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APPROXIMATION THEOREM FOR CONTRACTION C-SEMIGROUPS

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Abstract. In this paper we establish approximation of contraction C-semigroups on the extrapolation space X^C, by showing the equicontinuity of contraction C-semigroups on X^C.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Given a linear operator $A : D(A) \subset X \to X$, consider the abstract Cauchy problem

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x. \tag{1}$$

The solution of (1) is closely related to operator semigroups. It is known that the integrated version of (1)

$$u(t) = A\int_0^t u(s)ds + x \tag{2}$$

has a unique solution for all $x \in X$ if and only if A generates a C_0 semigroup (see [3]). To be a generator of a C_0 semigroup, A is a closed linear operator satisfying certain conditions and having dense domain. When we want to deal with the abstract Cauchy problem with A satisfying weaker conditions or having nondense domain, we need generalizations of C_0 semigroups. If the integrated version of abstract Cauchy problem (2) has a solution for some subset of initial values and the set of these initial values is the range of some operator C, we can arrive at one of generalizations of C_0 semigroups.

Definition 1. *Let $C \in B(X)$ be an injection. A family $\{T(t) : t \geq 0\}$ of bounded linear operators on X is called a C-semigroup if*

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(a) $T(\cdot)x$ *is continuous in* $t \geq 0$ *for each* $x \in X$.
(b) $T(t+s)C = T(t)T(s)$ *for* $t,\ s \geq 0$ *and* $T(0) = C$.

A C*-semigroup* $\{T(t) : t \geq 0\}$ *is said to be of contraction if* $||T(t)x|| \leq ||Cx||$ *for* $x \in X$ *and* $t \geq 0$.

The generator A *of* $\{T(t) : t \geq 0\}$ *is defined by*

$$Ax = C^{-1}\left(\lim_{h \to 0^+} \frac{T(h)x - Cx}{h}\right)$$

with $D(A) = \{x \in X : \lim_{h \to 0^+} \frac{T(h)x - Cx}{h}$ *exists and is in* $R(C)\}$.

If A generates a C-semigroup, then the integrated version of (1) has a unique solution for $x \in R(C)$ and the solution is given by $u(t) = S(t)C^{-1}x$. For C-semigroup and its generator, we refer to [2, 5].

In this paper we establish the convergence of contraction C-semigroups on an extrapolation space X^C without any conditions on the generator A and C. In general, the convergence of Laplace transforms does not imply the convergence of original functions. Under the condition of equicontinuity of original functions, we have the equivalence between the convergence of Laplace transforms and original functions. We need the extrapolation space of X to have the equicontinuity of contraction C-semigroups. And we study to approximate the operator A by operators A_n, and then conclude that A is the generator of a contraction C-semigroup.

Throughout this paper, $B(X)$ is the space of all bounded linear operators on X and $C \in B(X)$ is injective. For an operator A, $D(A)$ and $R(A)$ are its domain and range, respectively.

2. Approximation

First, we will present some basic properties of C-semigroups and generators.

LEMMA 1. *Let* $\{T(t) : t \geq 0\}$ *be a contraction* C*-semigroup on* X *with generator* A. *Then*

(a) A *is closed,* $C^{-1}AC = A$ *and* $R(C) \subset \overline{D(A)}$, *the closure of* $D(A)$ *in* X.
(b) *For* $x \in D(A)$ *and* $t \geq 0$, $T(t)x \in D(A)$ *and* $AT(t)x = T(t)Ax$.
(c) *For* $x \in X$ *and* $t \geq 0$, $\int_0^t T(s)xds \in D(A)$ *and* $A\int_0^t T(s)xds = T(t)x - Cx$.

(d) *For $\lambda > 0$ and $n \in N$, $\lambda - A$ is injective, $R(C) \subset R(\lambda - A)$ and $(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}T(t)xdt$.*

(e) *For $x \in X$, $\lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1}Cx = Cx$.*

Now we construct the extrapolation space. For each $x \in X$, define $\|x\|_C = \|Cx\|$. Then $\|\cdot\|_C$ is a norm on X and $\|x\|_C \leq \|C\|\|x\|$ for $x \in X$. Let X^C be the completion of X with respect to $\|\cdot\|_C$.

Let $\{T(t) : t \geq 0\}$ be a contraction C-semigroup on X. We can extend $T(t)$ to X^C by defining

$$T^C(t)x = \lim_{n\to\infty} T(t)x_n \text{ in } X,$$

where $\{x_n\}$ is a sequence in X converging to x in X^C. Notice that $\|x_n - x\|_C \to 0$ and $\|T^C(t)x - T(t)x_n\| \to 0$ as $n \to \infty$. Let C^C be the extension of C. Then C^C is bounded linear and injective on X. By the definition of $T^C(t)$, we have $R(T^C(t)) \subset \overline{R(T(t))}$, the closure of $R(T(t))$ in X.

THEOREM 1. *$\{T^C(t) : t \geq 0\}$ is a contraction C^C-semigroup on X^C.*

Proof. It is not difficult to show that $\{T^C(t) : t \geq 0\}$ satisfies $T^C(0) = C^C$ and $C^C T^C(t+s) = T^C(t)T^C(s)$ for all t, $s \geq 0$.

For $x \in X$, $\|T^C(t)x\|_C = \|T(t)x\|_C = \|T(t)Cx\| \leq \|C^2x\| = \|Cx\|_C = \|C^C x\|_C$. Since X is dense in X^C, $\|T^C(t)x\|_C \leq \|C^C x\|_C$ for $x \in X^C$.

Next we will show the continuity of $T^C(t)x$ for $x \in X^C$. For $x \in X^C$, there exists $\{x_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ in X^C. By the contractivity of $\{T^C(t) : t \geq 0\}$, we have

$$\begin{aligned}
\| \ & T^C(t+h)x - T^C(t)x\|_C \\
\leq \ & \|T^C(t+h)x - T^C(t+h)x_n\|_C + \|T^C(t+h)x_n - T^C(t)x_n\|_C \\
& + \|T^C(t)x_n - T^C(t)x\|_C \\
\leq \ & 2\|C^C(x - x_n)\|_C + \|CT(t+h)x_n - CT(t)x\| \\
\leq \ & 2\|C^C(x - x_n)\|_C + \|C\|\|T(h)x_n - Cx_n\|.
\end{aligned}$$

By the continuity of $T(t)x_n$ in X, $T^C(t)x$ is continuous in $t \geq 0$ for $x \in X^C$. □

Now we present the convergence of contraction C-semigroups.

THEOREM 2. *Let* $\{T(t) : t \geq 0\}$ *and* $\{T_n(t) : t \geq 0\}$, $n \in N$, *be contraction* C*-semigroups on* X *with generators* A *and* A_n, *respectively. Suppose that*

$$\lim_{n\to\infty}(\lambda - A_n)^{-1}Cx = (\lambda - A)^{-1}Cx \text{ in } X$$

for all $\lambda > 0$ *and* $x \in X$. *Then*

$$\lim_{n\to\infty} T_n^C(t)x = T^C(t)x \quad \text{in} \quad X^C$$

for all $x \in X^C$ *and the convergence is uniform on compact subset of* $[0, \infty)$

Proof. By Lemma 1 (d), $(\lambda - A_n)^{-1}Cx = \int_0^\infty e^{-\lambda t}T_n(t)xdt$ is a Laplace transform of $T_n(t)x \in C([0, \infty), X)$. So we show that $\{T^C(t)x : t \geq 0\}$ is equicontinuous at t in X^C and then apply Theorem 2.2 in [7].

Let $x \in X$. Then

$$\begin{aligned}
\|T_n^C(t+h)x \; - \; & T_n^C(t)x\|_C \\
= \; & \|CT_n(t+h)x - CT_n(t)x\| \\
= \; & \|T_n(t)T_n(h)x - T_n(t)Cx\| \\
\leq \; & \|T_n(h)Cx - C^2x\| \\
\leq \; & \|T_n(h)Cx - T_n(h)(\lambda(\lambda - A)^{-1}Cx)\| \\
& + \|T_n(h)(\lambda(\lambda - A)^{-1}Cx) - T_n(h)(\lambda(\lambda - A_n)^{-1}Cx)\| \\
& + \|T_n(h)(\lambda(\lambda - A_n)^{-1}Cx) - C(\lambda(\lambda - A_n)^{-1}Cx)\| \\
& + \|C(\lambda(\lambda - A_n)^{-1}Cx) - C(\lambda(\lambda - A)^{-1}Cx)\| \\
& + \|C(\lambda(\lambda - A)^{-1}Cx) - C^2x\|
\end{aligned}$$

Since $\{T_n(t) : t \geq 0\}$ is a contraction C-semigroup,

$$\|T_n(h)Cx - T_n(h)(\lambda(\lambda - A)^{-1}Cx)\| \leq \|C(\lambda(\lambda - A)^{-1}Cx - Cx)\|$$

and

$$\begin{aligned}
\|T_n(h)(\lambda(\lambda - A)^{-1}Cx) \; - \; & T_n(h)(\lambda(\lambda - A_n)^{-1}Cx)\| \\
\leq \; & \lambda\|C((\lambda - A)^{-1}Cx - (\lambda - A_n)^{-1}Cx)\|
\end{aligned}$$

By Lemma 1 (c) and the above inequalities, we have

$$\begin{aligned} &\| \quad T_n^C(t+h)x - T_n^C(t)x\|_C \\ &= \quad 2\|C\|\|\lambda(\lambda - A)^{-1}Cx - Cx\| \\ &\quad +2\lambda\|C\|\|(\lambda - A_n)^{-1}Cx - (\lambda - A)^{-1}Cx\| \\ &\quad +\|A_n \int_0^h T_n(s)(\lambda(\lambda - A_n)^{-1}Cx)ds\| \end{aligned}$$

Let $\varepsilon > 0$ be given. By Lemma 1 (e), there exists $\lambda_0 > 0$ such that

$$2\|C\|\|\lambda_0(\lambda_0 - A)^{-1}Cx - Cx\| < \varepsilon/3.$$

Since $\lim_{n\to\infty}(\lambda - A_n)^{-1}Cx = (\lambda - A)^{-1}Cx$, there exists n_0 such that

$$2\lambda_0\|C\|\|(\lambda_0 - A_n)^{-1}Cx - (\lambda_0 - A)^{-1}Cx\| < \varepsilon/3$$

for $n \geq n_0$ and $\|(\lambda - A_n)^{-1}Cx\|$ is bounded. And we have

$$\begin{aligned} &\| \quad A_n \int_0^h T_n(s)(\lambda_0(\lambda_0 - A_n)^{-1}Cx)ds\| \\ &= \quad \|\int_0^h T_n(s)(\lambda_0^2(\lambda_0 - A_n)^{-1}Cx - \lambda_0 Cx)ds\| \\ &\leq \quad \|C\|(\lambda_0^2\|(\lambda_0 - A_n)^{-1}Cx\| + \lambda_0\|Cx\|)h < \varepsilon/3 \end{aligned}$$

for all sufficiently small h.

Since $T_n(t)x$ is continuous for $n < n_0$, $\{T_n(\cdot)x : n \in N\}$ is equicontinuous at t in X^C. By Theorem 2.2 in [7], we have $\lim_{n\to\infty} T_n(t)x = T(t)x$ in X^C for $x \in X$ and the convergence is uniform on compact subsets of $[0, \infty)$.

Since X is dense in X^C, the result follows for $x \in X^C$. □

If C is an isometry, then we have the convergence of C-semigroups in X and the converse is also true by dominated convergence theorem. That is, $\lim_{n\to\infty} T_n(t)x = T(t)x$ in X for $x \in X$ and the convergence is uniform on compact subsets of $[0, \infty)$ if and only if $\lim_{n\to\infty}(\lambda - A_n)^{-1}Cx = (\lambda - A)^{-1}Cx$ in X for $x \in X$. In particular, if $C = I$, then the above theorem is a Trotter-Kato Theorem for C_0 semigroups.

In the above theorem we had to assume that the limit operator A is already known to be a generator. We want to approximate the operator A by the operators A_n and show that A is a generator of a contraction C-semigroup.

Theorem 3. *Let* $\{T_n(t) : t \geq 0\}$, $n \in N$, *be a contraction* C-*semigroup on* X *with generator* A_n, *and let* $R_n(\lambda) = (\lambda - A_n)^{-1}Cx$ *for* $\lambda > 0$. *Suppose that* $\lim_{n\to\infty} R_n(\lambda)x$ *exists for* $x \in X$ *and* $\lambda > 0$. *Then there exists a closed linear operator* A *in* X *such that*

$$\lim_{n\to\infty} R_n(\lambda)x = (\lambda - A)^{-1}Cx$$

for $x \in X$ *and* $\lambda > 0$ *and the part* $A_{\overline{D(A)}}$ *of* A *in* $\overline{D(A)}$ *generates a contraction* C-*semigroup* $\{T(t) : t \geq 0\}$ *on* $\overline{D(A)}$.

Proof. Let $u_n(t) = \int_0^t T_n(s)xds$. Then $u_n(t)$ is Lipschitz continuous, $u_n(0) = 0$ and $R_n(\lambda)x = \int_0^\infty e^{-\lambda t}T_n(t)xdt = \int_0^\infty e^{-\lambda t}du_n(t)$, the Laplace - Stieltjes transform of $u_n(t)$. Since $\lim_{n\to\infty} R_n(\lambda)x$ exists, there exists $R(\lambda) \in B(X)$ and a Lipschitz continuous function $u : [0, \infty) \to X$ with $u(0) = 0$ such that $\lim_{n\to\infty} R_n(\lambda)x = R(\lambda)x$, $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subintervals of $[0, \infty)$, and $R(\lambda)x = \int_0^\infty e^{-\lambda t}du(t)$ by Theorem 2.1.1 in [1].

Suppose that $R(\lambda)x = 0$. Then $\lim_{n\to\infty} R_n(\lambda)x = 0$. So for any $\varepsilon > 0$, there exists n_0 such that $\|\lambda R_n(\lambda)x\| < \varepsilon/2$ for $n \geq n_0$. By Lemma 1 (e), there exists λ_0 such that $\|\lambda_0 R_n(\lambda_0)x - Cx\| < \varepsilon/2$. Thus we have $\|Cx\| \leq \|Cx - \lambda_0 R_n(\lambda_0)x\| + \|\lambda_0 R_n(\lambda_0)x\| < \varepsilon$. By the injectivity of C, $x = 0$ and so $R(\lambda)$ is one-to-one.

Define a closed linear operator A by $Ax = (\lambda - R(\lambda)^{-1}C)x$ with $x \in D(A) = C^{-1}(R(R(\lambda)))$. Then $\lambda - A$ is injective, $C^{-1}AC = A$ and $(\lambda - A)^{-1}C = R(\lambda)$ by Theorem 3.4 in [6].

Define $S(t)x = u(t) = \lim_{n\to\infty} u_n(t)$ for $x \in X$. Then $(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}du(t) = \lambda \int_0^\infty e^{-\lambda t}u(t)dt = \lambda \int_0^\infty e^{-\lambda t}S(t)xdt$. Thus $\{S(t) : t \geq 0\}$ is a strongly Lipschitz continuous family of linear operators on X such that

$$\lambda(\lambda - A)\int_0^\infty e^{-\lambda t}S(t)xdt = Cx$$

for $x \in X$. By Theorem 23 in [4], $\{S(t) : t \geq 0\}$ is a Lipschitz continuous 1-times integrated C-semigroup on X with generator A. Define

$$T(t)x = \frac{d}{dt}S(t)x \quad \text{for} \quad x \in D(A)$$

Then we obtain a C-semigroup $\{T(t) : t \geq 0\}$ on $\overline{D(A)}$ with generator $A_{\overline{D(A)}}$ by Theorem 18 in [4]. Since $u_n(t) = \int_0^t T_n(r)xdr$, $\|u_n(t) - u_n(s)\| = \|\int_s^t T_n(r)xdr\| \leq |t - s|\|Cx\|$ for t, $s \geq 0$. So $\|S(t + h)x - S(t)x\| \leq$

$h\|Cx\|$. This implies $\|T(t)x\| \leq \|Cx\|$ for $x \in \overline{D(A)}$. That is, $\{T(t) : t \geq 0\}$ is a contraction C-semigroup on $\overline{D(A)}$ with generator $A_{\overline{D(A)}}$. $\square$

Remark that if $D(A)$ is dense in X, then $\{T(t) : t \geq 0\}$ is a contraction C-semigroup on X with generator A. By Lemma 1 (a), if we assume that $R(C)$ is dense in X, then we obtain that A has a dense domain.

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