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On Some New Paranormed Difference Sequence Spaces Defined by Orlicz Functions

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Abstract. The main aim of this article is to introduce a new class of sequence spaces using the concept of n-norm and to investigate these spaces for some linear topological structures as well as examine these spaces with respect to derived (n-1)-norm. We use an Orlicz function, a bounded sequence of positive real numbers and some difference operators to construct these spaces so that they become more generalized and some other spaces can be derived under special cases. These investigations will enhance the acceptability of the notion of n-norm by giving a way to construct different sequence spaces with elements in n-normed spaces.

1. Introduction

Throughout the article w, ℓ_∞, c, c_0, ℓ_p denote the classes of *all*, *bounded*, *convergent*, *null* and *p-absolutely summable* sequences of complex numbers.

The notion of difference sequence space was introduced by Kizmaz [9], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [14], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy, Esi and Tripathy [15] generalized the above notion and unified these as follows:

Let m, s be non-negative integers, then for a given sequence space Z we have

$$Z(\Delta_m^s) = \{x = (x_k) \in w : (\Delta_m^s x_k) \in Z\},$$

where $\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, the

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difference operator is equivalent to the following binomial representation:

$$\Delta_m^s x_k = \sum_{\nu=0}^{s} (-1)^\nu \binom{s}{\nu} x_{k+mv}.$$

Taking m=1, we get the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$ studied by Et and Colak [1]. Taking s=1, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [14]. Taking m=s=1, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [9].

Let m, s be non-negative integers, then for a given sequence space Z we introduce

$$Z(\Delta_m^{(s)}) = \{x = (x_k) \in w : (\Delta_m^{(s)} x_k) \in Z\},$$

where $\Delta_m^{(s)} x = (\Delta_m^{(s)} x_k) = (\Delta_m^{(s-1)} x_k - \Delta_m^{(s-1)} x_{k-m})$ and $\Delta_m^{(0)} x_k = x_k$ for all $k \in N$, the difference operator is equivalent to the following binomial representation:

$$\Delta_m^{(s)} x_k = \sum_{\nu=0}^{s} (-1)^\nu \binom{s}{\nu} x_{k-mv},$$

where $x_k = 0$, for $k < 0$.

The concept of 2-normed spaces was initially developed by Gähler [3] in the mid of 1960's, while that of n-normed spaces can be found in Misiak [11]. Since then, many others have studied this concept and obtained various results, see for instance Gunawan [5, 6] and Gunawan and Mashadi [8].

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real valued function $||., ..., .||$ on X^n satisfying the following four conditions

(1) $||x_1, x_2, ..., x_n|| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependant,

(2) $||x_1, x_2, ..., x_n||$ is invariant under permutation,

(3) $||\alpha x_1, x_2, ..., x_n|| = |\alpha| ||x_1, x_2, ..., x_n||$, for any $\alpha \in R$,

and

(4) $||x + x', x_2, ..., x_n|| \leq ||x, x_2, ..., x_n|| + ||x', x_2, ..., x_n||$

is called an n-*norm* on X, and the pair $(X, ||., ..., .||)$ is called an n-normed space.

As an example of an n-normed space we may take $X = R^n$ being equipped with the n-norm $||x_1, x_2, ..., x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, ..., x_n$, which may be given explicitly by the formula

$$||x_1, x_2, ..., x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, ..., x_{in}) \in R^n$ for each $i = 1, 2, ..., n$.

Let $(X, ||., ..., .||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then the following function $||., ..., .||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, ..., x_n||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, ..., n\}$$

defines an (n-1)-norm on X with respect to $\{a_1, a_2, ..., a_n\}$.

Let $n \in N$ and $(X, \langle ., . \rangle)$ be a real inner product space of dimension $d \geq n$. Then the following function $||., ..., .||_S$ on $X \times \cdots \times X$ (n factors) defined by

$$||x_1, x_2, ..., x_n||_S = [\det(\langle x_i, x_j \rangle)]^{\frac{1}{2}}$$

is an n-norm on X, which is known as standard n-norm on X. If we take $X = R^n$, then this n-norm is exactly the same as the Euclidean n-norm $||., ..., .||_E$ mentioned earlier. For n=1, this n-norm reduces to usual norm $||x_1|| = \langle x_1, x_1 \rangle^{\frac{1}{2}}$ (for further details one may refer to Gunawan and Mashadi [8]).

A sequence (x_k) in an n-normed space $(X, ||., ..., .||)$ is said to *converge* to some $L \in X$ in the n-norm if

$$\lim_{k \to \infty} ||x_k - L, z_1, ..., z_{n-1}|| = 0, \quad \text{for every } z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) in an n-normed space $(X, ||., ..., .||)$ is said to be *Cauchy* with respect to the n-norm if

$$\lim_{k,l \to \infty} ||x_k - x_l, z_1, ..., z_{n-1}|| = 0, \quad \text{for every } z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

We procure the following results those will help in establishing some results of this article.

Lemma 1 (Gunawan and Mashadi [8], Corollary 2.2)**.** *A standard n-normed space is complete if and only if it is complete with respect to the usual norm* $||.|| = \langle ., . \rangle^{\frac{1}{2}}$.

Lemma 2 (Gunawan and Mashadi [8], Fact 2.3)**.** *On a standard n-normed space X, the derived* $(n-1)$*-norm* $||., ..., .||_\infty$*, defined with respect to orthonormal set* $\{e_1, e_2, ..., e_n\}$*, is equivalent to the standard* (n-1)*-norm* $||., ..., .||_S$*. Precisely, we have for all* $x_1, x_2, ..., x_{n-1}$

$$||x_1, x_2, ..., x_{n-1}||_\infty \leq ||x_1, x_2, ..., x_{n-1}||_S \leq \sqrt{n}||x_1, x_2, ..., x_{n-1}||_\infty,$$

where $||x_1, x_2, ..., x_{n-1}||_\infty = \max\{||x_1, x_2, ..., x_{n-1}, e_i|| : i = 1, 2, ..., n\}$.

An Orlicz function is a function M:[0, ∞) $\to$ [0, ∞), which is continuous, non-decreasing and convex with $M(0)$=0, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [10] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \{(x_k) \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0\}.$$

They proved that ℓ_M is a Banach space normed by

$$||(x_k)|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}.$$

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) < \lambda M(x)$ for all λ with $0 < \lambda < 1$.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, D=max$\{1, 2^{G-1}\}$. Then for all a_k, $b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1}$$

Let $(X, \|., ..., .\|)$ be a real n-normed space and $w(n - X)$ denotes the space of X-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers. Then for an Orlicz function M, we define the following sequence spaces

$$(M, \Delta_r^s, p, ||., ..., .||)_0$$
$$= \Big\{x = (x_k) \in w(n - X) : \lim_{k\to\infty} \left[M(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} = 0,$$
$$z_1, ..., z_{n-1} \in X \text{ and for some } \rho > 0\Big\},$$

$$(M, \Delta_r^s, p, ||., ..., .||)_1$$
$$= \Big\{x = (x_k) \in w(n - X) : \lim_{k\to\infty} \left[M(||\frac{\Delta_r^s x_k - L}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} = 0,$$
$$z_1, ..., z_{n-1} \in X \text{ and for some } \rho > 0 \text{ and } L \in X\Big\},$$

and

$$(M, \Delta_r^s, p, ||., ..., .||)_\infty$$
$$= \Big\{x = (x_k) \in w(n - X) : \sup_{k\geq 1} \left[M(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} < \infty,$$
$$z_1, ..., z_{n-1} \in X \text{ and for some } \rho > 0\Big\}.$$

If we replace the difference operator Δ_r^s by $\Delta_r^{(s)}$ in the above definitions, we will get the spaces $(M, \Delta_r^{(s)}, p, ||., ..., .||)_0$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_1$ and $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$ respectively.

For $s = 0$, we write the above spaces as $(M, p, ||., ..., .||)_0$, $(M, p, ||., ..., .||)_1$ and $(M, p, ||., ..., .||)_\infty$ respectively.

It is clear from the definition that $(M, \Delta_r^s, p, ||., ..., .||)_0 \subset (M, \Delta_r^s, p, ||., ..., .||)_1$. Further $(M, \Delta_r^s, p, ||., ..., .||)_1 \subset (M, \Delta_r^s, p, ||., ..., .||)_\infty$ follows from eq(1) and the following inequality:

$$\left[M(||\frac{\Delta_r^s x_k}{2\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} \leq \left[\frac{1}{2}M(||\frac{\Delta_r^s x_k - L}{\rho}, z_1, ..., z_{n-1}||) + \frac{1}{2}M(\frac{L}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k}.$$

Similarly, we have

$$(M, \Delta_r^{(s)}, p, ||., ..., .||)_0 \subset (M, \Delta_r^{(s)}, p, ||., ..., .||)_1 \subset (M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty.$$

2. Main results

In this section we investigate some linear topological structures of the spaces $(M, \Delta_r^s, p, ||., ..., .||)_0$, $(M, \Delta_r^s, p, ||., ..., .||)_1$, $(M, \Delta_r^s, p, ||., ..., .||)_\infty$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_0$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_1$ and $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$.

Theorem 1. *If* $\{\Delta_r^s x_k, z_1, z_2, ..., z_{n-1}\}$ *is a linearly dependent set in* $(X, ||., ..., .||)$ *for all but finite* k, *where* $x = (x_k) \in w(n - X)$ *and* $\inf_k p_k > 0$, *then*

(i) $\lim_{k\to\infty} \left[M(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} = 0,$ for every $\rho > 0$,

(ii) $\sup_{k\geq 1} \left[M(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} < \infty,$ for every $\rho > 0$.

Proof. (i) Assume that $\{\Delta_r^s x_k, z_1, z_2, ..., z_{n-1}\}$ is linearly dependent set in X for all but finite k. Then we have $||\Delta_r^s x_k, z_1, ..., z_{n-1}|| \to 0$ as $k \to \infty$.
Since M is continuous and $0 < p_k \leq \sup p_k < \infty$, for each k, we have

$$\lim_{k\to\infty} \left[M(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||)\right]^{p_k} = 0, \text{ for every } \rho > 0.$$

(ii) Proof of this part is similar to part (i). □

Note. Theorem 1 will hold good if we replace the difference operator Δ_r^s by the difference operator $\Delta_r^{(s)}$.

Theorem 2. *The classes of sequences* $(M, \Delta_r^s, p, ||., ..., .||)_0$, $(M, \Delta_r^s, p, ||., ..., .||)_1$, $(M, \Delta_r^s, p, ||., ..., .||)_\infty$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_0$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_1$ *and* $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$ *are linear spaces.*

Proof. We proof the result for the space $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$ only and for other spaces it will follow on applying similar arguments.

Let (x_k) and (y_k) be any two elements of the space $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$. Then there exist ρ_1 and $\rho_2 > 0$ such that

$$\left[M(||\frac{\Delta_r^{(s)} x_k}{\rho_1}, z_1, ..., z_{n-1}||)\right]^{p_k} < \infty$$

and

$$\left[M(||\frac{\Delta_r^{(s)} y_k}{\rho_2}, z_1, ..., z_{n-1}||)\right]^{p_k} < \infty,$$

for all $k \geq 1$. Let α, β be any scalars and let ρ_3=max$(2|\alpha|\rho_1, 2|\beta|\rho_2)$.
Then we have

$$\left[M(||\frac{\Delta_r^{(s)}(\alpha x_k + \beta y_k)}{\rho_3}, z_1, ..., z_{n-1}||)\right]^{p_k}$$
$$\leq \left[M(||\frac{\Delta_r^{(s)} \alpha x_k}{\rho_3}, z_1, ..., z_{n-1}||) + M(||\frac{\Delta_r^{(s)} \beta y_k}{\rho_3}, z_1, ..., z_{n-1}||)\right]^{p_k}$$
$$\leq D\left\{\left[M(||\frac{\Delta_r^{(s)} x_k}{\rho_1}, z_1, ..., z_{n-1}||)\right]^{p_k} + \left[M(||\frac{\Delta_r^{(s)} y_k}{\rho_2}, z_1, ..., z_{n-1}||)\right]^{p_k}\right\} < \infty$$

for all $k \geq 1$. Hence $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$ is a linear space. □

Theorem 3. *The spaces* $(M,\Delta_r^s,p,||.,...,.||)_0$, $(M,\Delta_r^s,p,||.,...,.||)_1$ *and* $(M,\Delta_r^s,p,||.,...,.||)_\infty$ *are paranormed spaces paranormed by g defined by*

$$(2)\quad g(x) = \sum_{k=1}^{rs} ||x_k, z_1, ..., z_{n-1}|| + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_k M\left(||\frac{\Delta_r^s x_k}{\rho}, z_1, ..., z_{n-1}||\right) \leq 1\right\},$$

where H=max$(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$ and $g(\theta) = 0$. Let (x_k) and (y_k) be any two sequences belong to any one of the spaces $(M, \Delta_r^s, p, ||., ..., .||)_0$, $(M, \Delta_r^s, p, ||., ..., .||)_1$, and $(M, \Delta_r^s, p, ||., ..., .||)_\infty$. Then we have $\rho_1, \rho_2 > 0$ such that

$$\sup_k M\left(||\frac{\Delta_r^s x_k}{\rho_1}, z_1, ..., z_{n-1}||\right) \leq 1 \quad \text{and} \quad \sup_k M\left(||\frac{\Delta_r^s y_k}{\rho_2}, z_1, ..., z_{n-1}||\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by the convexity of M, we have

$$\sup_k M\left(||\frac{\Delta_r^s(x_k + y_k)}{\rho}, z_1, ..., z_{n-1}||\right)$$
$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_k M\left(||\frac{\Delta_r^s x_k}{\rho_1}, z_1, ..., z_{n-1}||\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_k M\left(||\frac{\Delta_r^s y_k}{\rho_2}, z_1, ..., z_{n-1}||\right)$$
$$\leq 1.$$

Hence we have

$$g(x+y)=\sum_{k=1}^{rs}||x_k+y_k,z_1,...,z_{n-1}||+\inf\left\{\rho^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s(x_k+y_k)}{\rho},z_1,...,z_{n-1}||)\leq 1\right\}$$
$$\leq\sum_{k=1}^{rs}||x_k,z_1,...,z_{n-1}||+\inf\left\{\rho_1^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s x_k}{\rho_1},z_1,...,z_{n-1}||)\leq 1\right\}$$
$$+\sum_{k=1}^{rs}||y_k,z_1,...,z_{n-1}||+\inf\left\{\rho_2^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s y_k}{\rho_2},z_1,...,z_{n-1}||)\leq 1\right\}.$$

This implies

$$g(x+y)\leq g(x)+g(y).$$

The continuity of the scalar multiplication follows from the following equality:

$$g(\lambda x)=\sum_{k=1}^{rs}||\lambda x_k,z_1,...,z_{n-1}||+\inf\{\rho^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s\lambda x_k}{\rho},z_1,...,z_{n-1}||)\leq 1\}$$
$$=|\lambda|\sum_{k=1}^{rs}\|x_k,z_1,...,z_{n-1}\|+\inf\{(t|\lambda|)^{\frac{p_k}{H}}:\sup_k M(\|\frac{\Delta_r^s x_k}{t},z_1,...,z_{n-1}\|)\leq 1\},$$

where $t=\frac{\rho}{|\lambda|}$. □

In view of the above result we state the following result without proof.

Theorem 4. *The spaces* $(M,\Delta_r^{(s)},p,||.,...,.||)_0$, $(M,\Delta_r^{(s)},p,||.,...,.||)_1$ *and* $(M,\Delta_r^{(s)},p,||.,...,.||)_\infty$ *are paranormed spaces paranormed by* h *defined by*

$$h(x)=\inf\left\{\rho^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^{(s)}x_k}{\rho},z_1,...,z_{n-1}||)\leq 1\right\},\tag{3}$$

where $H=max(1,\sup_k p_k)$.

Theorem 5. *Let* $(X.||.,...,.||)$ *be an n-Banach space. Then* $(M,\Delta_r^s,p,||.,...,.||)_0$, $(M,\Delta_r^s,p,||.,...,.||)_1$ *and* $(M,\Delta_r^s,p,||.,...,.||)_\infty$ *are complete paranormed spaces paranormed by* g *as defined by* eq(2).

Proof. We consider only $(M,\Delta_r^s,p,||.,...,.||)_\infty$ and for other spaces it will follow on applying similar arguments.

Let (x^i) be any Cauchy sequence in $(M,\Delta_r^s,p,\|.,...,.\|)_\infty$. Let $x_0>0$ be fixed and $t>0$ be such that for a given ε $(0<\varepsilon<1)$, $\frac{\varepsilon}{x_0 t}>0$, and $x_0 t\geq 1$. Then there exists a positive integer n_0 such that

$$g(x^i-x^j)<\frac{\varepsilon}{x_0 t},\quad \text{for all } i,j\geq n_0.$$

Using the definition of paranorm, we get

$$(4)\qquad \sum_{k=1}^{rs}||x_k^i-x_k^j, z_1,..,z_{n-1}||+\inf\{\rho^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s(x_k^i-x_k^j)}{\rho}, z_1,..,z_{n-1}||)\leq 1\} < \frac{\varepsilon}{x_0 t}, \quad \text{for all } i,j\geq n_0.$$

Hence we have

$$\sum_{k=1}^{rs}||x_k^i-x_k^j, z_1,...,z_{n-1}|| < \varepsilon, \quad \text{for all } i,j\geq n_0.$$

This implies that

$$||x_k^i-x_k^j, z_1,...,z_{n-1}|| < \varepsilon, \quad \text{for all } i,j\geq n_0 \text{ and } 1\leq k\leq rs.$$

Hence (x_k^i) is a Cauchy sequence in X for $k=1,2,...,rs$.

Thus (x_k^i) is convergent in X for $k=1,2,...,rs$.

For simplicity, let

$$(5)\qquad \lim_{i\to\infty} x_k^i = x_k, \text{ say for } k=1,2,...,rs.$$

Again from eq(4), we have

$$\inf\{\rho^{\frac{p_k}{H}}:\sup_k M(||\frac{\Delta_r^s(x_k^i-x_k^j)}{\rho}, z_1,...,z_{n-1}||)\leq 1\} < \varepsilon, \quad \text{for all } i,j\geq n_0.$$

Then we get

$$\sup_k M(||\frac{\Delta_r^s(x_k^i-x_k^j)}{g(x^i-x^j)}, z_1,...,z_{n-1}||)\leq 1, \quad \text{for all } i,j\geq n_0.$$

It follows that

$$M(||\frac{\Delta_r^s(x_k^i-x_k^j)}{g(x^i-x^j)}, z_1,...,z_{n-1}||)\leq 1, \quad \text{for each } k\geq 1 \text{ and for all } i,j\geq n_0.$$

For $t>0$ with $M(\frac{tx_0}{2})\geq 1$, we have

$$M\left(||\frac{\Delta_r^s(x_k^i-x_k^j)}{g(x^i-x^j)}, z_1,...,z_{n-1}||\right)\leq M\left(\frac{tx_0}{2}\right).$$

Then we have

$$||\Delta_r^s(x_k^i-x_k^j), z_1,...,z_{n-1}||\leq \frac{tx_0}{2}.\frac{\varepsilon}{tx_0}=\frac{\varepsilon}{2}.$$

Hence $(\Delta_r^s x_k^i)$ is a Cauchy sequence in X for all $k \in N$.
This implies that $(\Delta_r^s x_k^i)$ is convergent in X for all $k \in N$. Let $\lim_{i\to\infty} (\Delta_r^s x_k^i) = y_k$ exist for each $k \in N$.

Let $k = 1$. Then we have

$$\lim_{i\to\infty} (\Delta_r^s x_1^i) = \lim_{i\to\infty} \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{1+rv}^i = y_1 \tag{6}$$

We have that by eq(5) and eq(6)

$$\lim_{i\to\infty} x_{rs+1}^i = x_{rs+1} \text{ exists.}$$

Proceeding in this way inductively, we have $\lim_{i\to\infty} x_k^i = x_k$ exists for each $k \in N$.

Now we have for all $i, j \geq n_0$,

$$\sum_{k=1}^{rs} ||x_k^i - x_k^j, z_1, ..., z_{n-1}|| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(||\frac{\Delta_r^s (x_k^i - x_k^j)}{\rho}, z_1, ..., z_{n-1}||) \leq 1\} < \varepsilon$$

This implies that

$$\lim_{j\to\infty} \Big\{ \sum_{k=1}^{rs} ||x_k^i - x_k^j, z_1, .., z_{n-1}|| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(||\frac{\Delta_r^s (x_k^i - x_k^j)}{\rho}, z_1, .., z_{n-1}||) \leq 1\} \Big\}$$
$$< \varepsilon, \quad \text{for all } i \geq n_0.$$

Since M and n-norms are continuous functions, we have

$$\sum_{k=1}^{rs} ||x_k^i - x_k, z_1, .., z_{n-1}|| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(||\frac{\Delta_r^s (x_k^i - x_k)}{\rho}, z_1, .., z_{n-1}||) \leq 1\} < \varepsilon,$$

for all $i \geq n_0$.
It follows that $(x^i - x) \in (M, \Delta_r^s, p, ||., ..., .||)_\infty$ and $(M, \Delta_r^s, p, ||., ..., .||)_\infty$ is a linear space, so we have $x = x^i - (x^i - x) \in (M, \Delta_r^s, p, ||., ..., .||)_\infty$.
This completes the proof of the Theorem. □

In view of Theorem 5, we state the following result without proof.

Theorem 6. *Let $(X.||., ..., .||)$ be an n-Banach space. Then $(M, \Delta_r^{(s)}, p, ||., ..., .||)_0$, $(M, \Delta_r^{(s)}, p, ||., ..., .||)_1$ and $(M, \Delta_r^{(s)}, p, ||., ..., .||)_\infty$ are complete paranormed spaces paranormed by h as defined by* eq(3).

Remark. It is obvious that $(x_k) \in (M, \Delta_r^s, p, ||., ..., .||)_Y$ if and only if $(x_k) \in (M, \Delta_r^{(s)}, p, ||., ..., .||)_Y$, for $Y = 0, 1$ and ∞. Also it is clear that paranorms g and h are equivalent. Hence we state the following Corollary.

Corollary 7. *Let $(X.||., ..., .||)$ be an n-Banach space. Then*

(i) $(M,\Delta_r^s,p,||.,...,.||)_0$, $(M,\Delta_r^s,p,||.,...,.||)_1$ *and* $(M,\Delta_r^s,p,||.,...,.||)_\infty$ *are complete paranormed spaces paranormed by* h, *given by* eq(3).

(ii) $(M,\Delta_r^{(s)},p,||.,...,.||)_0$, $(M,\Delta_r^{(s)},p,||.,...,.||)_1$ *and* $(M,\Delta_r^{(s)},p,||.,...,.||)_\infty$ *are complete paranormed spaces paranormed by* g, *given by* eq(2).

Remark. In view of Lemma 1, we can replace the phrase"x is an n-Banach space" by"X is a Banach space" in Theorem 2.5, Theorem 2.6 and Corollary 2.7 if X is assumed to be equipped with standard n-norm.

We state the following Theorem in view of Lemma 2.

Theorem 8. *Let* X *be a standard* n*-norm space and* $\{e_1,e_2,...,e_n\}$ *be an orthonormal set in* X. *Then*

$$(M,\Delta_r^s,p,||.,...,.||_\infty)_0=(M,\Delta_r^s,p,||.,...,.||_{(n-1)})_0,$$
$$(M,\Delta_r^s,p,||.,...,.||_\infty)_1=(M,\Delta_r^s,p,||.,...,.||_{(n-1)})_1$$

and

$$(M,\Delta_r^s,p,||.,...,.||_\infty)_\infty=(M,\Delta_r^s,p,||.,...,.||_{(n-1)})_\infty,$$

where $||.,...,.||_\infty$ *is the derived* $(n-1)$*-norm defined with respect to* $\{e_1,e_2,...,e_n\}$ *and* $||.,...,.||_{(n-1)}$ *is the standard* $(n-1)$*-norm on* X.

Note 2. Theorem 8 holds good if we replace the difference operator Δ_r^s by the difference operator $\Delta_r^{(s)}$.

Theorem 9. (i) *The spaces* $S[(M,\Delta_r^s,p,||.,...,.||)_Y]$ *and* $(M,p,||.,...,.||)_Y$ *are equivalent as topological spaces, where* $S[(M,\Delta_r^s,p,||.,...,.||)_Y]=\{x=(x_k): x\in(M,\Delta_r^s,p,||.,...,.||)_Y, x_1=...=x_{rs}=0\}$ *is a subspace of* $(M,\Delta_r^s,p,||.,...,.||)_Y$, *for* $Y=0,1$ *and* ∞.
(ii) *The spaces* $(M,\Delta_r^{(s)},p,||.,...,.||)_Y$ *and* $(M,p,||.,...,.||)_Y$ *are equivalent as topological spaces, for* $Y=0,1$ *and* ∞.

Proof. (i) Let us consider the mapping
$T:S[(M,\Delta_r^s,p,||.,...,.||)_Y]\longrightarrow(M,p,||.,...,.||)_Y$ defined by

$$Tx=(\Delta_r^s x_k),\quad \text{for every } x\in S[(M,\Delta_r^s,p,||.,...,.||)_Y].$$

Then clearly T is a linear homeomorphism and the proof follows.

(ii) In this case we consider a mapping $T':(M,\Delta_r^{(s)},p,||.,...,.||)_Y\longrightarrow(M,p,||.,...,.||)_Y$ defined by

$$T'x=(\Delta_r^{(s)}x_k),\quad \text{for every } x\in(M,\Delta_r^{(s)},p,||.,...,.||)_Y$$

Then clearly T' is a linear homeomorphism and the proof follows. □

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