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STABILITY OF THE RECIPROCAL DIFFERENCE AND ADJOINT FUNCTIONAL EQUATIONS IN THREE VARIABLES

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Abstract. In this paper, we prove stabilities of the reciprocal difference functional equation

$$r\left(\frac{x+y+z}{3}\right) - r(x+y+z) = \frac{2r(x)r(y)r(z)}{r(x)r(y)+r(y)r(z)+r(z)r(x)}$$

and the reciprocal adjoint functional equation

$$r\left(\frac{x+y+z}{3}\right) + r(x+y+z) = \frac{4r(x)r(y)r(z)}{r(x)r(y)+r(y)r(z)+r(z)r(x)}$$

with three variables. Stabilities of the reciprocal difference functional equation and the reciprocal adjoint functional equation in two variables was proved by K. Ravi, J. M. Rassias and B. V. Senthil Kumar. We extend their results to three variables in similar types.

1. Introduction

In 1940, Ulam [16] proposed the Ulam stability problem of the functional equation:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

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In 1941, Hyers [7] answered the Ulam's question for the case of the additive mapping on the Banach spaces as following:

Let G_1 and G_2 be Banach spaces. Assume that a mapping $f : G_1 \to G_2$ satisfies the inequality

$$||f(x + y) - f(x) - f(y)|| \le \varepsilon$$

for all $x, y \in G_1$. Then the limit $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G_1$ and g is the unique additive mapping such that

$$||f(x) - g(x)|| \le \varepsilon, \quad \forall x \in G_1.$$

In 1949, the above result was improved Bourgin [4], with the following result:

Let $\varepsilon > 0$, $s \in S$:compact, $x \in C(S)$. If $T : S \to C$ satisfies

$$||T(xy) - T(x)T(y)|| \le \delta \min |x(s)| \min |y(s)|,$$

then T is multiplicative.

Next year, T. Aoki [1] has improved Hyers' result, which is following:

Let E_1 be a Banach space and G_2 a Banach space. Assume that a mapping $f : G_1 \to G_2E$ satisfies

$$||f(x + y) - f(x) - f(y)|| \le \varepsilon(||x||^p + ||y||^p)$$

for all $x, y \in G_1$, $\varepsilon > 0$ and $p < 1$. Then the limit $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G_1$ and g is the unique additive mapping such that

$$||f(x) - g(x)|| \le \frac{2\varepsilon}{2 - 2^p}||x||^p, \quad \forall x \in G_1.$$

Aoki's result was improved by Th. M. Rassias [14] and P. Gâvruta [6] in 1978 and 1994, respectively.

During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [2]-[15]).

K. Ravi, J. M. Rassias and B. V. Senthil Kumar [15] proved some interesting results on stabilities of a reciprocal difference functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

and a reciprocal adjoint functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}.$$

We can know that a solution of the above two equations is a reciprocal function, namely, $r(x) = 1/x$. Motivated by this, we have found the following reciprocal equations consisted with three variables.

In this paper, we prove the stabilities of the reciprocal difference functional equation

$$(1)\quad r\left(\frac{x+y+z}{3}\right) - r(x+y+z) = \frac{2r(x)r(y)r(z)}{r(x)r(y)+r(y)r(z)+r(z)r(x)}$$

and the reciprocal adjoint functional equation

$$(2)\quad r\left(\frac{x+y+z}{3}\right) + r(x+y+z) = \frac{4r(x)r(y)r(z)}{r(x)r(y)+r(y)r(z)+r(z)r(x)}$$

with three variables in the sense of Gâvruta.

2. Stabilities

THEOREM 1. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

(3)
$$\left|f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}\right| \le \phi(x,y,z)$$

for all $x, y, z \in X$, where $\phi : X^3 \to Y$ is a function such that for all $x, y, z \in X$

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{3^i}\phi\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}, \frac{z}{3^{i+1}}\right) < \infty.$$

Then there exists an unique reciprocal difference mapping $r : X \to Y$ which satisfies (1) and

$$|f(x) - r(x)| \leq \Phi(x,x,x)$$

for all $x \in X$.

Proof. Replacing (x, y, z) by $\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right)$ in (3), we obtain

$$\left|\frac{1}{3}f\left(\frac{x}{3}\right) - f(x)\right| \leq \phi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right). \tag{4}$$

Again replacing x by $\frac{x}{3}$ in (4) and dividing by 3, we get

$$\left|\frac{1}{3^2}f\left(\frac{x}{3^2}\right) - f(x)\right| \leq \phi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) + \frac{1}{3}\phi\left(\frac{x}{3^2}, \frac{x}{3^2}, \frac{x}{3^2}\right).$$

Again replacing x by $\frac{x}{3}$ in the above inequality and dividing by 3, we get

$$\left|\frac{1}{3^3}f\left(\frac{x}{3^3}\right) - f(x)\right| \leq \sum_{i=0}^{2} \frac{1}{3^i}\phi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right).$$

Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}\left|\frac{1}{3^n}f\left(\frac{x}{3^n}\right) - f(x)\right| &\leq \sum_{i=0}^{n-1} \frac{1}{3^i}\phi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right) \\ &\leq \sum_{i=0}^{\infty} \frac{1}{3^i}\phi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right)\end{aligned} \tag{5}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{1}{3^n}f\left(\frac{x}{3^n}\right)\right\}$, replace x by $\frac{x}{3^p}$ in (5) and divide by 3^p , we find that for

$n > p > 0$

$$\left|\frac{1}{3^p}f(\frac{x}{3^p}) - \frac{1}{3^{n+p}}f(\frac{x}{3^{n+p}})\right| = \frac{1}{3^p}\left|f(\frac{x}{3^p}) - \frac{1}{3^n}f(\frac{x}{3^{n+p}})\right|$$
$$\leq \sum_{i=0}^{\infty}\frac{1}{3^{p+i}}\phi\left(\frac{x}{3^{p+i+1}}, \frac{x}{3^{p+i+1}}, \frac{x}{3^{p+i+1}}\right)$$
$$\leq \sum_{i=p}^{\infty}\frac{1}{3^i}\phi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right).$$

Allow $p \to \infty$, the right-hand side of the above inequality tends to 0. Thus the sequence $\left\{\frac{1}{3^n}f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence. Thus we may define a mapping $r : X \to Y$ by

$$r(x) := \lim_{n\to\infty}\frac{1}{3^n}f\left(\frac{x}{3^n}\right), \quad \forall x \in X.$$

By (5) with $n \to \infty$, we have

$$|f(x) - r(x)| \leq \Phi(x, x, x), \quad \forall x \in X.$$

Replacing (x, y, z) by $\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)$ in (3) and dividing by 3^n, we obtain

$$\frac{1}{3^n}\left|f\left(\frac{1}{3^n}(\frac{x+y+z}{3})\right) - f\left(\frac{1}{3^n}(x+y+z)\right)\right.$$
$$\left. - \frac{2f(\frac{x}{3^n})f(\frac{y}{3^n})f(\frac{z}{3^n})}{f(\frac{x}{3^n})f(\frac{y}{3^n}) + f(\frac{y}{3^n})f(\frac{z}{3^n}) + f(\frac{z}{3^n})f(\frac{x}{3^n})}\right|$$
$$\leq \frac{1}{3^n}\phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)$$

for all $(x, y, z) \in X^3$. Letting $n \to \infty$, we have

$$r\left(\frac{x+y+z}{3}\right) - r(x+y+z) = \frac{2r(x)r(y)r(z)}{r(x)r(y) + r(y)r(z) + r(z)r(x)}$$

for all $(x, y, z) \in X^3$. Now let $B : X \to Y$ be another function which satisfies the equation (1) and $|f(x) - B(x)| \leq \Phi(x, x, x)$ for all $x \in X$.

Since $3^n B(x) = B(\frac{x}{3^n})$ and $3^n r(x) = r(\frac{x}{3^n})$, we have

$$\begin{aligned}|B(x) - r(x)| &= \frac{1}{3^n}\left|B(\frac{x}{3^n}) - r(\frac{x}{3^n})\right| \\ &\le \frac{1}{3^n}\left(\left|B(\frac{x}{3^n}) - f(\frac{x}{3^n})\right| + \left|f(\frac{x}{3^n}) - r(\frac{x}{3^n})\right|\right) \\ &\le 2\sum_{i=0}^{\infty}\frac{1}{3^{n+i}}\phi\left(\frac{x}{3^{n+i+1}}, \frac{x}{3^{n+i+1}}, \frac{x}{3^{n+i+1}}\right)\end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in the above inequality, we find that A is unique. This completes the proof of the theorem. □

Corollary 2. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\left|f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y) + f(y)f(z) + f(z)f(x)}\right| \le \delta$$

for all $x, y, z \in X$. Then there exists a unique reciprocal difference mapping $r : X \to Y$ which satisfies (1) and

$$|f(x) - r(x)| \le \frac{3}{2}\delta$$

for all $x \in X$.

Proof. In (3), putting $\phi(x, y, z) = \delta$ for all $x, y, z \in X$. Then, by Theorem 1, $\Phi(x) = \sum_{i=0}^{\infty} \frac{\delta}{3^i} = \frac{3\delta}{2}$ for all $x \in X$. This completes the proof of the corollary. □

Theorem 3. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

(6)
$$\left|f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y) + f(y)f(z) + f(z)f(x)}\right| \le \psi(x, y, z)$$

for all $x, y, z \in X$, *where* $\psi : X^3 \to Y$ *is a function such that for all* $x, y, z \in X$

$$\Psi(x,y,z) := \sum_{i=0}^{\infty} 3^{i+1}\psi\left(3^i x, 3^i y, 3^i z\right) < \infty.$$

Then there exists a unique reciprocal difference mapping $r : X \to Y$ *which satisfies (1) and*

$$|f(x) - r(x)| \leq \Psi(x,x,x)$$

for all $x \in X$.

Proof. Replacing (x,y,z) by (x,x,x) in (6) and multiplying by 3, we obtain

$$|f(x) - 3f(3x)| \leq 3\psi(x,x,x). \tag{7}$$

Again replacing x by $3x$ in (7) and multiplying by 3, we get

$$\left|f(x) - 3^2 f\left(3^2 x\right)\right| \leq 3\psi(x,x,x) + 3^2\psi(3x,3x,3x).$$

Again replacing x by $3x$ in the above inequality and multiplying by 3, we get

$$\left|f(x) - 3^3 f\left(3^3 x\right)\right| \leq \sum_{i=0}^{2} 3^{i+1}\psi\left(3^i x, 3^i x, 3^i\right).$$

Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} |f(x) - 3^n f(3^n x)| &\leq \sum_{i=0}^{n-1} 3^{i+1}\psi\left(3^i x, 3^i x, 3^i x\right) \\ &\leq \sum_{i=0}^{\infty} 3^{i+1}\psi\left(3^i x, 3^i x, 3^i x\right) \end{aligned} \tag{8}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{3^n f(3^n x)\}$, replacing x by $3^p x$ in (8) and multiplying 3^p , we find that for $n > p > 0$

$$\begin{aligned}\left|3^p f(3^p x) - 3^{n+p} f(3^{n+p} x)\right| &= 3^p \left|f(3^p x) - 3^n f(3^{n+p} x)\right| \\ &\leq \sum_{i=0}^{n-p} 3^{p+i+1} \psi\left(3^{p+i} x, 3^{p+i} x, 3^{p+i} x\right) \\ &\leq \sum_{i=p}^{\infty} 3^{i+1} \psi\left(3^i x, 3^i x, 3^i x\right).\end{aligned}$$

Allow $p \to \infty$, the right-hand side of the above inequality tends to 0. Thus the sequence $\{3^n f(3^n x)\}$ is a Cauchy sequence. Thus we may define a mapping $r : X \to Y$ by

$$r(x) := \lim_{n\to\infty} 3^n f(3^n x)$$

for all $x \in X$. By (8) with $n \to \infty$, we have

$$|f(x) - r(x)| \leq \Psi(x, x, x)$$

for all $x \in X$. Replacing (x, y, z) by $(3^n x, 3^n y, 3^n z)$ in (6) and multiplying by 3^n, we obtain

$$\begin{aligned}3^n \Bigg| f\left(3^n\left(\frac{x+y+z}{3}\right)\right) &- f\left(3^n(x+y+z)\right) \\ &- \frac{2f(3^n x)f(3^n y)f(3^n z)}{f(3^n x)f(3^n y) + f(3^n y)f(3^n z) + f(3^n z)f(3^n x)}\Bigg| \\ \leq \frac{1}{3} 3^{n+1} \psi(3^n x, 3^n y, 3^n z)&\end{aligned}$$

for all $(x, y, z) \in X^3$. Since $\Psi(x, y, z)$ is finite, letting $n \to \infty$, we have

$$r\left(\frac{x+y+z}{3}\right) - r(x+y+z) = \frac{2r(x)r(y)r(z)}{r(x)r(y) + r(y)r(z) + r(z)r(x)}$$

for all $(x, y, z) \in X^3$. Now let $B : X \to Y$ be another function which satisfies the equation (1) and $|f(x) - B(x)| \leq \Psi(x, x, x)$ for all $x \in X$.

Since $B(3^n x) = \frac{1}{3^n} B(x)$ and $r(3^n x) = \frac{1}{3^n} r(x)$ we have

$$\begin{aligned} |B(x) - r(x)| &= 3^n \, |B(3^n x) - r(3^n x)| \\ &\le 3^n \left(|B(3^n x) - f(3^n x)| + |f(3^n x) - r(3^n x)| \right) \\ &\le 2 \sum_{i=0}^{\infty} 3^{n+i+1} \psi \left(3^{n+i} x, 3^{n+i} x, 3^{n+i} x \right) \end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in the above inequality, we find that A is unique. This completes the proof of the theorem. □

THEOREM 4. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

(9)
$$\left| f\left(\frac{x+y+z}{3} \right) + f(x+y+z) - \frac{4f(x)f(y)f(z)}{f(x)f(y) + f(y)f(z) + f(z)f(x)} \right| \le \phi(x,y,z)$$

for all $x, y, z \in X$, where $\phi : X^3 \to Y$ is a function such that for all $x, y, z \in X$

$$\Phi(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{3^i} \phi \left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}, \frac{z}{3^{i+1}} \right) < \infty.$$

Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ which satisfies (2) and

$$|f(x) - r(x)| \le \Phi(x,x,x)$$

for all $x \in X$.

Proof. Replacing (x,y,z) by $\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right)$ in (9), we obtain

$$\left| \frac{1}{3} f\left(\frac{x}{3}\right) - f(x) \right| \le \phi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3} \right),$$

which states nothing else but (4) in Theorem 1. Hence, the same processing as the proof of Theorem 1 completes the proof. □

COROLLARY 5. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\left| f\left(\frac{x+y+z}{3}\right) + f(x+y+z) - \frac{4f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)} \right| \leq \delta$$

for all $x, y, z \in X$. Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ which satisfies (2) and the inequality

$$|f(x) - r(x)| \leq \frac{3}{2}\delta$$

for all $x \in X$.

THEOREM 6. *Let X and Y be spaces of non-zero real numbers. Assume that $f : X \to Y$ satisfies the functional inequality*

(10)
$$\left| f\left(\frac{x+y+z}{3}\right) + f(x+y+z) - \frac{4f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)} \right| \\ \leq \psi(x,y,z)$$

for all $x, y, z \in X$, where $\psi : X^3 \to Y$ is a function such that for all $x, y, z \in X$

$$\Psi(x,y,z) := \sum_{i=0}^{\infty} 3^{i+1}\psi\left(3^i x, 3^i y, 3^i z\right) < \infty.$$

Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ which satisfies (2) and the inequality

$$|f(x) - r(x)| \leq \Psi(x,x,x)$$

for all $x \in X$.

Proof. Replacing (x, y, z) by (x, x, x) in (10) and multiplying 3, we obtain

$$|f(x) - 3f(3x)| \leq 3\psi(x,x,x),$$

which states nothing else but (7) in Theorem 3. Hence, the same processing as the proof of Theorem 3 completes the proof.

By the same method of proof in Theorem 3, we complete the proof.□

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