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FOURTH ORDER ELLIPTIC BOUNDARY VALUE PROBLEM WITH SQUARE GROWTH NONLINEARITY

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Abstract. We give a theorem for the existence of at least three solutions for the fourth order elliptic boundary value problem with the square growth variable coefficient nonlinear term. We use the variational reduction method and the critical point theory for the associated functional on the finite dimensional subspace to prove our main result. We investigate the shape of the graph of the associated functional on the finite dimensional subspace, $(P.S.)$ condition and the behavior of the associated functional in the neighborhood of the origin on the finite dimensional reduction subspace.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $a : \overline{\Omega} \to R$ be a continuous function which changes sign on Ω, that is, the subsets of Ω

$$\Omega^+ = \{x \in \Omega |\ a(x) > 0\}, \qquad \Omega^- = \{x \in \Omega |\ a(x) < 0\}$$

are nonempty. Let us set

$$a^+ = a \cdot \chi_{\Omega^+}, \qquad a^- = -a \cdot \chi_{\Omega^-},$$

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so that $a = a^+ - a^-$. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. Let $c \in R$. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \Lambda u \qquad \text{in } \Omega,$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many eigenvalues $\Lambda_j = \lambda_j(\lambda_j - c)$, $j \geq 1$, and corresponding eigenfunctions ϕ_j, $j \geq 1$, where λ_j, $j \geq 1$ are the eigenvalues and ϕ_j, $j \geq 1$, are the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product and each eigenvalue λ_j is repeated as often as its multiplicity, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

We note that $\Lambda_1 < \Lambda_2$, $\lim_{k\infty} \Lambda_k = \infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. In this paper we study the following variable coefficient nonlinear fourth order elliptic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = a(x)g(u) \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

For the constant coefficient nonlinear case Choi and Jung [1,2] show that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \Lambda_1$ and $s > 0$). The authors obtained these results by use of the variational reduction method. The authors [3] also proved that when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$, (1.2) has at least three nontrivial solutions by use of the degree theory. Tarantello [6] also studied the problem

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1) \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

She show that if $c < \lambda_1$ and $b \geq \Lambda_1$, then (1.3) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [4] also proved that if $c < \lambda_1$ and $b \geq \Lambda_2$, then (1.3) has at least four solutions by the variational linking theorem and Leray-Schauder degree theory.

We assume that g satisfies the following conditions:
(g1) $g \in C^1(R, R)$ with $g(0) = 0$,
(g2) There exist $\alpha < \beta$ such that

$$\alpha \leq a^+(x)g'(u) \leq \beta.$$

(g3) Let $\Lambda_{j+1}, \Lambda_{j+2}, \Lambda_{j+m}, m \geq 1$, be all eigenvalues within $[\alpha, \beta]$(without loss of generality, we may assume that α, β are not the eigenvalues Λ_i, $i \geq 1$). Suppose that there exist γ and C such that $\Lambda_{j+m} < \gamma < \beta$ and

$$a^+(x)G(u) \geq \frac{1}{2}\gamma\|u\|^2_{L^2(\Omega)} - C, \qquad \forall u \in R,$$

where $G(\xi) = \int_0^\xi g(t)dt$.
(g4) There exists eigenvalue $\Lambda_l \in [\Lambda_{j+1}, \Lambda_{j+m})$ such that

$$\Lambda_l < a^+(x)g'(0) < \Lambda_{l+1}.$$

In this paper we are trying to find the weak solutions of (1.1), that is,

$$\int_\Omega \Delta^2 u \cdot v + c\Delta u \cdot v - a(x)g(u)v = 0, \qquad \forall v \in H,$$

where H is introduced in section 2.

Our main result is the following.

THEOREM 1.1. *Assume that $\lambda_j < c < \lambda_{j+1}$. We also assume that g satisfies the conditions (g1)-(g4), and there exists a small number $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Then (1.1) has at least three nontrivial solutions.*

For the proof of main theorem we use the finite dimensional reduction method to reduce the theory on the infinite dimensional space to the one on the finite dimensional subspace. So we obtain the critical points results of the functional on the infinite dimensional space H from the critical points results of the corresponding functional $\tilde{I}(v)$ on the finite dimensional reduction subspace. By these reasons we are trying to find the critical points for $\tilde{I}(v)$ by investigating the $(P.S.)_c$ condition and the shape of the graph of the functional $\tilde{I}$ and applying the critical point theory for the functional $\tilde{I}(v)$. The outline of the proof is as follows: In section 2 we introduce the Hilbert space H and show that the corresponding functional $I(u)$ of (1.1) is in $C^1(H, R)$, *Fréchet* differentiable. In section 3, we show that $\tilde{I}(v)$ satisfies the Palais-Smale condition. prove Theorem 1.1.

2. Variational formulation

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_j \phi_j \quad \text{with} \quad \sum h_j^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\Lambda_j| h_j^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\Lambda_j| h_j^2]^{\frac{1}{2}}.$$

Since $\Lambda_j \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
which is proved in [1].

From now on we assume that $\lambda_j < c < \lambda_{j+1}$. Let us define the subspaces of H as follows:

$H_+ = \text{span}\{$eigenfunctions whose corresponding eigenvalues Λ_j are positive.$\}$,

$H_- = \text{span}\{$eigenfunctions whose corresponding eigenvalues Λ_j are negative.$\}$.

Then $H = H_- \oplus H_+$, for $u \in H$, $u = u_- + u_+ \in H_- \oplus H_+$. Let P_+ be the orthogonal projection from H onto H_+ and P_- be the orthogonal projection from H onto H_-. We can write $P_+ u = u_+$, $P_- u = u_-$, for $u \in H$.

LEMMA 2.1. *Assume that g satisfies the conditions $(g1)$-$(g4)$. Then the solutions in $L^2(\Omega)$ of*

$$\Delta^2 u + c\Delta u = a(x)g(u) \qquad \text{in } L^2(\Omega)$$

belong to H.

Proof. Let $a(x)g(u) = \sum a(x) h_k \phi_k \in L^2(\Omega)$. Then

$$(\Delta^2 + c\Delta)^{-1}(a(x)g(u)) = \sum \frac{1}{\Lambda_k} a(x) h_k \phi_k.$$

Hence we have

$$\|(\Delta^2 + c\Delta)^{-1} a(x)g(u)\|^2 = \sum |\Lambda_k| \frac{1}{\Lambda_k^2} (a(x)h_k)^2 \leq C \sum (a(x)h_k)^2$$

for some $C > 0$, which means that

$$\|(\Delta^2 + c\Delta)^{-1}a(x)g(u)\| \leq C_1\|a(x)g(u)\|_{L^2(\Omega)}.$$

□

We are looking for the weak solutions of (1.1). The weak solutions of (1.1) coincide with the critical points of the associated functional

$$I(u) \in C^1(H, R),$$

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \int_\Omega a(x)G(u)]dx. \qquad (2.1) \\ &= \frac{1}{2}(\|P_+u\|^2 - \|P_-u\|^2) - \int_\Omega a(x)G(u)dx, \end{aligned}$$

where $G(\psi) = \int_0^\psi g(t)dt$.

By $(g1)$ and $(g2)$, I is well defined. By the following Lemma 2.2, $I \in C^1(H, R)$ and I is *Fréchet* differentiable in H:

LEMMA 2.2. *Assume that* $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$, *and* g *satisfies* $(g1) - (g4)$. *Then* $I(u)$ *is continuous and Fréchet differentiable in* H *with Fréchet derivative*

$$\nabla I(u)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - a(x)g(u)h]dx. \qquad (2.2)$$

If we set

$$K(u) = \int_\Omega a(x)G(u)dx,$$

then $K'(u)$ *is continuous with respect to weak convergence,* $K'(u)$ *is compact, and*

$$K'(u)h = \int_\Omega a(x)g(u)hdx \qquad \text{for all } h \in H,$$

this implies that $I \in C^1(H, R)$ *and* $K(u)$ *is weakly continuous.*

The proof of Lemma 2.2 has the same process as that of the proof in Appendix B in [5].

Now we shall give a lemma to obtain the critical points of the functional on the infinite dimensional space H from that of the reduced functional on the finite dimensional reduction one.

Let V be a m dimensional subspace of H spanned by $\phi_{j+1},\ldots,\phi_{j+m}$ whose eigenvalues are $\Lambda_j,\ldots, \Lambda_{j+m}$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ be the orthogonal projection of H onto

V and $I-P : H \to W$ denote that of H onto W. Then every element $u \in L^2(\Omega)$ is expressed by $u = v + z$, $v \in Pu$, $z = (I-P)u$. Then (1.1) is equivalent to the two systems in the two unknowns v and z:

$$\Delta^2 v + c\Delta v = P(a(x)g(v+z)) \qquad \text{in } \Omega,$$

$$\Delta^2 z + c\Delta z = (I-P)(a(x)g(v+z)) \qquad \text{in } \Omega,$$

$$v = 0, \qquad \Delta v = 0 \qquad \text{on } \partial\Omega,$$

$$z = 0, \qquad \Delta z = 0 \qquad \text{on } \partial\Omega.$$

Let W_1 be a subspace of W spanned by eigenvalues $\Lambda_1, \ldots, \Lambda_j$ and W_2 be a subspace of W spanned by eigenvalues Λ_i, $i \geq j+m+1$. Let $v \in V$ be fixed and consider the function $h : W_1 \times W_2 \to R$ defined by

$$h(w_1, w_2) = I(v + w_1 + w_2).$$

The function h has continuous partial *Fréchet* derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_i h(w_1, w_2)(y_i) = DI(v + w_1 + w_2)(y_i)$$

for $y_i \in W_i$, $i = 1, 2$.

LEMMA 2.3. *Assume that $\lambda_j < c < \lambda_{j+1}$. We also assume that g satisfies the conditions $(g1)$-$(g4)$ and there exists a small number $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Then*
(i) there exists $m_1 < 0$ such that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$(D_1 h(w_1, w_2) - D_1 h(y_1, w_2))(w_1 - y_1) \leq m_1 \|w_1 - y_1\|^2_{L^2(\Omega)},$$

(ii) there exists $m_2 > 0$ such that if w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$(D_2 h(w_1, w_2) - D_2 h(w_1, y_2))(w_2 - y_2) \geq m_2 \|w_2 - y_2\|^2_{L^2(\Omega)}.$$

(iii) there exists a unique solution $z \in W$ of the equation

$$\Delta^2 z + c\Delta z = (I-P)(a(x)g(v+z)) \qquad \text{in } W. \tag{2.3}$$

If we put $z = \theta(v)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). Moreover

$$DI(v + \theta(v))(w) = 0 \qquad \text{for all } w \in W.$$

(iv) If $\tilde{I} : V \to R$ is defined by $\tilde{I}(v) = I(v+\theta(v))$, then $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to v, and

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \qquad \text{for all } v, h \in V.$$

(v) If $v_0 \in V$ is a critical point of $\tilde{I}$ if and only if $v_0 + \theta(v_0)$ is a critical point of I.

Proof. (i) According to the variational characterization of the eigenvalues $\{\Lambda_j\}_{j=1}^{\infty}$ we have

$$\|w_1\|^2 \le \Lambda_j \|w_1\|^2_{L^2(\Omega)} \tag{2.4}$$

for all $w_1 \in W_1$ and

$$\|w_2\|^2 \ge \Lambda_{j+m+1} \|w_1\|^2_{L^2(\Omega)} \tag{2.5}$$

for all $w_2 \in W_2$. If w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned} & (D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1) \\ = & \int_\Omega |\Delta(w_1 - y_1)|^2 - c|\nabla(w_1 - y_1)|^2 - a^+(x)(g(v + w_1 + w_2) \\ - & \ g(v + y_1 + w_2))(w_1 - y_1) \\ + & \int_\Omega a^-(x)(g(v + w_1 + w_2) - g(v + y_1 + w_2))(w_1 - y_1)dx. \end{aligned}$$

Since $a^+(x)(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) > \alpha(\xi_2 - \xi_1)^2$ and (2.4) holds, we see that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$(D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1) \le m\|w_1 - y_1\|^2_{L^2(\Omega)}$$

$$+ \max |(g(v + w_1 + w_2) - g(v + y_1 + w_2))(w_1 - y_1)| \int_{\Omega^-} a^-(x)dx.$$

where $m = 1 - \frac{\alpha}{\Lambda_j} < 0$. Since there exists a small number $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$, We can choose a small number ϵ' such that

$$m\|w_1 - y_1\|^2_{L^2(\Omega)} + \max |(g(v+w_1+w_2) - g(v+y_1+w_2))(w_1 - y_1)| \int_{\Omega^-} a^-(x)dx$$

$$< m\|w_1 - y_1\|^2_{L^2(\Omega)} + \epsilon' < m_1\|w_1 - y_1\|^2_{L^2(\Omega)}$$

with $m_1 < 0$.
(ii) Similarly, we have that if w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$\begin{aligned}
& (D_2h(w_1,w_2) - D_2h(w_1,y_2))(w_2-y_2) \\
= & \int_\Omega |\Delta(w_2-y_2)|^2 - c|\nabla(w_2-y_2)|^2 - a^+(x)(g(v+w_1+w_2) \\
- & g(v+w_1+y_2))(w_2-y_2) \\
+ & \int_\Omega a^-(x)(g(v+w_1+w_2) - g(v+w_1+y_2))(w_2-y_2)dx.
\end{aligned}$$

Since $a^+(x)(g(\xi_2)-g(\xi_1))(\xi_2-\xi_1) < \beta(\xi_2-\xi_1)^2$ and (2.5) holds, we see that if w_1 is in W_1 and w_2 and $y_2 \in W_2$, then

$$(D_2h(w_1,w_2) - D_2h(w_1,y_2))(w_2-y_2) \geq m'\|w_2-y_2\|^2_{L^2(\Omega)}$$
$$+\min|(g(v+w_1+w_2) - g(v+w_1+y_2)(w_2-y_2)| \int_\Omega a^-(x)dx,$$

where $m' = 1 - \frac{\beta}{\Lambda_{j+m+1}} > 0$. We can choose a small number $\epsilon'' > 0$ such that

$$m'\|w_2-y_2\|^2_{L^2(\Omega)} + \min|(g(v+w_1+w_2)-g(v+w_1+y_2)(w_2-y_2)| \int_\Omega a^-(x)dx$$
$$> m'\|w_2-y_2\|^2_{L^2(\Omega)} + \epsilon'' > m_2\|w_2-y_2\|^2_{L^2(\Omega)}$$

with $m_2 > 0$.
(iii) Let $\delta = \frac{\alpha+\beta}{2}$. If $g_1(\xi) = g(\xi) - \delta\xi$, the equation (2.3) is equivalent to

$$z = (\Delta^2 + c\Delta - \delta)^{-1}(I-P)(a(x)g_1(v+z)). \tag{2.6}$$

Since $(\Delta^2 + c\Delta - \delta)^{-1}(I-P)$ is self adjoint, compact and linear map from $(I-P)L^2(\Omega)$ into itself, the eigenvalues of $(\Delta^2 + c\Delta - \delta)^{-1}(I-P)$ are $(\Lambda_l - \delta)^{-1}$, $l \leq j$ or $l \geq j+m+1$. Therefore its L_2 norm is $(\min\{|\Lambda_j - \delta|, |\Lambda_{j+m+1} - \delta|\}^{-1}$. We also have that $|a(x)(g_1(\xi_2) - g_1(\xi_1))| \leq |a^+(x)(g_1(\xi_2) - g_1(\xi_1))| + |a^-(x)(g_1(\xi_2) - g_1(\xi_1))| \leq \max\{|\alpha-\delta|, |\beta-\delta|\}|\xi_2-\xi_1| + |a^-(x)(g_1(\xi_2)-g_1(\xi_1))|$. Since $\int_{\Omega^-} a^-(x)dx < \epsilon$, there exists a small number ϵ_1 such that

$$\begin{aligned}
\max\{|\alpha-\delta|, |\beta-\delta|\}|\xi_2-\xi_1| + |a^-(x)(g_1(\xi_2)-g_1(\xi_1))| \\
< \max\{|\alpha-\delta|, |\beta-\delta|\}|\xi_2-\xi_1| + \epsilon_1|\xi_2-\xi_1)|
\end{aligned}$$

and $r = (\min\{|\Lambda_j - \delta|, |\Lambda_{j+m+1} - \delta|\}^{-1}(\max\{|\alpha-\delta|, |\beta-\delta|\} + \epsilon_1) < 1$. it follows that the right-hand side of (2.6) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)L^2(\Omega)$ into itself with Lipschitz constant

$r < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I-P)L^2(\Omega)$ which satisfies (2.6). If $\theta(v)$ denote the unique $z \in (I-P)L^2(\Omega)$ which solves (2.3), then θ is continuous and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). In fact, if $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$, then

$$\begin{aligned}
&\|z_1 - z_2\|_{L^2(\Omega)} \\
&= \|(\Delta^2 + c\Delta - \delta)^{-1}(I-P)a(x)(g_1(v_1+z_1) - g_1(v_2+z_2))\|_{L^2(\Omega)} \\
&\leq r\|(v_1+z_1) - (v_2+z_2)\|_{L^2(\Omega)} \\
&\leq r(\|v_1 - v_2\|_{L^2(\Omega)} + \|z_1 - z_2\|_{L^2(\Omega)}) \leq r\|v_1 - v_2\| + r\|z_1 - z_2\|.
\end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq C\|v_1 - v_2\|, \qquad C = \frac{r}{1-r}. \tag{2.7}$$

Let $u = v + z$, $v \in V$ and $z = \theta(v)$. If $w \in (I-P)L^2(\Omega) \cap H$, then from (2.3) we see that

$$\int_\Omega [\Delta z \cdot \Delta w - c\nabla z \cdot \nabla w - (I-P)(a(x)g(v+z)w)]dx = 0.$$

Since

$$\int_\Omega \Delta z \cdot \Delta w = 0 \qquad \text{and} \quad \int_\Omega \nabla v \cdot \nabla w = 0,$$

we have

$$DI(v + \theta(v))(w) = 0. \tag{2.8}$$

(iv) Since the functional I has a continuous *Fréchet* derivative DI, $\tilde{I}$ has a continuous *Fréchet* derivative $D\tilde{I}$ with respect to v.
(v) Suppose that there exists $v_0 \in V$ such that $D\tilde{I}(v_0) = 0$. From $D\tilde{I}(v)(h) = DI(v + \theta(v))(h)$ for all $v, h \in V$, $DI(v_0 + \theta(v_0))(h) = 0$ for all $h \in V$. Since $DI(v+\theta(v))(w)$ for all $w \in W$ and H is the direct sum of V and W, it follows that $DI(v_0 + \theta(v_0)) = 0$. Thus $v_0 + \theta(v_0)$ is a solution of (1.1). Conversely if u is a solution of (1.1) and $v = Pu$, then $D\tilde{I}(v) = 0$.

□

3. Proof of Theorem 1.1

Now we shall show that $\tilde{I}(v)$ satisfies the $(P.S.)_c$ condition.

Proposition 3.1. *(Palais-Smale condition)*
Assume that $\lambda_j < c < \lambda_{j+1}$. We also assume that g satisfies $(g1)-(g4)$ and there exists a small number $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx$. Then $\tilde{I}(v)$, $v \in V$, satisfies the Palais-Smale condition.

Proof. Let $u(v) = u_- + v + u_+$, $u_- \in H_-$, $v \in V$, $u_+ \in H_+$. Then we have

$$\begin{aligned}
\tilde{I}(v) &= I(u(v)) = I(u_- + v + u_+) \\
&= \int_\Omega [\frac{1}{2}|\Delta u(v)|^2 - \frac{c}{2}|\nabla u(v)|^2]dx - \int_\Omega a(x)G(u(v))dx \\
&= \int_\Omega [\frac{1}{2}|\Delta z|^2 - \frac{c}{2}|\nabla z|^2]dx - \int_\Omega a(x)G(z)dx \\
&+ \{\int_\Omega [\frac{1}{2}|\Delta u(v)|^2 - \frac{c}{2}|\nabla u(v)|^2 - \frac{1}{2}|\Delta z|^2 \\
&+ \frac{c}{2}|\nabla z|^2]dx - \int_\Omega a(x)[G(u(v)) - G(z)]dx\},
\end{aligned}$$

where $z = u_- + v$. The terms in the bracket

$$\int_\Omega [\frac{1}{2}|\Delta u(v)|^2 - \frac{c}{2}|\nabla u(v)|^2 - \frac{1}{2}|\Delta z|^2 + \frac{c}{2}|\nabla z|^2]dx - \int_\Omega a(x)[G(u(v)) - G(z)]dx$$

$$= \frac{1}{2}\int_\Omega (\Delta^2 + c\Delta)(u_+ + z)u_+ dx - \int_\Omega \int_0^1 a(x)g(su_+ + z)u_+ dsdx.$$

Integrating by parts, we have that

$$\begin{aligned}
&\int_\Omega \int_0^1 a(x)g'(su_+ + z)u_+u_+ sdsdx \\
&= \int_\Omega a(x)g(u_+ + z)u_+ dx - \int_\Omega \int_0^1 a(x)g(su_+ + z)u_+ dsdx \\
&= \int_\Omega (\Delta^2 + c\Delta)(u_+ + z)u_+ dx - \int_\Omega \int_0^1 a(x)g(su_+ + z)u_+ dsdx.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
& \frac{1}{2}\int_\Omega(\Delta^2+c\Delta)(u_++z)u_+dx-\int_\Omega\int_0^1 a(x)g(su_++z)u_+dsdx \\
= & \int_\Omega\int_0^1 a(x)g'(su_++z)u_+u_+sdsdx-\frac{1}{2}\int_\Omega(\Delta^2+c\Delta)(u_+)u_+dx \\
= & \int_\Omega\int_0^1 a^+(x)g'(su_++z)u_+u_+sdsdx-\int_\Omega\int_0^1 a^-(x)g'(su_++z)u_+u_+sdsdx \\
- & \frac{1}{2}\int_\Omega(\Delta^2+c\Delta)(u_+)u_+dx.
\end{aligned}$$

Since

$$\int_\Omega\int_0^1 a^+(x)g'(su_++z)u_+u_+sdsdx-\int_\Omega\int_0^1 a^-(x)g'(su_++z)u_+u_+sdsdx<0$$

and $-\frac{1}{2}\int_\Omega(\Delta^2+c\Delta)(u_+)u_+dx<0$,

$$\int_\Omega\int_0^1 a^+(x)g'(su_++z)u_+u_+sdsdx-\int_\Omega\int_0^1 a^-(x)g'(su_++z)u_+u_+sdsdx$$

$$-\frac{1}{2}\int_\Omega(\Delta^2+c\Delta)(u_+)u_+dx<0.$$

Thus

$$\begin{aligned}
\tilde{I}(v) &\leq \int_\Omega[\frac{1}{2}|\Delta z|^2-\frac{c}{2}|\nabla z|^2]dx-\int_\Omega a^+(x)G(z)dx+\max|G(z)|\int_\Omega a^-(x)dx \\
&\leq \frac{1}{2}(\Lambda_{j+m}-\gamma)\|z\|^2+\max|G(z)|\int_\Omega a^-(x)dx.
\end{aligned}$$

Since $\Lambda_{j+m}<\gamma<\beta$ and $\int_\Omega a^-(x)dx<\epsilon.$ we can choose ϵ_3 such that

$$\tilde{I}(v)\leq\frac{1}{2}(\Lambda_{j+m}-\gamma)\|z\|^2+\epsilon_3<0.$$

Thus $\tilde{I}(v)\to-\infty$ as $\|z\|\to\infty$. So $-\tilde{I}(v)$ is bounded from below and satisfies the Palais-Smale condition. □

PROOF OF THEOREM 1.1

By Lemma 3.1, $\tilde{I}(v)\to-\infty$ as $\|z\|\to\infty$, so $-\tilde{I}(v)$ is bounded from below and satisfies the (P.S.) condition and We claim that 0 is neither a minimum nor degenerate. In fact, we note that $0=0+\theta(0)$, $\theta(0)=$

0. Since $I+\theta$ is continuous, I is identity map, there exists a small neighborhood B of 0 such that if $v \in B$, then, by $(g4)$,

$$\frac{1}{2}\int_\Omega (\Delta^2 v + c\Delta v)v dx - \frac{\Lambda}{2}\int_\Omega v^2 dx + o(\|v\|^2) \le \tilde{I}(v)$$
$$\le \frac{1}{2}\int_\Omega (\Delta^2 v + c\Delta v)v dx - \frac{\bar{\Lambda}}{2}\int_\Omega v^2 dx + o(\|v\|^2),$$

where $(\Lambda, \bar{\Lambda}) \subset (\Lambda_l, \Lambda_{l+1})$. Thus $\tilde{I}(v)$ has at least three nontrivial weak solutions.

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