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EAKIN-NAGATA THEOREM FOR COMMUTATIVE RINGS WHOSE REGULAR IDEALS ARE FINITELY GENERATED

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Abstract. Let R be a commutative ring with identity, $T(R)$ be the total quotient ring of R, and D be a ring such that $R \subseteq D \subseteq T(R)$ and D is a finite R-module. In this paper, we show that each regular ideal of R is finitely generated if and only if each regular ideal of D is finitely generated. This is a generalization of the Eakin-Nagata theorem that R is Noetherian if and only if D is Noetherian.

1. Introduction

Let R be a commutative ring with identity, $T(R)$ be the total quotient ring of R, and D be a ring between R and $T(R)$. A regular element is an element which is not a zero divisor, while a regular ideal is an ideal containing a regular element. We say that R is an *r-Noetherian ring* if each regular ideal of R is finitely generated. Clearly, Noetherian rings are r-Noetherian, but r-Noetherian rings need not be Noetherian (see Example 1).

It is well known that if R is Noetherian, then $R[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n \in T(R)$, is also Noetherian. Moreover, if $R[x_1, \ldots, x_n]$ is integral over R, then $R[x_1, \ldots, x_n]$ is a finite R-module [6, Theorem 17], and thus R is Noetherian if and only if $R[x_1, \ldots, x_n]$ is Noetherian by the Eakin-Nagata theorem ([4, Theorem 2] or [7]) that if D is a finite R-module, then R is Noetherian if and only if D is Noetherian. However, if $R[x_1, \ldots, x_n]$ is not integral over R, then $R[x_1, \ldots, x_n]$ being Noetherian does not imply that R is Noetherian. For example, let $\mathbb{Q}$ (resp., $\mathbb{C}$) be the field of rational (resp., complex) numbers, X be an indeterminate over $\mathbb{C}$, $\mathbb{C}[\![X]\!]$ be the power series ring over $\mathbb{C}$, and $R = \mathbb{Q} + X\mathbb{C}[\![X]\!]$ be

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a subring of $\mathbb{C}[\![X]\!]$. Then $R[\frac{1}{X}] = T(R)$, and hence $R[\frac{1}{X}]$ is a Noetherian domain, but R is not a Noetherian domain [1, Theorem 2.1]. Let t be an indeterminate over R and $R[t]$ be the polynomial ring over R. Then t is a regular element of $R[t]$ and $R[t]/tR[t] \cong R$. So if $R[t]$ is an r-Noetherian ring, then R is Noetherian. In [3, Lemma 4], Chang and Kang showed that if R is an r-Noetherian ring, then $R[x]$ is also r-Noetherian for each $x \in T(R)$, and hence $R[x_1, \dots, x_n]$ is r-Noetherian for any $x_1, \dots, x_n \in T(R)$ (see Proposition 2).

Let D be a finite R-module. In [2, Lemma 3], Chang proved that if R is an r-Noetherian ring, then D is also an r-Noetherian ring. In this paper, we show that if D is an r-Noetherian ring, then R is an r-Noetherian ring, which is an r-Noetherian ring analog of the Eakin-Nagata theorem ([4, Theorem 2] or [7]). The proofs of our results in this paper heavily depend on those of the Eakin's results in [4]. For any undefined terminology and notation, see [6].

2. Main result

Throughout this paper, R denotes a commutative ring with identity, $T(R)$ is the total quotient ring of R, and D is a ring between R and $T(R)$. For any ideal A of R (resp., D), we mean by A^e (resp., A^c) the ideal AD of D (resp., $A \cap R$ of R). Also, $A^{ec} = AD \cap R$.

We begin this section with an example of r-Noetherian rings that are not Noetherian.

Example 1. Let $X_1, X_2, \dots, X_n$ be indeterminates over $\mathbb{Q}$, $\mathbb{Q}[X_1, X_2, \dots, X_n]$ be the polynomial ring over $\mathbb{Q}$, and M be the maximal ideal of $\mathbb{Q}[X_1, X_2, \dots, X_n]$ generated by $X_1, X_2, \dots, X_n$. Put $A = \mathbb{Q}[X_1, X_2, \dots, X_n]_M$ and $R = A/X_1^3A$.

(1) $\bar{R}$, the integral closure of R, is not a Noetherian ring.
(2) If $n = 2$, then each overring of R is an r-Noetherian ring.
(3) If $n = 3$, then $\bar{R}$ is an r-Noetherian ring.

Proof. We first note that $A = \mathbb{Q}[X_1, X_2, \dots, X_n]_M$ is a local Noetherian domain of $\dim(A) = n$, where $\dim(A)$ is the Krull dimension of A. Hence R is a local Noetherian ring of $\dim(R) = n - 1$.

(1) Since $\bar{X}_1 := X_1 + X_1^3A$ is a nonzero nilpotent element of R, $\bar{R}$ is not a Noetherian ring [5, Lemma 11.1]. (2) If $n = 2$, then $\dim(R) = 1$, and thus each overring of R is an r-Noetherian ring [5, Theorem 12.6].

(3) If $n = 3$, then $\dim(R) = 2$. Thus $\bar{R}$ is an r-Noetherian ring [5, Theorem 11.6]. □

The regular height of a regular prime ideal P of R, denoted by reg-ht$P$, is defined to be the supremum of the length of chains consisting of regular prime ideals contained in P plus 1. The regular dimension of R, denoted by reg-dim(R), is $\sup\{\text{reg-ht}P | P \text{ is a regular prime ideal of } R\}$. Clearly, if R is an integral domain, then reg-dim(R) is just the Krull dimension of R.

For more examples of r-Notherian rings that are not Noetherian, recall that a ring R is called a Marot ring if each regular ideal of R is generated by a set of regular elements. Examples of Marot rings include Noetherian rings and overrings of a Marot ring [5, Theorem 7.2 and Corollary 7.3]. It is known that if R is an r-Noetherian ring of reg-dim$(R) \leq 1$, then each overring of R is an r-Noetherian ring [2, Theorem 2]. Also, if R is a Marot r-Noetherian ring of reg-dim$(R) = 2$, then the integral closure of R is an r-Noetherian ring [2, Theorem 7].

The next result is an extension of the Chang's result [2, Lemma 3] that if D is a finite R-module and R is an r-Noetherian ring, then D is an r-Noetherian ring.

Proposition 2. *If R is an r-Noetherian ring, then $R[x_1, \ldots, x_n]$ is also an r-Noetherian ring for any $x_1, \ldots, x_n \in T(R)$.*

Proof. It is known that $R[x_1]$ is an r-Noetherian ring [3, Lemma 4]. Also, note that $R[x_1, \ldots, x_i] = R[x_1, \ldots, x_{i-1}][x_i]$ for $i = 2, \ldots, n$. Thus, by induction on n, $R[x_1, \ldots, x_n]$ is an r-Noetherian ring. □

Lemma 3. (cf. [4, Lemma 2]) *Let D be an r-Noetherian ring, and assume that D is a finite R-module. If A is a proper regular ideal of R with $AD \cap R = A$, then there exist regular prime ideals $P_1, \ldots, P_n$, not necessarily distinct, of R such that $(P_1 \cdots P_n)^{ec} \subseteq A$.*

Proof. Note that A^e is a regular ideal of D; so D/A^e is Noetherian. Hence there exist regular prime ideals $Q_1, \ldots, Q_n$, not necessarily distinct, of D such that $Q_1 \cdots Q_n \subseteq A^e$. Let $P_i = Q_i \cap R$ for $i = 1, \ldots, n$. Then each P_i is a regular prime ideal of R and $(P_1 \cdots P_n)^e \subseteq Q_1 \cdots Q_n \subseteq A^e$. Thus $P_1 \cdots P_n \subseteq (P_1 \cdots P_n)^{ec} = (P_1 \cdots P_n)^e \cap R \subseteq A^{ec} = AD \cap R = A$. □

Lemma 4. (cf. [4, Lemma 3]) *Let D be an r-Noetherian ring, and assume that D is a finite R-module. Then R/P is a Noetherian ring* *for each regular prime ideal P of R if and only if $R/(P_1 \cdots P_n)^{ec}$ is a Noetherian ring for regular prime ideals $P_1, \ldots, P_n$, not necessarily distinct, of R.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) We prove the lemma by induction on n.

Step 1. $n = 1$. Note that D is integral over R [6, Theorem 17], so there is a prime ideal Q of D such that $Q \cap R = P_1$, and hence $P_1^{ec} = P_1$. Thus R/P_1^{ec} is Noetherian by assumption.

Step 2. Suppose that the result holds for $n-1 \geq 1$. Let P be a prime ideal of R such that $(P_1 \cdots P_n)^{ec} \subseteq P$. By Cohen's theorem [6, Theorem 8], we have only to show that $P/(P_1 \cdots P_n)^{ec}$ is finitely generated. Note that since $P_1 \cdots P_n \subseteq P$, at least one of the P_i, say, P_n, is contained in P. If $P_n/(P_1 \cdots P_n)^{ec}$ is finitely generated, then, since R/P_n is Noetherian, P/P_n is finitely generated, and thus $P/(P_1 \cdots P_n)^{ec}$ is finitely generated. So it suffices to show that $P_n/(P_1 \cdots P_n)^{ec}$ is finitely generated.

Note that $P_n^{ec} = P_n$, and so $P_n/(P_1 \cdots P_n)^{ec} \subseteq P_n^e/(P_1 \cdots P_n)^e$. Since D is r-Noetherian and $(P_1 \cdots P_n)^e$ is regular, $P_n^e/(P_1 \cdots P_n)^e$ is a finitely generated D-module, and hence $P_n^e/(P_1 \cdots P_n)^e$ is a finite $D/(P_1 \cdots P_{n-1})^e$-module because $(P_1 \cdots P_{n-1})^e P_n^e = (P_1 \cdots P_n)^e$. Clearly, $D/(P_1 \cdots P_{n-1})^e$ is a finite $R/(P_1 \cdots P_{n-1})^{ec}$-module, and thus $P_n^e/(P_1 \cdots P_n)^e$ is a finite $R/(P_1 \cdots P_{n-1})^{ec}$-module. Note that $R/(P_1 \cdots P_{n-1})^{ec}$ is Noetherian by induction hypothesis, and $P_n/(P_1 \cdots P_n)^{ec}$ is a $R/(P_1 \cdots P_{n-1})^{ec}$-submodule of $P_n^e/(P_1 \cdots P_n)^e$. Thus $P_n/(P_1 \cdots P_n)^{ec}$ is finitely generated over $R/(P_1 \cdots P_{n-1})^{ec}$ [8, Theorem 18, p.158]. Thus P_n is a finitely generated R-module modulo $(P_1 \cdots P_n)^{ec}$, and it follows that $P_n/(P_1 \cdots P_n)^{ec}$ is a finitely generated ideal of $R/(P_1 \cdots P_n)^{ec}$. □

We are now ready to prove the main result.

THEOREM 5. *If D is a finite R-module, then R is an r-Noetherian ring if and only if D is an r-Noetherian ring.*

Proof. ($\Rightarrow$) This follows form Proposition 2.

($\Leftarrow$) Let P be a regular prime ideal of R. Note that D is integral over R [6, Theorem 17]; so there is a prime ideal Q of D such that $Q \cap R = P$. Since P is regular, Q is also regular, and hence D/Q is Noetherian. Note also that $R/P \hookrightarrow D/Q$ and D/Q is a finite R/P-module, and hence R/P is Noetherian [4, Theorem 2].

Let I be a regular ideal of R. Choose a regular element $a \in I$, and put $A = aR$. If R/A Noetherian, then I/A is finitely generated, and thus I is finitely generated. So it suffices to show that R/A is Noetherian. Since D is a finite R-module, there is a regular element $d \in R$ such that $dD \subseteq R$. So AdD is a common ideal of R and D such that $AdD \subseteq A$ and AdD is regular. By Lemma 3, there are some regular prime ideals $P_1, \ldots, P_n$ of R such that $(P_1 \cdots P_n)^{ec} \subseteq AdD \subseteq A$. Since $R/(P_1 \cdots P_n)^{ec}$ is Noetherian by Lemma 4,

$$R/A = (R/(P_1 \cdots P_n)^{ec})/(A/(P_1 \cdots P_n)^{ec})$$

is Noetherian. □

It is clear that an r-Noetherian integral domain is a Noetherian domain. Thus, by Theorem 5, we have

Corollary 6. *If D is a finite R-module, then R is a Noetherian domain if and only if D is a Noetherian domain.*

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