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GENERALIZATION OF THE SIGN REVERSING INVOLUTION ON THE SPECIAL RIM HOOK TABLEAUX

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Abstract. Eğecioğlu and Remmel [1] gave a combinatorial interpretation for the entries of the inverse Kostka matrix K^{-1}. Using this interpretation Sagan and Lee [8] constructed a sign reversing involution on special rim hook tableaux. In this paper we generalize Sagan and Lee's algorithm on special rim hook tableaux to give a combinatorial partial proof of $K^{-1}K = I$.

1. Introduction

Let λ, μ be partitions of a nonnegative integer n. Kostka number $K_{\lambda,\mu}$ is the number of column strict tableaux T of shape $\mathrm{sh}(T) = \lambda$ and $\mathrm{content}(T) = \mu$. For fixed n, we collect these numbers into the Kostka matrix $K = (K_{\lambda,\mu})$. If we use the reverse lexicographic order on partitions, K is an upper unitriangular matrix, and so K is invertible.

In [1] Eğecioğlu and Remmel gave a combinatorial interpretation for the entries of the inverse Kostka matrix K^{-1} and used the combinatorial interpretation to give a proof of the fact that $KK^{-1} = I$ using a sign reversing involution, but were not able to do the same thing for the identity $K^{-1}K = I$.

In [8] Sagan and Lee constructed an algorithmic sign-reversing involution which proves that the last column of $K^{-1}K = I$ is correct. Parts of Sagan and Lee' procedure are reminiscent of the lattice path involution of Lindström [5] and Gessel-Viennot [3, 4] as well as the rim hook Robinson-Schensted algorithm of White [11] and Stanton-White [10].

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In this paper we generalize Sagan and Lee's algorithm on special rim hook tableaux, which gives a combinatorial partial proof of $K^{-1}K = I$.

2. Definitions and combinatorial interpretation for $K^{-1}_{\mu,\lambda}$

In this section we describe some definitions necessary for later. See [2], [6], [7] or [9] for definitions and notations not described here.

Definition 2.1. A *partition* λ of a positive integer n, denoted $\lambda \vdash n$, is a weakly decreasing sequence of positive integers summing to n. We say each term λ_i is a *part* of λ and the number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. In addition, we will use the notation $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ which means that the integer j appears m_j times in λ.

Definition 2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition. The *Ferrers diagram* D_λ of λ is the array of cells or boxes arranged in rows and columns, λ_1 in the first row, λ_2 in the second row, etc., with each row left-justified. That is,

$$D_\lambda = \{(i,j) \in \mathbf{Z}^2 \mid 1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i\},$$

where we regard the elements of D_λ as a collection of boxes in the plane with matrix-style coordinates.

Definition 2.3. If λ, μ are partitions with $D_\lambda \supseteq D_\mu$, the *skew shape* $D_{\lambda/\mu}$ or just λ/μ is defined as the set-theoretic difference $D_\lambda \setminus D_\mu$. Thus

$$D_{\lambda/\mu} = \{(i,j) \in \mathbf{Z}^2 \mid 1 \le i \le \ell(\lambda), \mu_i < j \le \lambda_i\}.$$

Figure 2.1 shows the Ferrers diagram D_λ and skew shape $D_{\lambda/\mu}$, respectively, when $\lambda = (5,4,2,1) \vdash 12$ and $\mu = (2,2,1) \vdash 5$.

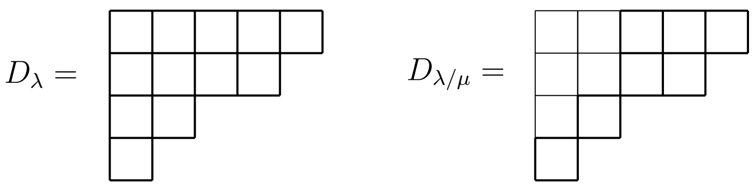


Figure 2.1

DEFINITION 2.4. Let λ be a partition. A *tableau* T of shape λ is an assignment $T : D_\lambda \to \mathbf{P}$ of positive integers to the cells of λ. The *content* of the tableau T, denoted by content(T), is the finite nonnegative vector whose ith component is the number of entries i in T.

A tableau T of shape λ is said to be *column strict* if it satisfies the following two conditions:

(i) $T(i,j) \le T(i,j+1)$, i.e., the entries increase weakly along the rows of λ from left to right.
(ii) $T(i,j) < T(i+1,j)$, i.e., the entries increase strictly along the columns of λ from top to bottom.

In Figure 2.2, T is a tableau of shape $(5,4,2,1)$ and S is a column strict tableau of shape $(5,4,2,1)$ and of content $(3,3,1,2,2,1)$.

$T =$

1	3	1	8	3
5	2	2	5	
6	7			
4				

$S =$

1	1	1	4	6
2	2	2	5	
3	5			
4				

Figure 2.2

DEFINITION 2.5. For partitions λ and μ of a positive integer n, the *Kostka number* $K_{\lambda,\mu}$ is the number of column strict tableaux of shape λ and content μ.

If we use the reverse lexicographic order on the set of partitions of a fixed n, the *Kostka matrix* $K = (K_{\lambda,\mu})$ becomes upper unitriangular so that K is invertible.

DEFINITION 2.6. A *rim hook* H is a skew shape which is connected and contains no 2×2 square of cells. The *size of* H is the number of cells it contains. The *leg length of rim hook* H, $\ell(H)$, is the number of vertical edges in H when viewed as in Figure 2.3. We define the *sign* of a rim hook H to be $\epsilon(H) = (-1)^{\ell(H)}$.

Figure 2.3 shows the rim hook H of size 6 with $\ell(H) = 2$ and $\epsilon(H) = (-1)^2 = 1$.

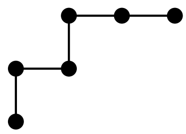
$H =$ 

Figure 2.3

Definition 2.7. A *rim hook tableau* T of shape λ is a partition of the diagram of λ into rim hooks. The *type* of T is $\text{type}(T) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_k is the number of rim hooks in T of size k. We now define the *sign* of a rim hook tableau T as

$$\epsilon(T) = \prod_{H \in T} \epsilon(H).$$

A rim hook tableau S is called *special* if each of the rim hooks contains a cell from the first column of λ. We use nodes for the Ferrers diagram and connect them if they are adjacent in the same rim hook as S in Figure 2.4.

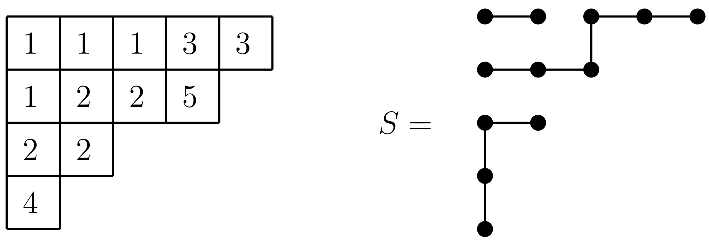
$T =$ $S =$ 

Figure 2.4

In Figure 2.4, T is a rim hook tableau of shape $(5, 4, 2, 1)$, $\text{type}(T) = (1^2, 2, 4^2)$ and $\epsilon(T) = (-1)^1 \cdot (-1)^1 \cdot (-1)^0 \cdot (-1)^0 \cdot (-1)^0 = 1$, while S is a special rim hook tableau with shape $(5, 3, 2, 1, 1)$, $\text{type}(S) = (2, 4, 6)$ and $\epsilon(S) = (-1)^0 \cdot (-1)^1 \cdot (-1)^2 = -1$.

We can now state Eğecioğlu and Remmel's interpretation for the entries of the inverse of Kostka matrix.

Theorem 2.8 (Eğecioğlu and Remmel[1]). *The entries of the inverse Kostka matrix are given by*

$$K^{-1}_{\mu,\lambda} = \sum_S \epsilon(S)$$

where the sum is over all special rim hook tableaux S with shape λ and type μ. $\square$

3. Sagan and Lee's sign reversing involution

In this section we introduce Sagan and Lee's sign reversing involution on the special rim hook tableaux. See [8] for details.

Let S be a special rim hook tableau with $t(S) = \mu$, and T be a standard Young tableau of the same shape as S, where μ is a partition of n. Sagan and Lee exhibited a sign reversing involution I on such pairs (S, T).

If the cell of n in T corresponds to a hook of size one in S, I can be clearly defined by induction. So for the rest of this section assume that the cell containing n in T corresponds to a cell in a hook of at least two cells in S.

To describe I under this assumption, a *rooted Ferrers diagram* is defined as a Ferrers diagram where one of the nodes has been marked. Marked cell will be indicated in the figures by making the distinguished node a square.

Now associate with any pair (S, T) a rooted special rim hook tableau $\dot{S}$ by rooting S at the node where the entry n occurs in T. A sign reversing involution ι will be defined on the set of rooted special rim hook tableaux of given type which are obtainable in this way. In addition, ι will have the property that if $\iota(\dot{S}) = \dot{S}'$ and $\dot{S}, \dot{S}'$ have roots r, r' respectively, then

$$\mathrm{sh}(\dot{S}) - r = \mathrm{sh}(\dot{S}') - r' \tag{1}$$

where the minus sign represents set-theoretic difference of diagrams. The full involution $I(S, T) = (S', T')$ will then be the composition

$$(S, T) \longrightarrow \dot{S} \xrightarrow{\iota} \dot{S}' \longrightarrow (S', T')$$

where S' is obtained from $\dot{S}'$ by forgetting about the root and T' is obtained by replacing the root of $\dot{S}'$ by n and leaving the numbers $1, 2, \ldots, n-1$ in the same positions as they were in T. Note that (1) guarantees that T' is well defined. Furthermore, it is clear from construction that I will be a sign reversing involution because ι is. Even though ι has not been fully defined, an example of the rest of the algorithm can be given as follows. See [8] for the definition of ι. Given (S, T), Figure 3.1 shows how a sign reversing involution I works on (S, T).

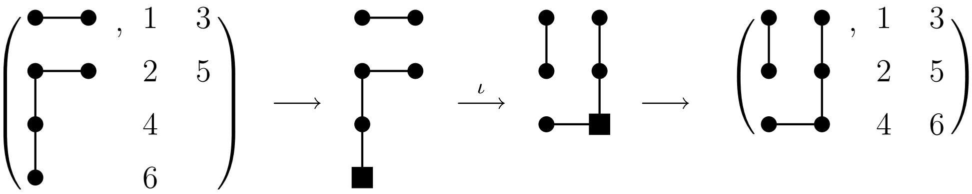


Figure 3.1

Theorem 3.1 (Sagan and Lee[8]). *Let μ be a partition of n with $\mu \neq 1^n$. Let*

$$\Gamma = \{\, (S,T) \mid t(S) = \mu, \mathrm{sh}(S) = \mathrm{sh}(T)\, \},$$

where S is a special rim hook tableau and T is a standard Young tableau. Then I defined in the above gives a sign reversing involution on Γ. □

4. Generalization of the Sagan and Lee's sign reversing involution

In this section we generalize Sagan and Lee's algorithm on special rim hook tableaux to get a combinatorial partial proof of $K^{-1}K = I$.

We first define a linear extension tableau $e(T)$ for a column strict tableau T.

Definition 4.1. Let T be a column strict tableau. The *linear extension tableau* $e(T)$ is the standard Young tableau of the same shape as T defined in the following way.

(i) If $T(i,j) < T(k,l)$, define $e(T)(i,j) < e(T)(k,l)$.
(ii) Assume $T(i,j) = T(k,l)$. Define $e(T)(i,j) < e(T)(k,l)$ if $i < k$ or $i = k, j < l$.

See Figure 4.1 for an example of the linear extension tableau $e(T)$ of T.

$T =$

1	1	2	3	3
2	4	5	5	
3	5			
6				

$e(T) =$

1	2	3	5	6
4	8	9	10	
7	11			
12				

Figure 4.1

We are now ready to describe our main theorem.

THEOREM 4.2. *Let μ and $\nu = (1^{m_1}, 2^{m_2})$ be partitions of n. Then*

$$\sum_{(S,T)} \epsilon(S) = \delta_{\mu,\nu} \tag{2}$$

the sum being all pairs (S,T) where S is a special rim hook tableau with $t(S) = \mu$ and T is a column strict tableau of content $(T) = \nu$ with the same shape as S, and where $\delta_{\mu,\nu}$ is the Kronecker's delta.

Proof. We will prove this identity by exhibiting a sign reversing involution I^* on such pairs (S,T), where $(S,T) \neq (S_0,T_0)$. Here I is the involution defined in Section 3 and

$$(S_0,T_0) = \left(\begin{array}{ll} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ & \vdots \\ \bullet & \bullet \\ \bullet & \\ \bullet & \\ \vdots & \\ \bullet & \end{matrix}, & \begin{matrix} 1 & 1 \\ 2 & 2 \\ & \vdots \\ k & k \\ k+1 & \\ k+2 & \\ \vdots & \\ m & \end{matrix} \end{array}\right)$$

Figure 4.2

Suppose first that the cell of the biggest entry m in T corresponds to a hook of size one in S. Then since S is special, this cell is at the end of the first column. In this case, remove that cell from both S and T to form $\overline{S}$ and $\overline{T}$ respectively. Now we can assume, by induction, that $I^*(\overline{S},\overline{T}) = (\overline{S}', \overline{T}')$ has been defined. So let $I^*(S,T) = (S',T')$ where S' is $\overline{S}'$ with a hook of size 1 added to the end of the first column and T' is $\overline{T}'$ with a cell labeled m added to the end of the first column. Clearly this will result in a sign reversing involution as long as this was true for pairs with $n-1$ cells. So for the rest of this section we will also assume that the cell containing the biggest entry in T corresponds to a cell in a hook of at least two cells in S.

With these assumptions let (S'',T'') be the image of $(S,e(T))$ under the involution I, i.e., $(S'',T'') = I(S,e(T))$. We divide into the following two cases.

Case 1 If there is a column strict tableau T' of content ν such that $e(T') = T''$, define $I^*(S,T) = (S',T')$ with $S' = S''$. See Figure 4.3.

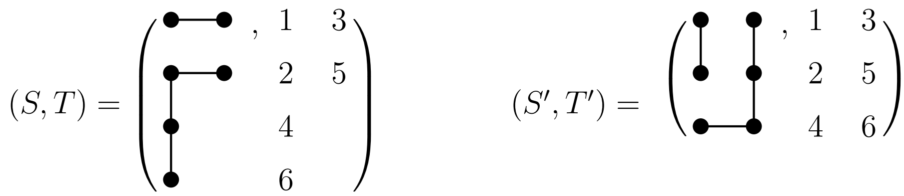


Figure 4.3

Case 2 Assume now there is no column strict tableau T' of content ν such that $e(T') = T''$. Under this assumption let $b_{n-1} = (i,j)$ and $b_n = (k,l)$ be the cells in $e(T)$ whose entries are $n-1$ and n, respectively.

(2–a) If $i < k$, $j \neq l$ since there is no column strict tableau T' such that $e(T') = T''$. Let T_1 be the standard Young tableau obtained from $e(T)$ by exchanging entries $n-1$ and n, and let $(S_1'', T_1'') = I(S, T_1)$. If T_1' is the column strict tableau of content ν such that $e(T_1') = T_1''$, define $I^*(S,T) = (S_1', T_1')$ with $S_1' = S_1''$. See Figure 4.4.

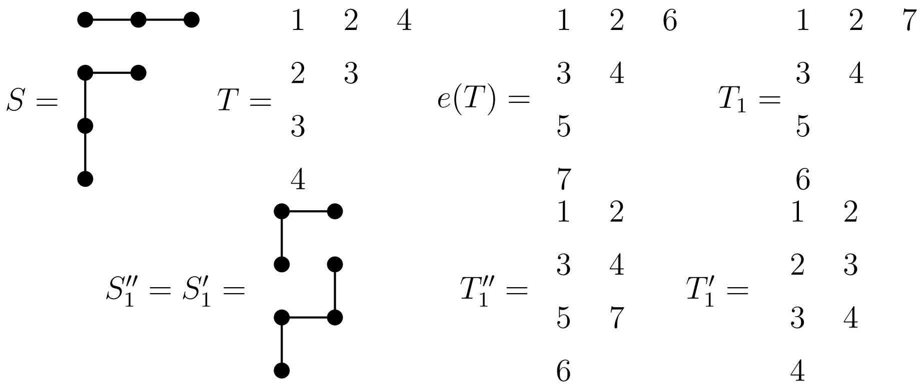


Figure 4.4

(2–b) If $i = k$, then $l = j+1$ and only two cells b_{n-1}, b_n are in the last row of $e(T)$. Hence cells of the entries $n-1$ and n in $e(T)$ corresponds to a hook κ of size two which are in last row of S. See Figure 4.5. In this case, remove last hook κ from both S to form $\overline{S}$, and remove two cells b_{n-1}, b_n from $e(T)$ to form $\overline{T}$.

Let $I(\overline{S}, \overline{T}) = (\overline{S}', \overline{T}')$. We define S_2 as $\overline{S}'$ with a hook of size 2 added to the end of the first column, and define T_2 as $\overline{T}'$ with two cells labeled $n-1, n$ added to the end of the first column. Finally let $I(S_2, T_2) = (S_2'', T_2'')$. If there is a column strict tableau T_2' such that $e(T_2') = T_2''$, define $I^*(S, T) = (S_2', T_2')$ with $S_2' = S_2''$.

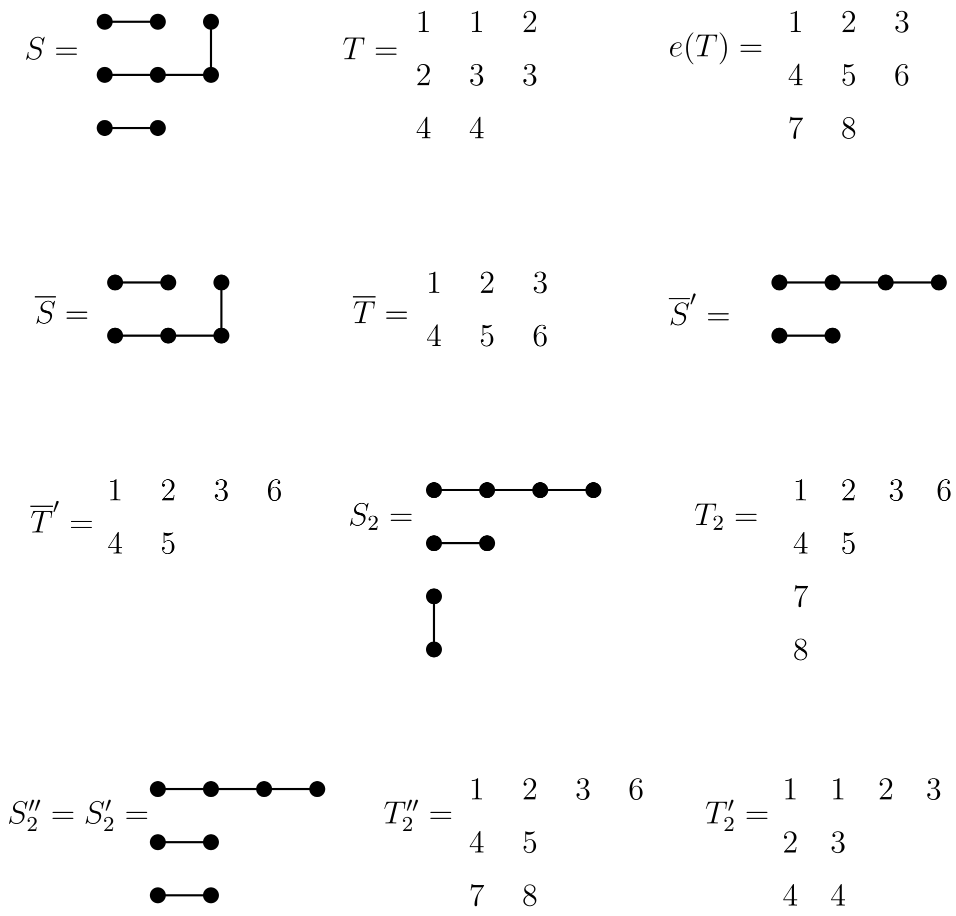


Figure 4.5

Clearly I^* is also a sign reversing involution since I is a sign reversing involution. Hence all terms $\epsilon(S)$ in the summation of (2) are cancelled out except $\epsilon(S_0)$, which is 1. This fact implies the identity in (2). $\square$

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