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Skew Difference Algebras

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Abstract. We modify the definition of difference algebra given by J. Meng to obtain a structure which is a directoid with sectional switching involutions with respect to the given partial order. Moreover, we show that this is a representation of our skew difference algebras because every such directoid can be converted into a skew difference algebra.

The concept of a difference algebra was introduced by J. Meng [5] in the sake to axiomatize structures useful in non-classical propositional logics in general and their implicational reducts in particular. A number of examples of these algebras is presented in [5] and [6]. The original Meng's definition is as follows

Definition 1. A structure $(A; *, \leq, 0)$ with a binary operation $*$, a nullary operation 0 and a binary relation $\leq$ is called a *difference algebra* if it satisfies the axioms

(D1) $(A; \leq)$ is a poset;

(D2) $x \leq y$ implies $x * z \leq y * z$;

(D3) $(x * y) * z \leq (x * z) * y$;

(D4) $0 \leq x * x$;

(D5) $x \leq y$ if and only if $x * y \leq 0$.

However, this structure has a rather weak properties as a poset. To improve this, a concept of the so-called representable difference algebra was introduced in [2] in the sake to obtain a semilattice structure. However, it turns out that the concept of a representable difference algebra is too strong. It was shown recently [1] that this concept coincides with a commutative BCK-algebra. The reason is that some of axioms are rather strong and they cannot be used together. In particular, the axiom (D3) (usually called the *exchange axiom*) is too strong and we are wonder what will happen if it would be deleted. However, (D3) has several nice consequences which

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should be used for our considerations. Hence, we replace (D3) by one consequence of it and the list of axioms complete by the so-called *quasi-commutativity axiom*

$$x * (x * y) = y * (y * x).$$

Then we are able to show that the modified difference algebra (called a skew difference algebra in the sequel) has a nice ordered structure (it is a commutative directoid) but it is not a semilattice and hence not a BCK-algebra. The definition is as follows:

Definition 2. By a *skew difference algebra* is meant a structure $\mathcal{S} = (S; *, \leq, 0)$ where $*$ is a binary operation, 0 is a nullary operation and $\leq$ is a binary relation such that

(S1) $(S; \leq)$ is a poset with the least element 0;

(S2) $x * x = 0$;

(S3) $x \leq y$ if and only if $x * y = 0$;

(S4) $(x * y) * x = 0$;

(S5) $x * (x * y) = y * (y * x)$.

Hence, every skew difference algebra satisfying (D3) becomes a difference algebra with the least element 0 and satisfying (S5). Conversely, if $\mathcal{A}$ is a difference algebra such that 0 is its least element and satisfying (S5) then $\mathcal{A}$ is a skew difference algebra.

Further, we need several concepts which were already used for semilattices and lattices when characterizing MV-algebras and their modifications by the author and his collaborators, see e.g. [3].

Definition 3. Let $(S; \leq, 0)$ be a poset with a least element 0. For $a \in S$, the interval $[0, a]$ is called a *section*. A mapping $f_a : [0, a] \to [0, a]$ is called a *sectional mapping*. Instead of $f_a(x)$, we will write briefly x^a. A sectional mapping is a *switching involution* if $x^{aa} = x$ for each $x \in [0, a]$ and $a^a = 0$, $0^a = a$. An ordered set $(S; \leq, 0)$ is said to have *sectional switching involutions* if for each $a \in S$ there exists a sectional switching involution $x \mapsto x^a$ on $[0, a]$.

The following concept was introduced by J. Ježek and R. Quackenbush [4].

Definition 4. By a *commutative directoid* is meant a grupoid $(A; \sqcap)$ satisfying the identities

(D1) $x \sqcap x = x$;

(D2) $x \sqcap y = y \sqcap x$;

(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

By a *commutative directoid with* **0** is meant a directoid $(A; \sqcap)$ where $0 \in A$ and satisfying

(D4) $x \sqcap 0 = 0$.

The following facts can be taken from [4]:

(I) Let $(A; \sqcap)$ be a commutative directoid. Define $\leq$ by the rule

$$x \leq y \quad \text{if and only if} \quad x \sqcap y = x.$$

Then $(A; \leq)$ is a poset. If $(A; \sqcap)$ is a commutative directoid with 0 then $0 \leq x$ for each $x \in A$.

(II) Let $(A; \leq)$ be an arbitrary ordered set with a least element 0. Denote by $L(x, y) = \{a \in A;\, a \leq x \text{ and } a \leq y\}$ the set of common lower bounds of x, y. Of course, $L(x, y) \neq \emptyset$ since $0 \in L(x, y)$. Let φ be an arbitrary mapping $\varphi : A \times A \to A$ satisfying the following

(a) $\varphi(x, y) \in L(x, y)$ and $\varphi(x, y) = \varphi(y, x)$;

(b) if $x \leq y$ then $\varphi(x, y) = x$.

Taking $x \sqcap y = \varphi(x, y)$, the derived structure $(A; \sqcap)$ is a commutative directoid with 0.

The following lemma will be useful for our computations.

Lemma 1. *Let* $\mathcal{S} = (S; *, \leq, 0)$ *be a skew difference algebra. Then*

(i) $0 * x = 0$;

(ii) $y * (y * (y * x)) = y * x$;

(iii) $x * 0 = x$.

Proof. (i) follows immediately from (S3) and the fact that $0 \leq x$ for each $x \in S$.
Prove (iii): Applying (S5) and (S2), we get

$$x * (x * 0) = 0 * (0 * x) = 0 * 0 = 0$$

due to (i). Hence, $x \leq x * 0$. However, (S4) yields $(x * 0) * x = 0$ whence $x * 0 \leq x$. Thus $x * 0 = x$.
Prove (ii): By (S5), (S4) and (iii) we infer

$$y * (y * (y * x)) = (y * x) * ((y * x) * y) = (y * x) * 0 = y * x. \qquad \square$$

Theorem 1. *Let* $\mathcal{S} = (S; *, \leq, 0)$ *be a skew difference algebra. Define* $x \sqcap y = x * (x * y)$ *as a term operation. Then* $(S; \sqcap)$ *is a commutative directoid with* 0.

Proof. By (S4) we get $(x * (x * y)) * x = 0$ thus $x * (x * y) \leq x$. Using (S5), we conclude $x * (x * y) = y * (y * x) \leq y$, hence $x * (x * y) \in L(x, y)$. Suppose now $x \leq y$. By (S3) we have $x * y = 0$ and, using (iii) of Lemma 1, we compute

$$x * (x * y) = x * 0 = x.$$

Hence, the mapping $\varphi(x, y) = x * (x * y)$ satisfies the afore mentioned conditions (a), (b) of (II) thus $(S; \sqcap)$ for $x \sqcap y = x * (x * y)$ is a commutative directoid. Moreover, $x \sqcap 0 = x * (x * 0) = x * x = 0$ thus this directoid is with 0. □

In the sequel, the commutative directoid $(S; \sqcap)$ reached in Theorem 1 will be referred as the induced directoid of $\mathcal{S} = (S; *, \leq, 0)$.

Lemma 2. *Let $\mathcal{S} = (S; *, \leq, 0)$ be a skew difference algebra and $(S; \sqcap)$ the induced directoid. Then the induced order of $(S; \sqcap)$ coincides with $\leq$.*

Proof. Assume $x \leq y$. Then $x * y = 0$ and hence $x \sqcap y = x * (x * y) = x * 0 = x$. Conversely, let $x \sqcap y = x$. Then $x * (x * y) = x$ and hence (by (S2) and (ii) of Lemma 1) $x * y = x * (x * (x * y)) = x * x = 0$ whence $x \leq y$. □

Theorem 2. *Let $\mathcal{S} = (S; *, \leq, 0)$ be a skew difference algebra. For $a \in S$ and every $x \in [0, a]$ define $x^a = a * x$. Then the mapping $x \mapsto x^a$ is a sectional switching involution on the section $[0, a]$ for each $a \in S$.*

Proof. Let $x \in [0, a]$. Then $x \leq a$. Due to (S3) we have $x^a * a = (a * x) * a = 0$ thus $x^a \leq a$, i.e. the mapping $x \mapsto x^a$ is a sectional mapping on the section $[0, a]$. Further, using the induced directoid, we obtain $x^{aa} = a * (a * x) = a \sqcap x = x$ and $a^a = a * a = 0, 0^a = a * 0 = a$, i.e. it is a switching involution on $[0, a]$. □

Corollary. *Let $\mathcal{S} = (S; *, \leq, 0)$ be a skew difference algebra. Define $x \sqcap y = x * (x * y)$ and for $x \leq a$, $x^a = a * x$. Then $(S; \sqcap)$ is a commutative directoid with 0 and with sectional switching involutions.*

Our next goal is to prove the converse.

Theorem 3. *Let $(S; \sqcap)$ be a commutative directoid with 0 and with sectional switching involutions. Define $x * y = (x \sqcap y)^x$. Let $\leq$ be the induced order of $(S; \sqcap)$. Then $\mathcal{S} = (S; *, \leq, 0)$ is a skew difference algebra.*

Proof. At first, $x \sqcap y \leq x$ thus $x \sqcap y \in [0, x]$ and hence the operation $x * y = (x \sqcap y)^x$ is correctly defined. As already mentioned in (I), $(S; \leq)$ is a poset with the least element 0. We have to prove (S2)–(S5).

(S2): $x * x = (x \sqcap x)^x = x^x = 0$.

(S3): Assume $x \leq y$. Then $x \sqcap y = x$ and hence $x * y = (x \sqcap y)^x = x^x = 0$. Conversely, if $x * y = 0$, then $(x \sqcap y)^x = 0$ and, since the mapping $z \mapsto z^x$ is an involution (and hence bijection) which is switching, we conclude $x \sqcap y = x$. Hence $x \leq y$.

(S4): Since $(x \sqcap y)^x \in [0, x]$, we have $x \leq (x \sqcap y)^x$ and hence $(x \sqcap y)^x \sqcap x = (x \sqcap y)^x$. Therefore $(x * y) * x = ((x \sqcap y)^x \sqcap x)^{(x \sqcap y)^x} = ((x \sqcap y)^x)^{(x \sqcap y)^x} = 0$.

(S5): Since the operation $\sqcap$ is commutative, we have $x * (x * y) = x \sqcap y = y \sqcap x = y * (y * x)$. $\square$

Example 1. We can illustrate the procedure described in Theorem 3 by the following example. Let $A = \{0, a, b, c, d, 1\}$ and $(A; \sqcap)$ be a commutative directoid as shown in Fig. 1.

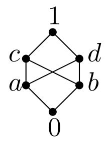

Fig. 1

Let the operation $\sqcap$ is defined as follows: if $x \wedge y$ exists then $x \sqcap y = x \wedge y$ and $c \sqcap d = a$. Sectional switching mappings are determined (in non-singleton sections) as follows.
Section [0,1]: $a^1 = d$, $d^1 = a$, $b^1 = c$, $c^1 = b$, $0^1 = 1$, $1^1 = 0$.
Section [0,c]: $c^c = 0$, $0^c = c$, $a^c = b$, $b^c = a$.
Section [0,d]: $d^d = 0$, $0^d = d$, $a^d = a$, $b^d = b$.
Section [0,a]: $a^a = 0$, $0^a = a$.
Section [0,b]: $b^b = 0$, $0^b = b$.

Then $(A; \sqcap)$ is a commutative directoid with 0 and with sectional switching involutions. Define $x * y = (x \sqcap y)^x$. Then $*$ is given by the table

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	0	0	0
b	b	b	0	0	0	0
c	c	b	a	0	b	0
d	d	a	b	a	0	0
1	1	d	c	b	a	0

One can easily check that $(A; *, \leq, 0)$ is really a skew difference algebra.

In the remaining part, we can exhibit two very characteristic examples showing among other things that the directoid $(S; \sqcap)$ need not be a semilattice and hence that skew difference algebras are far from commutative BCK-algebras.

Example 2. Consider the five element skew difference algebra whose table is

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	b
d	d	b	a	b	0

Then the induced ordered set is depicted in Fig. 2.

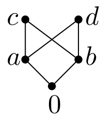

Fig. 2

In the induced directoid $(\{0, a, b, c, d\}; \sqcap)$ we have $x \sqcap y = x \wedge y$ whenever $x \wedge y$ exists and $c \sqcap d = a$. The switching involutions in non-trivial (i.e. with more than two element) sections are as follows:

[0,c]: $0^c = c$, $c^c = 0$, $a^c = b$, $b^c = a$;

[0,d]: $0^d = d$, $d^d = 0$, $a^d = b$, $b^d = a$.

Since there does not exist a greatest element of $L(c, d) = \{a, b, 0\}$, $(S; \sqcap)$, is not $\wedge$-semilattice and hence $(\{0, a, b, c, d\}, *, 0)$ is not a BCK-algebra.

Example 3. Consider the four element skew difference algebra whose table is

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	b
c	c	a	c	0

The induced order is visualized in Fig. 3.

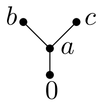

Fig. 3

The switching involutions in non-trivial sections are as follows:

[0,b]: $0^b = b$, $b^b = 0$, $a^b = a$;

[0,c]: $0^c = c$, $c^c = 0$, $a^c = a$.

Although there can be defined a $\wedge$-semilattice structure on the poset $(\{0, a, b, c\}; \leq)$, the induced directoid $(\{0, a, b, c\}; \sqcap)$ is not a semilattice since

$$b \sqcap c = b * (b * c) = b * b = 0.$$

References

[1] Ahn S.S., Lee K.J., *Remarks on "Representable difference algebras"*, Kyungpook Math. J., **46**(2006), 433-436.

[2] Chajda I., Emanovský P., *Representable difference algebras*, Kyungpook Math. J., **44**(2004), 335-342.

[3] Chajda I., Halaš R., Kühr J., *Distributive lattices with sectionally antitone involutions*, Acta Sci. Math. (Szeged), **71**(2005), 19-33.

[4] Ježek J., Quackenbush R., *Directoids: algebraic models of up-directed sets*, Algebra Universalis, **27**(1990), 49-69.

[5] Meng J., *Difference algebras*, Selected Papers on BCK- and BCI-algebras, **1**(1992), 33-39.

[6] Roh E.H., Kim S.Y., Jun Y.B., Shim W.H., *On difference algebras*, Kyungpook Math. J., **43**(2003), 407-414.