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EXISTENCE AND MANN ITERATIVE METHODS OF POSITIVE SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

Jinbiao Hao and Shin Min Kang*

Abstract. In this paper, we study the first order nonlinear neutral difference equation:

$$\Delta\big(x(n) + px(n-\tau)\big) + f\big(n, x(n-c), x(n-d)\big) = r(n), \quad n \geq n_0.$$

Using the Banach fixed point theorem, we prove the existence of bounded positive solutions of the equation, suggest Mann iterative schemes of bounded positive solutions, and discuss the error estimates between bounded positive solutions and sequences generated by Mann iterative schemes.

1. Introduction and preliminaries

In this paper, we are concerned with the following first order nonlinear neutral difference equation:

$$\Delta\big(x(n)+px(n-\tau)\big)+f\big(n,x(n-c),x(n-d)\big) = r(n), \quad n \geq n_0, \tag{1.1}$$

where $p \in \mathbb{R}$, $\tau \in \mathbb{N}$, $n_0, c, d \in \mathbb{Z}$, $\{r(n)\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$ and $f : \mathbb{N}_{n_0} \times \mathbb{R}^2 \to \mathbb{R}$.

Some special cases of the equation (1.1) have been studied by several authors [1]-[4]. In particular, Chen and Zhang [2] and Lalli and Zhang

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*Corresponding author

[3] discussed the oscillation, the bounded oscillation and the existence of positive solutions of the first order neutral delay difference equation:

$$\Delta\big(x(n)+px(n-k)\big)-a(n)x(n-r)-b(n)x(n-l)=0, \quad n\ge n_0. \tag{1.2}$$

The aim of this paper is to prove the existence of bounded positive solutions of the equation (1.1), to construct Mann iterative schemes of bounded positive solutions, and to establish the error estimates between bounded positive solutions and sequences generated by Mann iterative schemes.

Throughout this paper, we assume that Δ stands for the forward difference operator: $\Delta x(n) = x(n+1) - x(n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and positive integers, respectively, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$,

$$\begin{aligned} \beta &= \inf\{n_0-\tau, n_0-c, n_0-d\},\\ \mathbb{Z}_\beta &= \{n : n\in\mathbb{Z} \text{ and } n\ge\beta\},\\ \mathbb{N}_{n_0} &= \{n : n\in\mathbb{N}_0 \text{ and } n\ge n_0\}, \end{aligned}$$

l^∞_β denote the Banach space of all bounded real sequences on $\mathbb{Z}_\beta$ with norm $\|x\| = \sup\{|x(n)| : n\ge\beta\}$ for any $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta$ and

$$A(1,2) = \big\{x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in X : 1\le x(n)\le 2,\ n\ge\beta\big\}.$$

It is easy to see that $A(1,2)$ is a bounded closed subset of l^∞_β. A sequence $\{x(n)\}_{n\in\mathbb{Z}_\beta}$ is said to be a *solution* of the equation (1.1) if there exists $H\ge n_0+\max\{\tau,|c|,|d|\}+|\beta|$ such that the equation (1.1) holds for $n\ge H$.

Put

(A_1) there exist two sequences $\{p(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying

$$\begin{aligned} &|f(n,u,v)-f(n,\bar u,\bar v)|\\ &\le p(n)\max\{|u-\bar u|,|v-\bar v|\}, \quad \forall n\in\mathbb{N}_{n_0},\ u,\bar u,v,\bar v\in[1,2]; \end{aligned}$$

$$|f(n,u,v)|\le q(n), \quad \forall n\in\mathbb{N}_{n_0},\ u,v\in[1,2];$$

$$\sum_{s=n_0}^{\infty}\max\{p(s),q(s),|r(s)|\}<+\infty. \tag{1.3}$$

Consider

(A_2) $\{a_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with $\sum_{m=0}^{\infty} a_m = +\infty$.

2. Existence of bounded positive solutions and Mann iterative algorithms

Now, we are ready to give our main theorems.

THEOREM 2.1. *Let (A_1) and (A_2) hold and $p \in [0,1)$. Then there exist $\theta \in (0,1)$ and $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_m(n)\}_{n\in\mathbb{Z}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the following scheme:*

$$x_{m+1}(n) = \begin{cases} (1-a_m)x_m(n) + a_m\Big\{\frac{3(1+p)}{2} - px_m(n-\tau) \\ \quad + \sum_{s=n}^{\infty}\big[f\big(s, x_m(s-c), x_m(s-d)\big) - r(s)\big]\Big\}, \\ \quad n \geq H,\ m \geq 0, \\ x_{m+1}(H), \quad \beta \leq n < H,\ m \geq 0 \end{cases} \tag{2.1}$$

converges to a bounded positive solution $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ of the equation (1.1) and has the following error estimate:

$$\|x_{m+1} - x\| \leq e^{-(1-\theta)\sum_{i=0}^{m} a_i}\|x_0 - x\|, \quad \forall m \geq 0. \tag{2.2}$$

Proof. It follows from (1.3) and $p \in [0,1)$ that there exist $\theta \in (0,1)$ and $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ satisfying

$$\theta = p + \sum_{s=H}^{\infty} p(s), \tag{2.3}$$

$$\sum_{s=H}^{\infty}[q(s) + |r(s)|] \leq \frac{1-p}{2}. \tag{2.4}$$

Define a mapping $G : A(1,2) \to l_\beta^\infty$ by

$$Gx(n) = \begin{cases} \frac{3(1+p)}{2} - px(n-\tau) \\ \quad + \sum_{s=n}^{\infty}\big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big], \\ \quad n \geq H, \\ Gx(H), \quad \beta \leq n < H \end{cases} \tag{2.5}$$

for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$. It follows from (2.4) and (2.5) that for any $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ and $n \ge H$,

$$\begin{aligned} Gx(n) &= \frac{3(1+p)}{2} - px(n-\tau) + \sum_{s=n}^{\infty} \big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big] \\ &\le \frac{3(1+p)}{2} - p + \sum_{s=n}^{\infty}[q(s) + |r(s)|] \\ &\le \frac{3(1+p)}{2} - p + \frac{1-p}{2} \\ &\le 2 \end{aligned}$$

and

$$\begin{aligned} Gx(n) &= \frac{3(1+p)}{2} - px(n-\tau) + \sum_{s=n}^{\infty} \big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big] \\ &\ge \frac{3(1+p)}{2} - 2p - \sum_{s=n}^{\infty}[q(s) + |r(s)|] \\ &\ge \frac{3(1+p)}{2} - 2p - \frac{1-p}{2} \\ &\ge 1, \end{aligned}$$

which give that $G(A(1,2)) \subseteq A(1,2)$. In light of (A_1), (2.2) and (2.4), we conclude that for any $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ and $n \ge H$,

$$\begin{aligned} &|Gx(n) - Gy(n)| \\ &= \Bigg| \frac{3(1+p)}{2} - px(n-\tau) \\ &\qquad + \sum_{s=n}^{\infty} \big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big] - \frac{3(1+p)}{2} \\ &\qquad + py(n-\tau) + \sum_{s=n}^{\infty} \big[f\big(s, y(s-c), y(s-d)\big) - r(s)\big] \Bigg| \\ &\le p|x(n-\tau) - y(n-\tau)| \\ &\quad + \sum_{s=n}^{\infty} \big|f\big(s, x(s-c), x(s-d)\big) - f\big(s, y(s-c), y(s-d)\big)\big| \end{aligned}$$

$$\begin{aligned}&\le p\|x-y\| + \sum_{s=n}^{\infty} p(s)\max\big\{|x(s-c)-y(s-c)|,\\&\qquad |x(s-d)-y(s-d)|\big\}\\&\le p\|x-y\| + \sum_{s=H}^{\infty} p(s)\|x-y\|,\end{aligned}$$

which implies that

$$\|Gx - Gy\| \le \theta\|x-y\|, \quad \forall x, y \in A(1,2). \tag{2.6}$$

That is, G is a contraction mapping in the bounded closed subset $A(1,2)$ of the Banach space l_β^∞. Hence the Banach fixed point theorem ensures that G has a unique fixed point $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$, which means that

$$\begin{aligned}x(n) &= \frac{3(1+p)}{2} - px(n-\tau)\\&\quad + \sum_{s=n}^{\infty}\big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big], \quad \forall n \ge H,\end{aligned}$$

which implies that

$$\Delta\big(x(n) + px(n-\tau)\big) = -f\big(n, x(n-c), x(n-d)\big) + r(n), \quad \forall n \ge H,$$

that is, the fixed point $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ of G is a bounded positive solution of the equation (1.1). Note that (2.1) and (2.6) yield that for any $m \ge 0$ and $n \ge H$,

$$\begin{aligned}&|x_{m+1}(n) - x(n)|\\&= \Bigg|(1-a_m)x_m(n) + a_m\Bigg\{\frac{3(1+p)}{2} - px_m(n-\tau)\\&\qquad + \sum_{s=n}^{\infty}\big[f\big(s, x_m(s-c), x_m(s-d)\big) - r(s)\big]\Bigg\} - x(n)\Bigg|\\&\le (1-a_m)|x_m(n) - x(n)| + a_m|Gx_m(n) - Gx(n)|\\&\le (1-a_m)|x_m(n) - x(n)| + a_m\theta|x_m(n) - x(n)|\\&\le \big(1-(1-\theta)a_m\big)\|x_m - x\|\\&\le e^{-(1-\theta)\sum_{i=0}^{m} a_i}\|x_0 - x\|, \quad \forall m \ge 0,\end{aligned}$$

which yields that

$$\|x_{m+1} - x\| \le e^{-(1-\theta)\sum_{i=0}^{m} a_i}\|x_0 - x\|, \quad \forall m \ge 0,$$

that is, (2.2) holds. Thus (1.3) and (2.3) guarantee that $\lim_{m\to\infty} x_m = x$. This completes the proof. □

THEOREM 2.2. *Let (A_1) and (A_2) hold and $p > 1$. Then there exist $\theta \in (0,1)$ and $H \ge n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_m(n)\}_{n\in\mathbb{Z}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the following scheme:*

$$x_{m+1}(n) = \begin{cases} (1-a_m)x_m(n) + a_m\Big\{\frac{3}{2}\big(1+\frac{1}{p}\big) - \frac{1}{p}x_m(n+\tau) \\ \quad +\frac{1}{p}\sum_{s=n+\tau}^{\infty}\big[f\big(s, x_m(s-c), x_m(s-d)\big) \\ \quad\quad -r(s)\big]\Big\}, \quad n \ge H,\ m \ge 0, \\ x_{m+1}(H), \quad \beta \le n < H,\ m \ge 0 \end{cases} \tag{2.7}$$

converges to a bounded positive solution $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ of the equation (1.1) and has the error estimate (2.2).

Proof. It follows from (A_1) and $p > 1$ that there exist $\theta \in (0,1)$ and $H \ge n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ satisfying

$$\theta = \frac{1}{p}\left(1 + \sum_{s=H}^{\infty} p(s)\right), \tag{2.8}$$

$$\sum_{s=H}^{\infty}[q(s) + |r(s)|] \le \frac{1}{2}(p-1). \tag{2.9}$$

Define a mapping $G : A(1,2) \to l_\beta^\infty$ by

$$Gx(n) = \begin{cases} \frac{3}{2}\big(1+\frac{1}{p}\big) - \frac{x(n+\tau)}{p} \\ \quad +\frac{1}{p}\sum_{s=n+\tau}^{\infty}\big[f\big(s, x(s-c), x(s-d)\big) - r(s)\big], \\ \quad n \ge H, \\ Gx(H), \quad \beta \le n < H \end{cases} \tag{2.10}$$

for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$. It follows from (2.9) and (2.10) that for any $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ and $n \geq H$,

$$\begin{aligned} Gx(n) &= \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{x(n+\tau)}{p} \\ &\quad + \frac{1}{p}\sum_{s=n+\tau}^{\infty}\left[f\big(s, x(s-c), x(s-d)\big) - r(s)\right] \\ &\leq \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{\infty}[q(s)+|r(s)|] \\ &\leq \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{1}{p} + \frac{1}{2p}(p-1) \\ &\leq 2 \end{aligned}$$

and

$$\begin{aligned} Gx(n) &= \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{x(n+\tau)}{p} \\ &\quad + \frac{1}{p}\sum_{s=n+\tau}^{\infty}\left[f\big(s, x(s-c), x(s-d)\big) - r(s)\right] \\ &\geq \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{2}{p} - \frac{1}{p}\sum_{s=n+\tau}^{\infty}[q(s)+|r(s)|] \\ &\geq \frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{2}{p} - \frac{1}{2p}(p-1) \\ &\geq 1, \end{aligned}$$

which imply that $G(A(1,2)) \subseteq A(1,2)$. In view of (A_1), (2.8) and (2.10), we obtain that for any $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ and $n \geq H$,

$$\begin{aligned} &|Gx(n) - Gy(n)| \\ &= \Bigg|\frac{3}{2}\left(1+\frac{1}{p}\right) - \frac{x(n+\tau)}{p} \\ &\quad + \frac{1}{p}\sum_{s=n+\tau}^{\infty}\left[f\big(s, x(s-c), x(s-d)\big) - r(s)\right] - \frac{3}{2}\left(1+\frac{1}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{y(n+\tau)}{p}-\frac{1}{p}\sum_{s=n+\tau}^{\infty}\big[f\big(s,y(s-c),y(s-d)\big)-r(s)\big]\Big| \\
\le\ & \frac{1}{p}|x(n+\tau)-y(n+\tau)| \\
&+\frac{1}{p}\sum_{s=n+\tau}^{\infty}\big|f\big(s,x(s-c),x(s-d)\big)-f\big(s,y(s-c),y(s-d)\big)\big| \\
\le\ & \frac{1}{p}\|x-y\|+\frac{1}{p}\sum_{s=n+\tau}^{\infty}p(s)\max\big\{|x(s-c)-y(s-c)|, \\
&\quad |x(s-d)-y(s-d)|\big\} \\
\le\ & \frac{1}{p}\|x-y\|+\frac{1}{p}\sum_{s=H+\tau}^{\infty}p(s)\|x-y\| \\
\le\ & \theta\|x-y\|,
\end{aligned}
$$

which implies (2.6). Thus $G : A(1,2) \to A(1,2)$ is a contraction mapping and it has a unique fixed point $x \in A(1,2)$, which yields that

$$
\begin{aligned}
x(n) = &\ \frac{3}{2}\left(1+\frac{1}{p}\right)-\frac{x(n+\tau)}{p} \\
&+\frac{1}{p}\sum_{s=n+\tau}^{\infty}\big[f\big(s,x(s-c),x(s-d)\big)-r(s)\big], \quad \forall n \ge H,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&x(n+\tau)+px(n) \\
&=\frac{3(p+1)}{2}+\sum_{s=n+\tau}^{\infty}\big[f\big(s,x(s-c),x(s-d)\big)-r(s)\big], \quad \forall n \ge H,
\end{aligned}
$$

which means that

$$
\Delta\big(x(n)+px(n-\tau)\big) = -f\big(n,x(n-c),x(n-d)\big)+r(n), \quad \forall n \ge H+\tau,
$$

which is a bounded positive solution of the equation (1.1). It follows from (2.6), (2.7) and (2.10) that for any $m \geq 0$ and $n \geq H$,

$$\begin{aligned}
&|x_{m+1}(n) - x(n)| \\
&= \bigg|(1-a_m)x_m(n) + a_m\bigg\{\frac{3}{2}\Big(1+\frac{1}{p}\Big) - \frac{1}{p}x_m(n+\tau) \\
&\qquad + \frac{1}{p}\sum_{s=n+\tau}^{\infty}\big[f\big(s, x_m(s-c), x_m(s-d)\big) - r(s)\big]\bigg\} - x(n)\bigg| \\
&\leq (1-a_m)|x_m(n) - x(n)| + a_m|Gx_m(n) - Gx(n)| \\
&\leq (1-a_m)|x_m(n) - x(n)| + a_m\theta|x_m(n) - x(n)| \\
&\leq \big(1-(1-\theta)a_m\big)\|x_m - x\| \\
&\leq e^{-(1-\theta)\sum_{i=0}^{m} a_i}\|x_0 - x\|, \quad \forall m \geq 0,
\end{aligned}$$

which implies that (2.2) holds and $\lim_{m\to\infty} x_m = x$ by (A_2). This completes the proof. □

Theorem 2.3. *Let* (A_1) *and* (A_2) *hold and* $p \in (-1, 0)$. *Then there exist* $\theta \in (0,1)$ *and* $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ *such that for each* $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_m(n)\}_{n\in\mathbb{Z}_\beta}\}_{m\in\mathbb{N}_0}$ *generated by the scheme* (2.1) *converges to a bounded positive solution* $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ *of the equation* (1.1) *and has the error estimate* (2.2).

Proof. It follows from (1.3) and $p \in (-1,0)$ that there exist $\theta \in (0,1)$ and $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ satisfying

$$\theta = -p + \sum_{s=H}^{\infty} p(s), \tag{2.11}$$

$$\sum_{s=H}^{\infty}[q(s) + |r(s)|] \leq \frac{1+p}{2}. \tag{2.12}$$

The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

THEOREM 2.4. *Let (A_1) and (A_2) hold and $p < -1$. Then there exist $\theta \in (0,1)$ and $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_m(n)\}_{n\in\mathbb{Z}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme (2.7) converges to a bounded positive solution $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(1,2)$ of the equation (1.1) and has the error estimate (2.2).*

Proof. It follows from (A_1) and $p < -1$ that there exist $\theta \in (0,1)$ and $H \geq n_0 + \max\{\tau, |c|, |d|\} + |\beta|$ satisfying

$$\theta = \frac{-1}{p}\left(1 + \sum_{s=H}^{\infty} p(s)\right), \tag{2.13}$$

$$\sum_{s=H}^{\infty} [q(s) + |r(s)|] \leq \frac{1}{2}(-p-1). \tag{2.14}$$

The rest of the proof is similar to that of Theorem 2.2 and is omitted. This completes the proof. □

Finally we construct an example to illustrate our results.

EXAMPLE 2.1. Consider the first order nonlinear neutral delay difference equation:

$$\begin{aligned}\Delta\big(x(n) + px(n-\tau)\big) &+ \frac{nx_{n-3} - \sqrt{n}x_{n+5}}{1 + n^3 + x_{n-3}^2} \\ &= \frac{(-1)^n \sin(n^3 - 3n + 1)}{1 + n\ln^2 n}, \quad n \geq 1,\end{aligned} \tag{2.15}$$

where $\tau \in \mathbb{N}$ and $p \in \mathbb{R}\setminus\{\pm 1\}$. Let $\{a_m\}_{m\geq 0}$ be an arbitrary sequence in $[0,1]$ satisfying (A_2), $c = 3$, $d = -5$ and

$$\begin{aligned}
f(n,u,v) &= \frac{nu - \sqrt{n}v}{1 + n^3 + u^2},\\
p(n) &= \frac{(n + \sqrt{n})(1 + n^3) + 4n + 20\sqrt{n}}{(2 + n^3)^2}, \quad q(n) = \frac{2(n + \sqrt{n})}{2 + n^3},\\
r(n) &= \frac{(-1)^n \sin(n^3 - 3n + 1)}{1 + n\ln^2 n}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times [1,2] \times [1,2].
\end{aligned}$$

It is clear that (A_1) and (A_2) hold. It follows from one of Theorems 2.1∼2.4 that for any $p \in \mathbb{R} \setminus \{\pm 1\}$, the equation (2.15) possesses a bounded positive solution in $A(1, 2)$ and the Mann iterative sequence $\{x_n\}_{n \geq 0}$ generated by one of (2.1) and (2.7) converges to the bounded positive solution.

None of the known results can be applied to the above example.

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Department of Mathematics
Liaoning Normal University
Dalian, Liaoning 116029
People's Republic of China
E-mail: haojinbiao@yahoo.cn

Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail: smkang@gnu.ac.kr