Korean J. Math. **18** (2010), No. 3, pp. 277–288

NONLINEAR BIHARMONIC PROBLEM WITH VARIABLE COEFFICIENT EXPONENTIAL GROWTH TERM

Q-Heung Choi* and Tacksun Jung

Abstract. We consider the nonlinear biharmonic equation with coefficient exponential growth term and Dirichlet boundary condition. We show that the nonlinear equation has at least one bounded solution under the suitable conditions. We obtain this result by the variational method, generalized mountain pass theorem and the critical point theory of the associated functional.

1. Introduction and statement of main results

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $a : \overline{\Omega} \to R$ be a continuous function which changes sign in Ω and Δ^2 be the biharmonic operator. Let $c \in R$. In this paper we study the following nonlinear biharmonic equation with variable coefficient exponential growth nonlinear term and Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = a(x)g(u) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

We assume that g satisfies the following conditions:
$(g1)$ $g \in C(R, R)$,

Received August 5, 2010. Revised September 4, 2010. Accepted September 10, 2010.

2000 Mathematics Subject Classification: 35J20, 35J25.

Key words and phrases: biharmonic boundary value problem, variable coefficient exponential growth term, variational method, generalized mountain pass geometry, critical point theory, $(P.S.)$ condition.

This work(Choi) was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (KRF- 2010-0015835).

*Corresponding author.

$(g2)$ there is a constant $A_0 > 0$ such that

$$|g(\xi)| \leq A_0^{\phi(\xi)} \qquad \text{for } \xi \in R,$$

where $\phi : R \to R$ is a function satisfying $\phi(\xi)\xi^{-2} \to 0$ as $|\xi| \to \infty$,
$(g3)$ there are constants $\mu > 2$ and $r_0 \geq 0$ such that

$$0 < \mu G(\xi) = \mu \int_0^\xi g(t)dt \leq \xi g(\xi) \qquad \text{for } |\xi| \geq r_0,$$

$(g4)$ there exist $0 < \alpha_1 \leq \alpha_2 < 2$, $A_1, A_2 > 0$, and $B_1, B_2 \geq 0$ such that

$$A_1 \exp^{|\xi|^{\alpha_1}} - B_1 \leq G(\xi) = \int_0^\xi g(t)dt \leq A_2 \exp^{|\xi|^{\alpha_2}} + B_2 \text{ for } \ \xi \in R,$$

where α_1, α_2 are further restricted by

$$\frac{2}{\alpha_2} - 2 > \frac{1}{\alpha_1}.$$

We note that the conditions $0 < \alpha_1 \leq \alpha_2 < 2$ and $\frac{2}{\alpha_2} - 2 > \frac{1}{\alpha_1}$ imply $\alpha_2 < \frac{1}{2}$. Khanfir and Lassoued [4] showed the existence of at least one solution for the nonlinear elliptic boundary problem when $f = 0$ and g is locally *Hölder* continuous on R_+. Choi and Jung [2] show that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.2}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$), where λ_i is the eigenvalue of $\Delta u + u = \lambda u$ with Dirichlet boundary condition. They obtained these results by using the variational reduction method. They [3] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.2) has at least three nontrivial solutions by use of the degree theory. Tarantello [7] also studied

$$\Delta^2 u + c\Delta u = b((u + 1)^+ - 1), \tag{1.3}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

She show that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.3) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [5] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.3) has at least four solutions by the vatiational linking theorem and

Leray-Schauder degree theory. In this paper we are looking for the weak solutions of (1.1), that is,

$$\int_\Omega (\Delta^2 u + c\Delta u - a(x)g(u))v dx = 0 \qquad \text{for } v \in H,$$

where the space H is introduced in section 2. We note that the weak solutions of (1.1) coincide with the critical points of the associated functional

$$I(u) \in C^1(H, R),$$

$$I(u) = \frac{1}{2}\int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \int_\Omega a(x)G(u)]dx.$$

$$= \frac{1}{2}(\|P_+u\|^2 - \|P_-u\|^2) - \int_\Omega a(x)G(u)dx.$$

Our main results is as follows:

THEOREM 1.1. *Assume that $\lambda_k < c < \lambda_{k+1}$ and g satisfies $(g1)$-$(g4)$. Then we have:*
(i) If $g(u)u - \mu G(u)$ is bounded, then (1.1) has at least one bounded solution.
(ii) If $g(u)u - \mu G(u)$ is not bounded and there exists a small $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$, then (1.1) has at least two solutions, (i) one of which is bounded and (ii) the other solution of which is large norm such that $\max_{x\in\Omega}|u(x)| > M$ for some M, where

$$a^+ = a \cdot \chi_{\Omega^+}, \qquad a^- = -a \cdot \chi_{\Omega^-}$$

with

$$\Omega^+ = \{x \in \Omega | a(x) > 0\}, \qquad \Omega^- = \{x \in \Omega |\ a(x) < 0\}.$$

In section 2, we obtain some results on the operator $\Delta(\Delta - c)$, introduce a Hilbert space H and investigate that $I(u)$ is continuous, *Fréchet* differentiable and satisfies the $(P.S.)$ condition. In section 3, we prove Theorem 1.1(i) and in section 4, we prove Theorem 1.1(ii) by using the variational method, the generalized mountain pass theorem and the critical point theory.

2. Some results on $\Delta(\Delta - c)$ and $I(u)$

Let $c \in R$. Throughout this paper we assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$. Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \ \sum |\lambda_k(\lambda_k - c)| < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
which is proved in [1].
Let

$$H_+ = \{u \in H | \ h_k = 0 \text{ if } \ \lambda_k(\lambda_k - c) < 0\},$$

$$H_- = \{u \in H | \ h_k = 0 \text{ if } \ \lambda_k(\lambda_k - c) > 0\}.$$

Then $H = H_- \oplus H_+$, for $u \in H$, $u = u^- + u^+ \in H_- \oplus H_+$. Let P_+ be the orthogonal projection on H_+ and P_- be the orthogonal projection on H_-. We can wtite $P_+ u = u^+$, $P_- u = u^-$, for $u \in H$. By $(g1)$ and $(g2)$, I is well defined. We note that $(g3)$ implies the existence of positive constants a_1, a_2, a_3 such that

$$\frac{1}{\mu}\left(\xi g(\xi) + a_1\right) \geq G(\xi) + a_2 \geq a_3 |\xi|^\mu \qquad \text{for } \xi \in R. \tag{2.1}$$

By the following Lemma 2.1, $I \in C^1(H, R)$ and I is *Fréchet* differentiable in H, which is proved in Appendix B in [9].:

LEMMA 2.1. *Assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, and g satisfies $(g1) - (g4)$. Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - a(x)g(u)h]dx. \tag{2.2}$$

If we set

$$K(u)=\int_\Omega a(x)G(u)dx,$$

then $K'(u)$ is continuous with respect to weak convergence, $K'(u)$ is compact, and

$$K'(u)h=\int_\Omega a(x)g(u)hdx \qquad \text{for all } h\in H,$$

this implies that $I\in C^1(H,R)$ and $K(u)$ is weakly continuous.

LEMMA 2.2. *Assume that $\lambda_k<c<\lambda_{k+1}$, $k\geq 1$, g satisfies $(g1)-(g4)$. If $g(u)u-\mu G(u)$ is bounded or there exists an $\epsilon>0$ such that $\int_{\Omega^-}a^-(x)dx<\epsilon$, then $I(u)$ satisfies the Palais-Smale condition.*

Proof. We assume that $g(u)u-\mu G(u)$ is bounded or there exists an $\epsilon>0$ such that $\int_{\Omega^-}a^-(x)dx<\epsilon$. Suppose that (u_m) is a sequence with $I(u_m)\leq M$ and $I'(u_m)\to 0$ as $m\to\infty$. Then by $(g2)$, $(g3)$, *Hölder* inequality and Sobolev Embedding Theorem, for large m and $\mu>2$ with $u=u_m$, we have

$$\begin{aligned}
M+\frac{1}{2}\|u\| &\geq I(u)-\frac{1}{2}I'(u)u=\int_\Omega[\frac{1}{2}a(x)g(u)u-a(x)G(u)]dx\\
&= \int_\Omega a^+(x)[\frac{1}{2}g(u)u-G(u)]-\int_\Omega a^-(x)[\frac{1}{2}g(u)u-G(u)]\\
&\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\mu\int_\Omega a^+(x)\cdot G(u)\\
&\quad -\max_\Omega|\frac{1}{2}g(u)u-G(u)|\int_{\Omega^-}a^-(x)dx\\
&\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\mu\int_\Omega a^+(x)\cdot\left(A_1\mathrm{e}^{|u|^{\alpha_1}}-B_1\right)\\
&\quad -\max_\Omega|\frac{1}{2}g(u)u-G(u)|\int_{\Omega^-}a^-(x)dx.
\end{aligned}$$

Thus if $\frac{1}{2}g(u)u-G(u)$ is bounded or there exists an $\epsilon>0$ such that $\int_{\Omega^-}a^-(x)<\epsilon$, then we have

$$1+\|u\|\geq M_1\int_\Omega \mathrm{e}^{|u|^{\alpha_1}}. \tag{2.3}$$

Moreover since

$$|I'(u_m)\varphi|\leq\|\varphi\| \tag{2.4}$$

for large m and all $\varphi \in H$, choosing $\varphi = u_m^+ \in H_+$ gives

$$\begin{aligned}\|u_m^+\|^2 &= \int_\Omega \big(\Delta^2 u_m + c\Delta u_m\big)\cdot u_m^+ = \int_\Omega a(x)g(u_m)u_m^+ \\ &\le \int_\Omega |a(x)||g(u_m)||u_m| \le \|a\|_\infty \int_\Omega A_0 \mathrm{e}^{\phi(u_m)}|u_m| \\ &\le C_1 \int_\Omega \mathrm{e}^{\phi(u_m)}|u_m|.\end{aligned}$$

Taking $\varphi = -u_m^-$ in (2.4) yields

$$\begin{aligned}\|u_m^-\|^2 &= \int_\Omega \big(\Delta^2 u_m + c\Delta u_m\big)\cdot(-u_m^-) \\ &= \int_\Omega a(x)g(u_m)\cdot(-u_m^-) \\ &\le \int_\Omega |a(x)||g(u_m)||u_m| \\ &\le \|a\|_\infty \int_\Omega \mathrm{e}^{\phi(u_m)}|u_m| \\ &\le C_2 \int_\Omega \mathrm{e}^{\phi(u_m)}|u_m|.\end{aligned}$$

Thus, by (2.3), we have

$$\begin{aligned}\|u_m\|^2 &= \|u_m^+\|^2 + \|u_m^-\|^2 \\ &\le M_2 \int_\Omega \mathrm{e}^{\phi(u_m)}|u_m| \\ &\le M_3 \int_\Omega (|u_m| + |u_m|(u_m^2\phi(u_m)u_m^{-2}) + \frac{u_m^4}{2}\phi_{u_m}^2 u_m^{-4} + \ldots) \\ &\le M_4 \int_\Omega \mathrm{e}^{|u_m|^{\alpha_1}} \\ &\le M_5\,(1 + \|u_m\|)\end{aligned}$$

since $u_m^2\phi(u_m)u_m^{-2}) + \frac{u_m^4}{2}\phi_{u_m}^2 u_m^{-4} + \ldots \to 0$ as $|u_m| \to \infty$, from which the boundedness of (u_m) follows. Thus (u_m) converges weakly in H. Since $P_\pm I'(u_m) = \pm P_\pm u_m + P_\pm \tilde{\mathcal{P}}(u_m)$ with $\tilde{\mathcal{P}}$ compact and the weak convergence of $P_\pm u_m$ imply the strong convergence of $P_\pm u_m$ and hence (PS) condition holds. $\square$

3. Proof of Theorem 1.1 (i)

Let H be a Hilbert space and let

$$H_k = \mathrm{span}\{\phi_1, \dots, \phi_k\}.$$

Then H_k is a subspace of H such that

$$H = \oplus_{k \in N} H_k \qquad \text{and} \quad H = H_k \oplus H_k^{\perp}.$$

Let

$$B_r = \{u \in H | \ \|u\| \le r\},$$

$$Q = (\bar{B}_R \cap H_k) \oplus \{re | \ 0 < r < R\}.$$

Now we recall the generalized mountain pass Theorem in [9] which is a crucial role for the proof of main results:

THEOREM 3.1. *(Generalized Mountain Pass Theorem) Let $H = V \oplus X$, where H is a real Banach space and $V \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(H, R)$, satisfies $(P.S.)$ condiion, and*
(i) there are constants ρ, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap X} \ge \alpha$, and
(ii) there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap V) \oplus \{re | \ 0 < r < R\}$, then $I|_{\partial Q} \le 0$.
Then I possesses a critical value $b \ge \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) | \ \gamma = id \ on \ \ \partial Q\}.$$

The following lemma show that $I(u)$ satisfies the generalized mountain pass geometrical assumptions:

LEMMA 3.1. *Assume that $\lambda_k < c < \lambda_{k+1}$ and g satisfies $(g1) - (g4)$. Then*
(i) there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_k^{\perp}} \ge \alpha$, and
(ii) there is an $e \in \partial B_1 \cap H_k^{\perp}$ and $R > \rho$ such that $I|_{\partial Q} \le 0$, and
(iii) there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \le 0$.

Proof. (i) Let $u \in H_k^\perp$. Then we have

$$\int_\Omega (\Delta^2 u + c\Delta u) u dx \geq \lambda_{k+1}(\lambda_{k+1} - c)\|u\|^2_{L^2(\Omega)} > 0.$$

Thus by $(g4)$, (2.1) and the *Hölder* inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \int_\Omega a(x)G(u) \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty \int_\Omega C_1|u|^\mu \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty C_1'\|u\|^\mu \end{aligned}$$

for $C_1, C_1' > 0$. Since $\mu > 2$, there exist $\rho > 0$ and $\alpha > 0$ such that if $u \in \partial B_\rho$, then $I(u) \geq \alpha$.
(ii) Let $u \in (\bar{B}_r \cap H_k) \oplus \{re| \ 0 < r\}$. Then $u = v + w$, $v \in B_r \cap H_k$, $w = re$. We note that

$$\text{if } \ v \in H_k, \ \int_\Omega (\Delta^2 v + c\Delta v) v dx \leq \lambda_k(\lambda_k - c)\|v\|^2_{L^2(\Omega)} < 0.$$

Thus we have

$$\begin{aligned} I(u) &= \frac{1}{2}r^2 - \frac{1}{2}\|P_- v\|^2 - \int_\Omega a(x)G(v + re) \\ &\leq \frac{1}{2}r^2 + \frac{1}{2}(\lambda_k(\lambda_k - c))\|v\|^2_{L^2(\Omega)} - \int_{\Omega^+} a(x)(A_1 e^{|v+re|^{\alpha_1}} - B_1). \end{aligned}$$

Since $\mu > 2$, there exists $R > 0$ such that if $u = v + re \in Q = (\bar{B}_R \cap H_k) \oplus \{re| \ 0 < r < R\}$, then $I(u) < 0$.
(iii) follows from (ii). □

Proof of Theorem 1.1 (i)
By Lemma 2.1 and Lemma 2.2, $I(u) \in C^1(H,\mathrm{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1, there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_k^\perp} \geq \alpha$, and there is an $e \in \partial B_1 \cap H_k^\perp$ and $R > \rho$ such that if $u \in Q = (\bar{B}_R \cap H_k) \oplus \{re| \ 0 < r < R\}$, then $I|_{u \in \partial Q}(u) \leq 0$, and there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. By the generalized mountain pass theorem, $I(u)$ has a critical value $b \geq \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) |\ \gamma = id \text{ on } \ \partial Q\}.$$

We denote by $\tilde{u}$ a critical point of I such that $I(\tilde{u}) = b$. We claim that there exists a constant $C > 0$ such that

$$\|a^+(x)^{\frac{1}{\mu}}\tilde{u}\|_{L^2(\Omega)} \leq C\left(1 + L\int_{\Omega^-} a^-(x)dx\right)^{\frac{1}{\mu}},$$

where $L = \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$.
In fact, we have

$$b \leq \max I(tu_0), \qquad 0 \leq t \leq 1,$$

and

$$\begin{aligned}
I(tu_0) &= t^2\left(\frac{1}{2}\|P_+u_0\|^2 - \frac{1}{2}\|P_-u_0\|^2\right) - \int_\Omega a(x)G(tu_0)dx \\
&\leq t^2\|u_0\|^2 - \int_\Omega a^+(x)G(tu_0)dx + \int_\Omega a^-(x)G(tu_0)dx \\
&\leq t^2\|u_0\|^2 - a_3t^\mu\int_\Omega a^+(x)u_0^\mu + a_4\int_\Omega a^+(x) + a_5t^\mu\int_\Omega a^-(x)u_0^\mu \\
&= Ct^2 - Ct^\mu + C + C't^\mu.
\end{aligned}$$

Since $0 \leq t \leq 1$, b is bounded: $b < \tilde{C}$.
By (2.1), we can write

$$\begin{aligned}
b &= I(\tilde{u}) - \frac{1}{2}I'(\tilde{u})\tilde{u} \\
&= \int_\Omega a(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \\
&= \int_\Omega a^+(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx - \int_\Omega a^-(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega a^+(x)g(\tilde{u})\tilde{u} - \max_\Omega |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})| \int_{\Omega^-} a^-(x)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu\int_\Omega a^+(x)\left(a_3|\tilde{u}|^\mu - a_4\right) - L\int_{\Omega^-} a^-(x)dx,
\end{aligned}$$

where $L = \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$. Thus we have

$$C\left(1 + L\int_{\Omega^-} a^-(x)dx\right) \geq \int_\Omega a^+(x)|\tilde{u}|^\mu$$

$$\geq \left[\int_\Omega \left(a^+(x)^{\frac{1}{\mu}} |\tilde{u}| \right)^2 \right]^{\frac{\mu}{2}}, \tag{3.1}$$

from which we can conclude that $\tilde{u}$ is bounded. In fact, we suppose that $\tilde{u}$ is not bounded. Then for any $R > 0$, $|\tilde{u}| \geq R$. Thus we have

$$\int_\Omega a^+(x)|\tilde{u}|^\mu \geq R^\mu \int_\Omega a^+(x)dx$$

for any R, which contradicts to the fact (3.1) and the proof of Theorem 1.1 (i) is completed.

4. Proof of Theorem 1.1 (ii)

Assume that $\frac{1}{2}g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) < \epsilon$. By Lemma 2.1 and Lemma 2.2, $I \in C^1(H,\mathrm{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1 and generalized mountain pass theorem, $I(u)$ has a critical value b with critical point $\tilde{u}$ such that $I(\tilde{u}) = b$. If $\int_{\Omega^-} a^-(x)dx$ is sufficiently small, by (3.1), we have

$$\int_\Omega a^+(x)|\tilde{u}|^\mu \leq C$$

for $C > 0$, from which we can conclude that $\tilde{u}$ is bounded and the proof of Theorem 1.2(i) is completed. Next we shall prove Theorem 1.2 (ii). We may assume that $R_n < R_{n+1}$ for all $n \in N$. Let us set $D_n = B_{R_n} \cap H_n$, $\partial D_n = \partial B_{R_n} \cap H_n$.

LEMMA 4.1. *Assume that g satisfies $(g1)$-$(g4)$. Then there exists an $R_n > 0$ such that*

$$I(u) \leq 0 \qquad \textit{for } \; u \in H_n \backslash B_{R_n}, \tag{4.1}$$

where $B_{R_n} = \{u \in H| \; \|u\| \leq R_n\}$.

Proof. Let us choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\mathrm{supp}(\psi) \subset \Omega^+$. Then, by $(g3)$, (2.1) and the *Hölder* inequality, we have

$$\begin{aligned} I(t\psi) &= \frac{1}{2}\|P_+ t\psi\|^2 - \frac{1}{2}\|P_- t\psi\|^2 - \int_\Omega a(x)G(t\psi) \\ &\leq \frac{1}{2}t^2 - \|a\|_\infty \int_\Omega C_1 t^\mu \psi^\mu + \|a\|_\infty C_2 \\ &\leq \frac{1}{2}t^2 - t^\mu \|a\|_\infty C_1' \psi^\mu + \|a\|_\infty C_2 \end{aligned}$$

for C_1, C_1' and $C_2 > 0$. Since $\mu > 2$, there exist t_n great enough for each n and an $R_n > 0$ such that $u_n = t_n\psi$ and $I(u_n) < 0$ if $u_n \in H_n \backslash B_{R_n}$ and $\|u_n\| > R_n$, so the lemma is proved □

Let us set

$$\Gamma_n = \{\gamma \in C([0,1], H)|\ \gamma(0) = 0 \text{ and } \ \gamma(1) = u_n\}$$

and

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{[0,1]} I(\gamma(u)) \qquad n \in N.$$

Proof of Theorem 1.2 (ii)

We assume that $g(u)u - \mu G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. By Lemma 2.1 and Lemma 2.2, $I \in C^1(H, R)$ and satisfies the Palais-Smale condition. By Lemma 4.1, there exists an $R_n > 0$ such that $I(u_n) \leq 0$ for $u_n \in H_n \backslash B_{R_n}$. We note that $I(0) = 0$. By Lemma 4.1 and the generalized mountain pass theorem, for n large enough, $b_n > 0$ is a critical value of I and $\lim_{n\to\infty} b_n = +\infty$. Let $\tilde{u_n}$ be a critical point of I such that $I(\tilde{u_n}) = b_n$. Then for each real number M, $\max_\Omega |\tilde{u_n}(x)| \geq M$. In fact, by contradiction , $\Delta^2 u + c\Delta u = a(x)g(u)$ and $\max_\Omega |\tilde{u_n}(x)| \leq K$ imply that

$$I(\tilde{u_n}) \leq \max_{|\tilde{u_n}| \leq K} (\frac{1}{2} g(\tilde{u_n})\tilde{u_n} - G(\tilde{u_n})) \int_\Omega ||a(x)|,$$

which means that b_n is bounded. This is absurd to the fact that $\lim_{n\to\infty} b_n = +\infty$. Thus we complete the proof.

References

[1] Choi, Q. H. and Jung, T., *Multiplicity of solutions and source terms in a fourth order nonlinear elliptic equation*, Acta Math. Sci. **19**(4)(1999), 361–374 .

[2] Choi, Q. H.and Jung, T., *Multiplicity results on nonlinear biharmonic operator*, Rocky Mountain J. Math. **29**(1)(1999), 141–164 .

[3] Jung, T. S. and Choi, Q. H., *Multiplicity results on a nonlinear biharmonic equation*, Nonlinear Anal. **30**(8)(1997), 5083–5092 .

[4] Khanfir, S. and Lassoued, L., *On the existence of positive solutions of a semi-linear elliptic equation with change of sign*, Nonlinear Anal. **22**(11)(1994), 1309–1314.

[5] Micheletti, A. M. and Pistoia, A., *Multiplicity results for a fourth-order semi-linear elliptic problem*, Nonlinear Anal. **31**(7)(1998), 895–908 .

[6] Rabinowitz and P. H., *Minimax methods in critical point theory with applications to differential equations*, CBMS. Regional conf. Ser. Math.**65**, Amer. Math. Soc., Providence, Rhode Island, 1986.

[7] Tarantello, *A note on a semilinear elliptic problem*, Differ. Integral. Equ. Appl. **5**(3)(1992), 561–565.

Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr

Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr