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The Fekete-Szegö Problem for a Generalized Subclass of Analytic Functions

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Abstract. In this present work, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disk for which $\frac{(1-\alpha)z(D^m_{\lambda,\mu}f(z))'+\alpha z(D^{m+1}_{\lambda,\mu}f(z))'}{(1-\alpha)D^m_{\lambda,\mu}f(z)+\alpha D^{m+1}_{\lambda,\mu}f(z)}$ $(\lambda \geq \mu \geq 0,\ m \in \mathbb{N}_0,\ \alpha \geq 0)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by Hadamard product (or convolution) are given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to generalize the Fekete-Szegö inequalities obtained by Srivastava *et al.*, Orhan *et al.* and Shanmugam *et al.*, by making use of the generalized differential operator $D^m_{\lambda,\mu}$.

1. Introduction and Definitions

Let A denote the class of all analytic functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let S be the subclass of A consisting of univalent functions. If the functions $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \ \big(z \in U\big).$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z)),\ z \in U$.

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,

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their convolution (or Hadamard product) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the open unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), (z \in U)$$

and let $C(\phi)$ be the class of functions $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), (z \in U)\,,$$

where $\prec$ denotes the subordination between analytic functions. These classes were defined and studied by Ma and Minda [3]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf' \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close-to convex functions, see the recent paper by Srivastava *et al.* [11].

We define a new *fractional differential operator* $D^{n,\alpha}_{\lambda,\mu} f$ as follows

$$D^0_{\lambda,\mu} f(z) = f(z),$$

$$D^1_{\lambda,\mu} f(z) = D_{\lambda,\mu}\left(f(z)\right) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z), \tag{1.2}$$

$$D^2_{\lambda,\mu} f(z) = D_{\lambda,\mu}\left(D^1_{\lambda,\mu} f(z)\right),$$

$$\vdots$$

$$D^m_{\lambda,\mu} f(z) = D_{\lambda,\mu}\left(D^{m-1}_{\lambda,\mu} f(z)\right), \tag{1.3}$$

where $\lambda \geq \mu \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
If f is given by (1.1) then from the definition of the operator$D^m_{\lambda,\mu} f(z)$it is easy to see that

$$D^m_{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^n. \tag{1.4}$$

It should be remarked that the $D^{n,\alpha}_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A$ we have the following:

1. $D^m_{1,0}f(z) \equiv D^m f(z)$ the operator investigated by Salagean [8].
2. $D^m_{\lambda,0}f(z) \equiv D^m_\lambda f(z)$ the operator studied by Al-Oboudi [1].
3. $D^m_{\lambda,\mu}f(z)$ the operator considered for $0 \le \mu \le \lambda \le 1$, by Raducanu and Orhan [6].

Now, by making use of $D^m_{\lambda,\mu}$, we define a new subclass of analytic functions.

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, with $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is said to be in the class $M^{m,\alpha}_{\lambda,\mu}(\phi)$ if

$$\frac{(1-\alpha)z(D^m_{\lambda,\mu}f(z))' + \alpha z(D^{m+1}_{\lambda,\mu}f(z))'}{(1-\alpha)D^m_{\lambda,\mu}f(z) + \alpha D^{m+1}_{\lambda,\mu}f(z)} \prec \phi(z) \ (\alpha \ge 0). \tag{1.5}$$

If we write $\Lambda^{m,\alpha}_{\lambda,\mu}f(z) = \{(1-\alpha)D^m_{\lambda,\mu} + \alpha D^{m+1}_{\lambda,\mu}\}f(z)$ then $f \in M^{m,\alpha}_{\lambda,\mu}(\phi) \Leftrightarrow \Lambda^{m,\alpha}_{\lambda,\mu}f \in S^*(\phi)$. Also, we have

$$\Lambda^{m,\alpha}_{\lambda,\mu}f(z) = \Psi^{m,\alpha}_{\lambda,\mu}(z) * f(z), \tag{1.6}$$

where

$$\Psi^{m,\alpha}_{\lambda,\mu}(z) = z + \sum_{n=2}^{\infty}[1+(\lambda\mu n+\lambda-\mu)(n-1)]^m[1+\alpha(\lambda\mu n+\lambda-\mu)(n-1)]z^n. \tag{1.7}$$

The class $M^{m,\alpha}_{\lambda,\mu}(\phi)$reduces to the following classes, defined earlier.

1. $M^{m,\alpha}_{1,0}(\phi) \equiv M_{\alpha,n}(\phi)$ introduced and studied by Orhan *et al* [4].
2. $M^{0,\alpha}_{1,0}(\phi) \equiv M_\alpha(\phi)$ introduced and worked by Shanmugam *et al.* [9].
3. $M^{0,0}_{0,0}(\phi) \equiv S^*(\phi)$ introduced and investigated by Ma *et al.* [3].
4. $M^{0,1}_{0,0}(\phi) \equiv C(\phi)$ introduced and studied by Ma *et al.* [3].

In order to derive our main results, we need the following lemma [3].

Lemma 1.2([3]). *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic functions with positive real part in U, then*

$$\left|c_2 - vc_1^2\right| \le \begin{cases} -4v+2 & \text{if } v \le 0, \\ 2 & \text{if } 0 \le v \le 1, \\ 4v-2 & \text{if } v \ge 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \qquad (0 \le \gamma \le 1),$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. The above upper bound is sharp. When $0 < v < 1$, it can be improved as follows:

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \leq 2 \qquad (0 < v \leq 1/2)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \leq 2 \qquad (1/2 < v \leq 1)\,.$$

In the present paper, we obtain the Fekete-Szegö inequality for a more general class of functions $M_{\lambda,\mu}^{m,\alpha}(\phi)$ that we defined upper. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider the class $M_{\lambda,\mu}^{m,\alpha,\gamma}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to generalize the Fekete-Szegö inequalities obtained by Srivastava *et al.* [10], Orhan *et al* [4] and Shanmugam *et al.* [9].

2. Fekete-Szegö problem

In this section, we will give some upper bounds for the Fekete-Szegö functional $\left|a_3 - \eta a_2^2\right|$.

In order to prove our main results we have to recall the following.

Firstly, the following information will be used in the proof of the Theorem 2.1. By geometric interpretation there exists a function w satisfying the conditions of the Schwarz lemma such that

$$\frac{z\left(\Lambda_{\lambda,\mu}^{m}f(z)\right)'}{\Lambda_{\lambda,\mu}^{m}f(z)} = \phi\left(w(z)\right) \quad (z \in U)\,. \tag{2.1}$$

Secondly, we introduce the following functions which will be used in the discussion of sharpness of our results.

Corresponding to the function $\Psi_{\lambda,\mu}^{m,\alpha}(z)$ defined by (1.7), we also consider the function $\Psi_{\lambda,\mu}^{m,\alpha}(z)^{(-1)}$ given by

$$\begin{aligned}\Psi_{\lambda,\mu}^{m,\alpha}(z)^{(-1)} &\\ = z + \sum_{n=2}^{\infty} &\frac{1}{[1+(\lambda\mu n+\lambda-\mu)(n-1)]^m[1+\alpha(\lambda\mu n+\lambda-\mu)(n-1)]}z^n,\end{aligned} \tag{2.2}$$

where inverse is taken with respect to Hadamard product.

Using standard procedure it can be deduce that $f \in M_{\lambda,\mu}^{m,\alpha}$ if and only if

$$f(z) = \Psi_{\lambda,\mu}^{m,\alpha}(z)^{(-1)} * \left\{z\exp\left(\int_0^z \frac{\phi(w(t))-1}{t}dt\right)\right\}$$

for some function $w(z)$ satisfying the conditions of the Schwarz Lemma.

Define the function G in U by

$$G(z) = \frac{1}{z}\left[\Psi_{\lambda,\mu}^{m,\alpha}(z)^{(-1)} * \left\{z \exp\left(\int_0^z \frac{\phi(\xi)-1}{\xi} d\xi\right)\right\}\right]. \tag{2.3}$$

Also we consider the following extremal function

$$K(z,\theta,\tau) = \Psi_{\lambda,\mu}^{m,\alpha}(z)^{(-1)} * z\exp\left(\int_0^z \left[\phi\left(\frac{e^{i\theta}\xi(\xi+\tau)}{1+\tau\xi}\right)-1\right]\frac{d\xi}{\xi}\right) \quad (0\le\theta\le 2\pi;\ 0\le\tau\le 1). \tag{2.4}$$

Note that $K(z,0,1) = zG(z)$ defined by (2.3) and $K(z,\theta,0)$ is an odd function.

Theorem 2.1. *Let $\alpha \geq 0$, $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1.1) belongs to the class $M_{\lambda,\mu}^{m,\alpha}(\phi)$, then*

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} - \frac{\eta B_1^2}{A_1^{2m}[1+\alpha(A_1-1)]^2} + \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]} & \text{if } \eta \le \sigma_1; \\ \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]} & \text{if } \sigma_1 \le \eta \le \sigma_2; \\ -\frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} + \frac{\eta B_1^2}{A_1^{2m}[1+\alpha(A_1-1)]^2} - \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]} & \text{if } \eta \ge \sigma_2, \end{cases} \tag{2.5}$$

where

$$\sigma_1 := \frac{A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2-B_1)+B_1^2\right\}}{2A_2^m[1+\alpha(A_2-1)]B_1^2},$$

$$\sigma_2 := \frac{A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2+B_1)+B_1^2\right\}}{2A_2^m[1+\alpha(A_2-1)]B_1^2}$$

and

$$A_1 := [1 + (2\lambda\mu + \lambda - \mu)]\,;\ A_2 := [1 + 2\,(3\lambda\mu + \lambda - \mu)]\,, \tag{2.6}$$

$$(\lambda \ge \mu \ge 0, m \in \mathbb{N}_0).$$

Each of the estimates in (2.5) is sharp for the function $K(z,\theta,\tau)$ given by (2.4).

Proof. For $f(z) \in M_{\lambda,\mu}^{m,\alpha}(\phi)$, let

$$p(z) = \frac{(1-\alpha)z(D_{\lambda,\mu}^m f(z))' + \alpha z(D_{\lambda,\mu}^{m+1} f(z))'}{(1-\alpha)(D_{\lambda,\mu}^m f(z)) + \alpha(D_{\lambda,\mu}^{m+1} f(z))} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.7}$$

From (2.6), we obtain

$$A_1^m[1+\alpha(A_1-1)]a_2 = b_1 \ \text{ and } \ 2A_2^m[1+\alpha(A_2-1)]a_3 = b_2 + 2A_1^{2m}[1+\alpha(A_1-1)]^2 a_2^2.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in U. We also have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right), \tag{2.8}$$

and thus, we get

$$b_1 = \tfrac{1}{2}B_1 c_1 \text{ and } b_2 = \tfrac{1}{2}B_1(c_2 - \tfrac{1}{2}c_1^2) + \tfrac{1}{4}B_2 c_1^2.$$

Hence, we have

$$a_3 - \eta a_2^2 = \frac{B_1}{4A_2^m[(1-\alpha)+\alpha A_2]}\left\{c_2 - v c_1^2\right\}, \tag{2.9}$$

where

$$v := \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2\eta A_2^m[1+\alpha(A_2-1)] - A_1^{2m}[1+\alpha(A_1-1)]^2}{A_1^{2m}[1+\alpha(A_1-1)]^2}B_1\right).$$

If $\eta \le \sigma_1$, then, according to Lemma 1.2, we get

$$\begin{aligned}\left|a_3 - \eta a_2^2\right| &= \frac{B_1}{4A_2^m[1+\alpha(A_2-1)]} \\ &\times\left|c_2 - c_1^2\left[\frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2\eta A_2^m[1+\alpha(A_2-1)] - A_1^{2m}[1+\alpha(A_1-1)]^2}{A_1^{2m}[1+\alpha(A_1-1)]^2}B_1\right)\right]\right|\end{aligned} \tag{2.10}$$

and thus,

$$\left|a_3 - \eta a_2^2\right| \le \frac{B_2}{2A_2^m\left[1+\alpha(A_2-1)\right]} - \frac{\eta B_1^2}{A_1^{2m}\left[1+\alpha(A_1-1)\right]^2} + \frac{B_1^2}{2A_2^m\left[1+\alpha(A_2-1)\right]},$$

which is the fist assertion of (2.5).
Next, if $\eta \ge \sigma_2$, by applying Lemma 1.2, we get

$$\left|a_3 - \eta a_2^2\right| \le -\frac{B_2}{2A_2^m\left[1+\alpha(A_2-1)\right]} + \frac{\eta B_1^2}{A_1^{2m}\left[1+\alpha(A_1-1)\right]^2} - \frac{B_1^2}{2A_2^m\left[1+\alpha(A_2-1)\right]}$$

which is the third assertion of (2.5).
If $\sigma_1 \le \eta \le \sigma_2$, by using again Lemma 1.2, we obtain

$$\left|a_3 - \eta a_2^2\right| \le \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}$$

which is the second part of the assertion (2.5).

We now obtain sharpness of the estimates in (2.5).

If $\eta < \sigma_1$ or $\eta > \sigma_2$, then equality holds in (2.5) if and only if equality holds in (2.10). This happens if and only if $c_1 = 2$ and $c_2 = 2$. Thus $w(z) = z$. It follows that the extremal function is of the form $K(z, 0, 1)$ defined by (2.4) or one of its rotations.

If $\eta = \sigma_2$, the equality holds if and only if $|c_2| = 2$.In this case, we have

$$\frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2\eta A_2^m[1+\alpha(A_2-1)] - A_1^{2m}[1+\alpha(A_1-1)]^2}{A_1^{2m}[1+\alpha(A_1-1)]^2} B_1\right) = 0.$$

Therefore the extremal function f is $K(z, \pi, \tau)$ or one of its rotations.

Similarly, $\eta = \sigma_1$ is equivalent to

$$p_1(z) = \frac{1+\tau}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-\tau}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \tau < 1;\ z \in U).$$

Thus the extremal function is $K(z, 0, \tau)$or one of its rotations.

Finally if $\sigma_1 \le \eta \le \sigma_2$, then equality holds if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$h(z) = \frac{1+\tau z^2}{1-\tau z^2} \quad (0 \le \tau \le 1;\ z \in U).$$

Therefore the extremal function f is $K(z, 0, 0)$ or one of its rotations. The proof of Theorem 2.1 is now completed. □

Remark 2.2.

1. For $\mu = 0, \lambda = 1$ in Theorem 2.1, we get the results obtained by Orhan et al.[4].
2. For $\mu = m = 0,\ \lambda = 1$ in Theorem 2.1, we obtain the results obtained by Shanmugam et al. [9].

Remark 2.3. If $\sigma_1 \le \eta \le \sigma_2$, then, in view of Lemma 1.2, Theorem 2.1 can be improved. Let σ_3 given by

$$\sigma_3 := \frac{A_1^{2m}[1+\alpha(A_1-1)]^2\left\{B_1^2 + B_2\right\}}{2A_2^m[1+\alpha(A_2-1)]B_1^2}.$$

If $\sigma_1 \le \eta \le \sigma_3$, then

$$\begin{aligned} &\left|a_3 - \eta a_2^2\right| \\ &+ \frac{A_1^{2m}[1+\alpha(A_1-1)]^2}{2A_2^m[1+\alpha(A_2-1)]B_1^2}\left[B_1 - B_2 + \frac{2\eta A_2^m[1+\alpha(A_2-1)] - A_1^{2m}[1+\alpha(A_1-1)]^2}{A_1^{2m}[1+\alpha(A_1-1)]^2} B_1^2\right]|a_2|^2 \\ &\le \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}. \end{aligned}$$

If $\sigma_3 \leq \eta \leq \sigma_2$, then

$$\left|a_3 - \eta a_2^2\right| + \frac{A_1^{2m}[1+\alpha(A_1-1)]^2}{2A_2^m[1+\alpha(A_2-1)]B_1^2}\left[B_1+B_2-\frac{2\eta A_2^m[1+\alpha(A_2-1)]-A_1^{2m}[1+\alpha(A_1-1)]^2}{A_1^{2m}[1+\alpha(A_1-1)]^2}B_1^2\right]|a_2|^2 \leq \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]},$$

where A_1 and A_2 are given by (2.6).

Proof. If $\sigma_1 \leq \eta \leq \sigma_3$, we have

$$\begin{aligned}
&\left|a_3 - \eta a_2^2\right| + (\eta - \sigma_1)\left|a_2\right|^2 \\
&= \frac{B_1}{4A_2^m[1+\alpha(A_2-1)]}\left|c_2 - vc_1^2\right| + (\eta-\sigma_1)\frac{B_1^2}{4A_1^{2m}[1+\alpha(A_1-1)]^2}\left|c_1\right|^2 \\
&= \frac{B_1}{4A_2^m[1+\alpha(A_2-1)]}\left|c_2 - vc_1^2\right| \\
&\quad + \left(\eta - \frac{A_1^{2m}[1+\alpha(A_1-1)]^2[B_1^2-B_1+B_2]}{2A_2^m[1+\alpha(A_2-1)]B_1^2}\right)\frac{B_1^2}{4A_1^{2m}[1+\alpha(A_1-1)]^2}\left|c_1\right|^2 \\
&= \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\left[\frac{1}{2}\left|c_2 - vc_1^2\right| x\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{2\eta A_2^m[1+\alpha(A_2-1)]B_1^2 - A_1^{2m}[1+\alpha(A_1-1)]^2[B_1^2-B_1+B_2]}{2A_1^{2m}[1+\alpha(A_1-1)]^2B_1}\right)\left|c_1\right|^2\right] \\
&= \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\left\{\frac{1}{2}\left[\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2\right]\right\} \\
&\leq \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}.
\end{aligned}$$

Similarly, if $\sigma_3 \leq \eta \leq \sigma_2$, we can write

$$\begin{aligned}
&\left|a_3 - \eta a_2^2\right| + (\sigma_2 - \eta)\left|a_2\right|^2 \\
&= \frac{B_1}{4A_2^m[1+\alpha(A_2-1)]}\left|c_2 - vc_1^2\right| + (\sigma_2-\eta)\frac{B_1^2}{4A_1^{2m}[1+\alpha(A_1-1)]^2}\left|c_1\right|^2 \\
&= \frac{B_1}{4A_2^m[1+\alpha(A_2-1)]}\left|c_2 - vc_1^2\right| \\
&\quad + \left(\frac{A_1^{2m}[1+\alpha(A_1-1)]^2[B_1^2+B_1+B_2]}{2A_2^m[1+\alpha(A_2-1)]B_1^2} - \eta\right)\frac{B_1^2}{4A_1^{2m}[1+\alpha(A_1-1)]^2}\left|c_1\right|^2 \\
&= \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\left[\frac{1}{2}\left|c_2 - vc_1^2\right|\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{A_1^{2m}[1+\alpha(A_1-1)]^2[B_1^2+B_1+B_2] - 2\eta A_2^m[1+\alpha(A_2-1)]B_1^2}{2A_1^{2m}[1+\alpha(A_1-1)]^2B_1}\right)\left|c_1\right|^2\right]
\end{aligned}$$

$$= \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\left\{\frac{1}{2}\left[\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2\right]\right\}$$
$$\leq \frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}.$$

Thus, the proof of Remark 2.3 is completed. □

3. Applications to functions defined by fractional derivatives

For fixed $g \in A$, let $M_{\lambda,\mu}^{m,\alpha,g}(\phi)$ be class of functions $f \in A$ for which $(f * g) \in M_{\lambda,\mu}^{m,\alpha}(\phi)$.
In order to introduce the class $M_{\lambda,\mu}^{m,\alpha,\gamma}(\phi)$, we need the following:

Definition 3.1([5]). Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order γ is defined by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\gamma} d\zeta (0 \leq \gamma < 1),$$

where the multiplicity of $(z-\zeta)^\gamma$ is removed by requiring that $\log(z-\zeta)$ is real for $z - \zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega : A \to A$ defined by

$$\Omega^\gamma f(z) = \Gamma(2-\gamma)z^\gamma D_z^\gamma f(z) \quad (\gamma \neq 2, 3, 4, \cdots).$$

The class $M_{\lambda,\mu}^{m,\alpha,\gamma}(\phi)$ consists of functions $f \in A$ for which $\Omega^\gamma f \in M_{\lambda,\mu}^{m,\alpha}(\phi)$. Note that $M_{0,0}^{0,0}(\phi) \equiv S^*(\phi)$ and $M_{\lambda,\mu}^{m,\alpha,\gamma}(\phi)$ is the special case of the class $M_{\lambda,\mu}^{m,\alpha,g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n. \tag{3.1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n (g_n > 0).$$

Since $D_{\lambda,\mu}^m f(z) \in M_{\lambda,\mu}^{m,\alpha,g}(\phi)$ if and only if $D_{\lambda,\mu}^m f(z) * g(z) \in M_{\lambda,\mu}^{m,\alpha}(\phi)$, we obtain the coefficient estimate for functions in the class $M_{\lambda,\mu}^{m,\alpha,g}(\phi)$, from the corresponding estimate for functions in the class $M_{\lambda,\mu}^{m,\alpha}(\phi)$.
Applying Theorem 2.1 for the function

$$D_{\lambda,\mu}^m f(z) * g(z) = z + \sum_{n=2}^{\infty} \left[1 + (\lambda\mu n + \lambda - \mu)(n-1)\right]^m a_n g_n z^n$$
$$= z + \left[1 + (2\lambda\mu + \lambda - \mu)(n-1)\right]^m a_2 g_2 z^2 + \cdots .$$

We get the following Theorem 3.2 after an obvious change of the parameter η :

Theorem 3.2. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If $D_{\lambda,\mu}^m f(z)$ defined by (1.4) belongs to the class $M_{\lambda,\mu}^{m,\alpha,g}(\phi)$, then*

$$|a_3 - \eta a_2| \leq \begin{cases} \frac{1}{g_3}\left[\frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} - \frac{\eta g_3 B_1^2}{A_1^{2m}[1+\alpha(A_1-1)]^2 g_2^2} + \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \eta \leq \sigma_1^*; \\ \frac{1}{g_3}\left[\frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \sigma_1^* \leq \eta \leq \sigma_2^*; \\ \frac{1}{g_3}\left[-\frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} + \frac{\eta g_3 B_1^2}{A_1^{2m}[1+\alpha(A_1-1)]^2 g_2^2} - \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \eta \geq \sigma_2^*, \end{cases}$$

where A_1 and A_2 are given by (2.6) *and*

$$\sigma_1^* := \frac{g_2^2 A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2 - B_1) + B_1^2\right\}}{2g_3 A_2^m[1+\alpha(A_2-1)]B_1^2},$$

$$\sigma_2^* := \frac{g_2^2 A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2 + B_1) + B_1^2\right\}}{2g_3 A_2^m[1+\alpha(A_2-1)]B_1^2}.$$

The result is sharp.

Since

$$(\Omega^\gamma D_{\lambda,\mu}^m f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}\left[1 + (\lambda\mu n + \lambda - \mu)(n-1)\right]^m z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}.$$

For g_2 and g_3 given by above equalities, Theorem 3.2 reduces to the following.

Theorem 3.3. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$* and $\alpha \geq 0$. *If $D_{\lambda,\mu}^m f(z)$ defined by* (1.4) *belongs to the class $M_{\lambda,\mu}^{m,\alpha,g}(\phi)$, then*

$$|a_3 - \eta a_2| \leq$$

$$\begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} - \frac{3(2-\gamma)\eta B_1^2}{2(3-\gamma)A_1^{2m}[1+\alpha(A_1-1)]^2} + \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \eta \leq \sigma_1^{**}; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_1}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \sigma_1^{**} \leq \eta \leq \sigma_2^{**}; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{B_2}{2A_2^m[1+\alpha(A_2-1)]} + \frac{3(2-\gamma)\eta B_1^2}{2(3-\gamma)A_1^{2m}[1+\alpha(A_1-1)]^2} - \frac{B_1^2}{2A_2^m[1+\alpha(A_2-1)]}\right] & \text{if } \eta \geq \sigma_2^{**}, \end{cases}$$

where A_1 *and* A_2 *are given by* (2.6) *and*

$$\sigma_1^{**} := \frac{(3-\gamma)A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2-B_1)+B_1^2\right\}}{3(2-\gamma)A_2^m[1+\alpha(A_2-1)]B_1^2},$$

$$\sigma_2^{**} := \frac{(3-\gamma)A_1^{2m}[1+\alpha(A_1-1)]^2\left\{(B_2+B_1)+B_1^2\right\}}{3(2-\gamma)A_2^m[1+\alpha(A_2-1)]B_1^2}.$$

The result is sharp.

Remark 3.4. When $\alpha = 0$, $\lambda = 0 = \mu$, $B_1 = 8/\pi^2$ and $B_2 = 16/3\pi^2$, Theorem 3.3 reduces to a result of Srivastava and Mishra [10, Theorem 8, p. 64] for a class of which $\Omega^\gamma f(z)$ is a parabolic starlike function (see [2], [7]).

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