{u_4}. If $x_u \geq x_{u_4}$, by Lemma 2.1, removing the k pendant paths to u, we have $\mu(B_2) < \mu(B_1)$. If $x_u < x_{u_4}$, by Lemma 2.1, deleting edges uu_1, uu_2 and adding edges u_4u_1, u_4u_2, we also have $\mu(B_2) < \mu(B_1)$. Similarly, for B_3, consider the eigencomponents corresponding to u and u_6, we have $\mu(B_3) < \mu(B_1)$.

For case (2), we claim that $max\{\mu(C_1), \mu(C_2), \mu(C_3), \mu(C_4)\} = \mu(C_1)$.

In fact, for C_2, consider the eigencomponents corresponding to u and u_2, say, x_u and x_{u_2}. If $x_u \geq x_{u_2}$, by Lemma 2.1, removing the k pendant paths to u, we have $\mu(C_2) < \mu(C_1)$. If $x_u < x_{u_2}$, by Lemma 2.1, deleting edges uu_3, uu_4 and adding edges u_2u_3, u_2u_4, we also have $\mu(C_2) < \mu(C_1)$. Similarly, for C_3, consider the eigencomponents corresponding to u and u_6, we have $\mu(C_3) < \mu(C_1)$; for C_4, consider the eigencomponents corresponding to u and u_4, we have $\mu(C_4) < \mu(C_1)$.

For case (3), we claim that $max\{\mu(D_1), \mu(D_2)\} = \mu(D_1)$.

This is similar to the above two cases.

At last, we claim that $max\{\mu(B_1), \mu(C_1), \mu(D_1)\} = \mu(D_1)$.

In fact, for C_1, by Lemma 2.3, contracting edge u_1u_5 and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph D_1 and $\mu(C_1)\} < \mu(D_1)$.

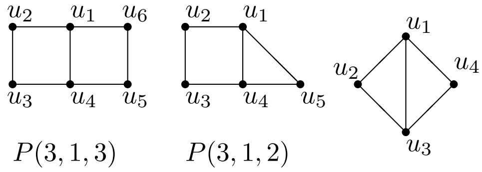
For B_1, contracting edge u_1u_5, u_3u_6 and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph D_1 and $\mu(B_1)\} < \mu(D_1)$. □

Suppose the vertices of the graphs $P(3,1,3)$, $P(3,1,2)$, $P(2,1,2)$ are labeled as in Fig. 2.

Let E_1 be the graph obtained from $P(2,1,2)$ by attaching k paths of almost equal lengths at u_3; E_2 be the graph obtained from $P(2,1,2)$ by attaching k paths of almost equal lengths at u_4.

Let F_1 be the graph on n vertices obtained from $P(3,1,2)$ by attaching k paths of almost equal lengths at u_3; F_2 be the graph on n vertices obtained from $P(3,1,2)$ by attaching k paths of almost equal lengths at u_4; F_3 be the graph obtained from $P(3,1,2)$ by attaching k paths of almost equal lengths at u_5.

Let G_1 be the graph on n vertices obtained from $P(3,1,3)$ by attaching k paths of almost equal lengths at u_4; G_2 be the graph on n vertices obtained from $P(3,1,3)$ by attaching k paths of almost equal lengths at u_3.



$P(3,1,3)$ $P(3,1,2)$ $P(2,1,2)$

Fig.2

Theorem 3.2. *Let G be a bicyclic graph in $\mathcal{B}_n^{++}(k)$. Then $\mu(G) \leq \mu(E_1)$. The equality holds if and only if $G \cong E_1$.*

Proof. Similar as in Theorem 3.1, we conclude that the following three cases holds.

(1). If $p = l = 2$ and $q = 1$, then $\mu(G) \leq max\{\mu(E_1), \mu(E_2)\}$.

(2). If $p = 3$, $l = 2$ and $q = 1$ or 2, then $\mu(G) \leq max\{\mu(F_1), \mu(F_2), \mu(F_3)\}$.

(3). If $p \geq l \geq 3$, $q \geq 1$, then $\mu(G) \leq max\{\mu(G_1), \mu(G_2)\}$.

For case (1), we claim that $max\{\mu(E_1), \mu(E_2)\} = \mu(E_1)$

In fact, in E_2, just consider the eigencomponents of u_3 and u_4, by Lemma 2.1, we can get the claim.

For case (2), we claim that $max\{\mu(F_1), \mu(F_2), \mu(F_3)\} = \mu(F_2)$.

In fact, in F_1, just consider the eigencomponents of u_3 and u_1, by Lemma 2.1, we get $\mu(F_1) \leq \mu(F_2)$; in F_3, consider the eigencomponents of u_1 and u_5, by Lemma 2.1, we get $\mu(F_3) \leq \mu(F_2)$, as claimed.

For case (3), we claim that $max\{\mu(G_1), \mu(G_2)\} = \mu(G_1)$.

In fact, in G_2, just consider the eigencomponents of u_3 and u_1, by Lemma 2.1, we can get the claim.

At last, we claim that $max\{\mu(E_1), \mu(F_2), \mu(G_1)\} = \mu(E_1)$.

In fact, for F_2, by Lemma 2.3, contracting edge u_2u_3 and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph E_1 and $\mu(F_2)\} < \mu(E_1)$.

For G_1, contracting edge u_2u_3, u_5u_6 and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph E_1 and $\mu(G_1)\} < \mu(E_1)$. □

Lemma 3.3([4]). *Let G be a graph on n vertices with at least one edge and the maximum degree of G be Δ. Then $\mu(G) \geq \Delta + 1$. The equality holds if and only if G is a star.*

Lemma 3.4([9]). *For a connected graph G, we have $\mu(G) \leq \max\{d_u + m_u : u \in V(G)\}$, where m_u satisfies $d_u m_u = \sum_{vu \in E(G)} d_v$. The equality holds if and only if G is regular or semiregular bipartite.*

Theorem 3.5. *Let G be a bicyclic graph in $\mathcal{B}_n(k)$. Then $\mu(G) \leq \mu(D_1)$, the equality holds if and only if $G = D_1$.*

Proof. By Theorems 3.1, 3.2, we have $\mu(G) \leq \max\{\mu(D_1), \mu(E_1)\}$. For D_1, by Lemma 3.3, we have $\mu(D_1) \geq k + 5$. By Lemma 3.4, $\mu(E_1) < k + 5$. This implies the result. □

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