Korean J. Math. **18** (2010), No. 3, pp. 243–252

PROJECTIVE PROPERTIES OF REPRESENTATIONS OF A QUIVER $Q = \bullet \to \bullet$ AS $R[x]$-MODULES

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Abstract. In this paper we extend the projective properties of representations of a quiver $Q = \bullet \to \bullet$ as left R-modules to the projective properties of representations of quiver $Q = \bullet \to \bullet$ as left $R[x]$-modules. We show that if P is a projective left R-module then $0 \to P[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules. And we show $0 \to L$ is a projective representation of $Q = \bullet \to \bullet$ as R-module if and only if $0 \to L[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules. Then we show if P is a projective left R-module then $P[x] \xrightarrow{id} P[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules. We also show that if $L \xrightarrow{id} L$ is a projective representation of $Q = \bullet \to \bullet$ as R-module, then $L[x] \xrightarrow{id} L[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of the

Received June 23, 2010. Revised September 6, 2010. Accepted September 11, 2010.

2000 Mathematics Subject Classification: 16E30, 13C11, 16D80.

Key words and phrases: quiver, projective module, injective module, projective representation, injective representation.

This study was supported by research funds from Dong-A University.

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quiver $Q = \bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2$, V_1 and V_2 are vector spaces and f is a linear map (morphism). Then we extend this representation to the left R-modules, a representation of the quiver $Q = \bullet \to \bullet$ is $M_1 \xrightarrow{\phi} M_2$, M_1 and M_2 are left R-modules and ϕ is an R-linear map.

If M is a left R-module, then the polynomial $M[x]$ is a left $R[x]$-module defined by

$$r(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = rm_0 + rm_1x + rm_2x^2 + \cdots + rm_ix^i$$

$$x(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = m_0x + m_1x^2 + m_2x^3 + \cdots + m_ix^{i+1}.$$

We call $M[x]$ as a polynomial module. Similarly we can define the power series $M[[x]]$ as a left $R[x]$-module and we call a power series module.

Now we can define R-linear maps between these two representations. R-linear maps of $M_1 \xrightarrow{f} M_2$ to $N_1 \xrightarrow{g} N_2$ are given by a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ N_1 & \xrightarrow[g]{} & N_2 \end{array}$$

with s_1, s_2 R-linear maps.

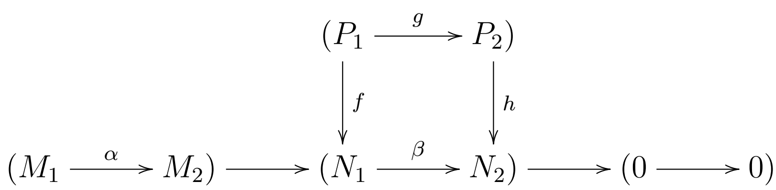
In ([3]) a homotopy of quivers was developed and in ([2]) cyclic quiver ring was studied. The theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]). Recently, in ([7]) injective covers and envelopes of representations of linear quivers was studied, and in ([6]) properties of multiple edges of quivers was studied.

Definition 1.1. ([8]) A left R-module P is said to be projective if given any surjective linear map $\sigma : M' \to M$ and any linear map $h : P \to M$, there is a linear map $g : P \to M'$ such that $\sigma \circ g = h$. That is

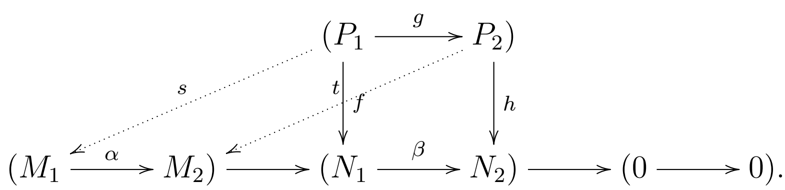
$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle h} & & \\ M' & \xrightarrow{\sigma} & M & \longrightarrow & 0 \end{array}$$

can always be completed to a commutative diagram.

Definition 1.2. ([4]) Let P_1, P_2, M_1, M_2, N_1 and N_2 be left R-modules. A representation $P_1 \longrightarrow P_2$ of a quiver $Q = \bullet \to \bullet$ is called a projective representation if every diagram of representations



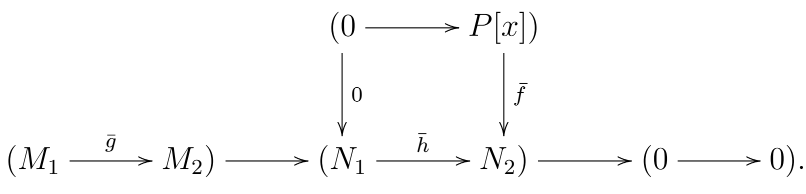
can be completed to a commutative diagram as follows:



2. Projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules

Theorem 2.1. If P is a projective left R-module, then $0 \to P[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

Proof. Let M_1, M_2, N_1 and N_2 be left $R[x]$-modules, and $\alpha : M_1 \to N_1$ and $\beta : M_2 \to N_2$ be onto $R[x]$-linear maps, and $\bar{f} : R[x] \to N_2$ be a $R[x]$-linear map. Consider the following diagram

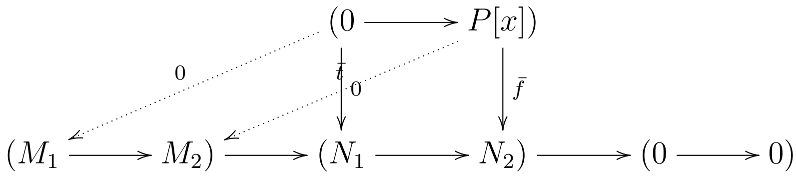


Since P is a projective left R-module, there exists an R-linear map $t : P \to M_2$ such that $\beta \circ t = \bar{f}|_P$.

Define $\bar{t} : P[x] \to M_2$ by $\bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n$. Then

$$\begin{aligned}
&\beta \circ \bar{t}(p_0 + p_1 x + \cdots + p_n x^n) \\
&= \beta(t(p_0) + t(p_1)x + \cdots + t(p_n)x^n) \\
&= (\beta \circ t)(p_0) + (\beta \circ t)(p_1)x + \cdots + (\beta \circ t)(p_n)x^n \\
&= \bar{f}|_P(p_0) + \bar{f}|_P(p_1)x + \cdots + \bar{f}|_P(p_n)x^n \\
&= \bar{f}(p_0 + p_1 x + \cdots + p_n x^n).
\end{aligned}$$

So we have $\beta \circ \bar{t} = \bar{f}$. Therefore, we can complete the following diagram



as a commutative diagram. Hence, $0 \to P[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

□

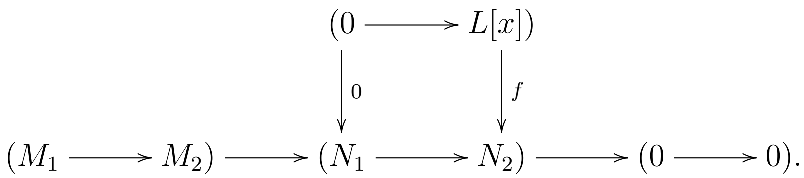
We can extend above result to the power series modules.

Corollary 2.2. If P is a projective left R-module, then $0 \to P[[x]]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

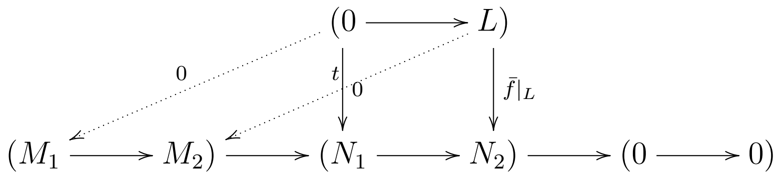
Example 2.3. Let $R = Z_6$, then $P = Z_2$ is a projective Z_6-module. $0 \to Z_2[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $Z_6[x]$-modules.

Theorem 2.4. $0 \to L$ is a projective representation of $Q = \bullet \to \bullet$ as R-modules if and only if $0 \to L[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

Proof. Let M_1, M_2, N_1 and N_2 be left $R[x]$-modules, and $\alpha : M_1 \to N_1$ and $\beta : M_2 \to N_2$ be onto $R[x]$-linear maps, and $f : L[x] \to N_2$ be a $R[x]$-linear map.
Consider the following diagram



Since $0 \to L$ is a projective representation, we can complete the following diagram

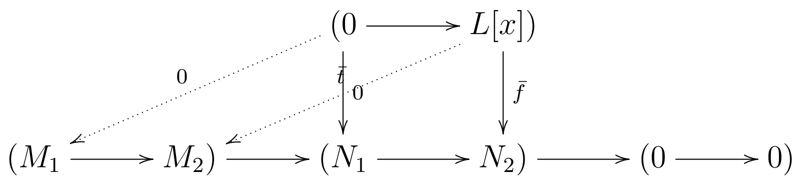


as a commutative diagram.

Define $\bar{t} : L[x] \to M_2$ by $\bar{t}(n_0 + n_1x + \cdots + n_nx^n) = t(n_0) + t(n_1)x + \cdots + t(n_n)x^n$. Then

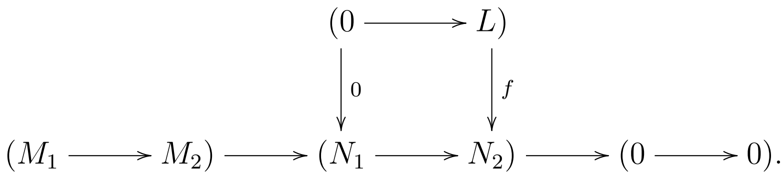
$$\begin{aligned}
&\beta \circ \bar{t}(n_0 + n_1x + \cdots + n_nx^n)\\
&= \beta(t(n_0) + t(n_1)x + \cdots + t(n_n)x^n)\\
&= (\beta \circ t)(p_0) + (\beta \circ t)(p_1)x + \cdots + (\beta \circ t)(p_n)x^n\\
&= f(n_0) + f(n_1)x + \cdots + f(n_n)x^n\\
&= f(n_0 + n_1x + \cdots + n_nx^n).
\end{aligned}$$

So we have the following diagram by $\bar{t} : L[x] \to M_2$

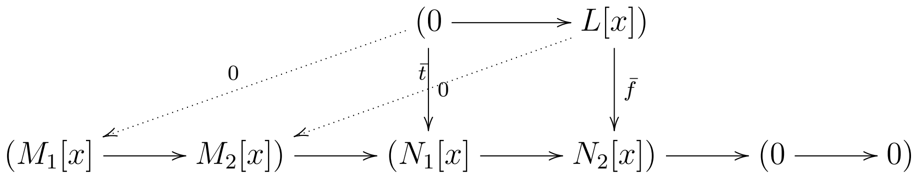


as a commutative diagram. Hence, $0 \to L[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

Conversely, Let M_1, M_2, N_1 and N_2 be left R-modules, and $\alpha : M_1 \to N_1$ and $\beta : M_2 \to N_2$ be onto R-linear maps, and $f : L \to N_2$ be a R-linear map. Consider the following diagram



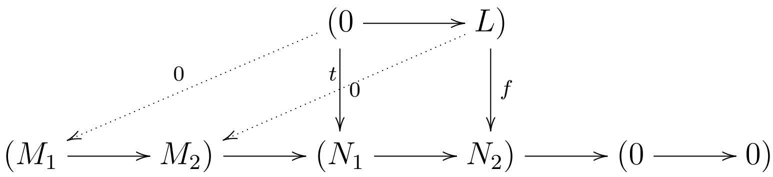
Since $0 \to L[x]$ is a projective representation, we can complete the following diagram by $\bar{t} : L[x] \to M_2[x]$



where $\bar{f} : L[x] \to M_2[x]$ by $\bar{f}(n_0 + n_1x + \cdots + n_jx^j) = f(n_0) + f(n_1)x + \cdots + f(n_j)x^j$. and $\bar{\beta} : M_2[x] \to N_2[x]$ by $\bar{\beta}(m_0 + m_1x + \cdots + m_ix^i) = \beta(m_0) + \beta(m_1)x + \cdots + \beta(m_i)x^i$. Define $t : L \to M_2$ by $t(n_0) = m_0$. Let $n_0 \in L$ then $\beta \circ t(n_0) = \beta(m_0)$. Since

$$\begin{aligned}&\bar{\beta} \circ \bar{t}(n_0)\\ &= \bar{\beta}(m_0 + m_1x + \cdots m_ix^i)\\ &= \beta(m_0) + \beta(m_1)x + \cdots \beta(m_i)x^i\\ &= \bar{f}(n_0) = f(n_o),\end{aligned}$$

$\beta(m_1), \cdots, \beta(m_i) = 0$. So $\beta(m_0) = f(n_0)$. Therefore $\beta \circ t(n_0) = f(n_0)$. So we have the following diagram



as a commutative diagram. Hence, $0 \to L$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as R-modules.

□

Corollary 2.5. $0 \to L$ is a projective representation of $Q = \bullet \to \bullet$ as R-modules if and only if $0 \to L[[x]]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

Remark 2.6. $P[x] \to 0$ is not a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules if $P \neq 0$, because the following diagram

$$
\begin{array}{ccccccc}
 & & (P[x] & \longrightarrow & 0) & & \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle 0} & & \\
(P[x] \xrightarrow{id} P[x]) & \longrightarrow & (P[x] & \longrightarrow & 0) & \longrightarrow & (0 \longrightarrow 0).
\end{array}
$$

can not be completed as a commutative diagram.

Similarly, $P[[x]] \to 0$ is not a projective representation of a quiver $Q = \bullet \longrightarrow\ \bullet$ as $R[x]$-modules if $P \neq 0$.

THEOREM 2.7. If P is a projective left R-module, then $P[x] \xrightarrow{id} P[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow\ \bullet$ as $R[x]$-modules.

Proof. Let M_1, M_2, N_1 and N_2 be left $R[x]$-modules and let $g : M_1 \to M_2$ and $h : N_1 \to N_2$ be $R[x]$-linear maps. Let $\alpha : M_1 \to N_1$, $\beta : M_2 \to N_2$ be onto $R[x]$-linear maps. Let $k : P[x] \to N_1$ be an $R[x]$-linear map and choose $h \circ k : P[x] \to N_2$ as an $R[x]$-linear map. And consider the following diagram:

$$
\begin{array}{ccccccc}
 & & (P[x] & \xrightarrow{id} & P[x]) & & \\
 & & \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle h\circ k} & & \\
(M_1 \xrightarrow{g} M_2) & \longrightarrow & (N_1 & \xrightarrow{h} & N_2) & \longrightarrow & (0 \longrightarrow 0)
\end{array}
$$

Since P is a projective left R-module, there exist R-linear maps $s : P \to M_1$ and $t : P \to M_2$ such that $\alpha \circ s = k|_P$ and $\beta \circ t = h \circ k|_P$.

Define $\bar{s} : P[x] \to M_1$ by $\bar{s}(p_0 + p_1 x + \cdots + p_n x^n) = s(p_0) + s(p_1)x + \cdots + s(p_n)x^n$. Then

$$
\begin{aligned}
&\alpha \circ \bar{s}(p_0 + p_1 x + \cdots + p_n x^n) \\
&= \alpha(s(p_0) + s(p_1)x + \cdots + s(p_n)x^n) \\
&= (\alpha \circ s)(p_0) + (\alpha \circ s)(p_1)x + \cdots + (\alpha \circ s)(p_n)x^n \\
&= k|_P(p_0) + k|_P(p_1)x + \cdots + k|_P(p_n)x^n \\
&= k(p_0 + p_1 x + \cdots + p_n x^n).
\end{aligned}
$$

Define $\bar{t} : P[x] \to M_2$ by $\bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n$. Then similarly we have $\beta \circ \bar{t}(p_0 + p_1 x + \cdots + p_n x^n) =$

$(h \circ k)(p_0 + p_1x + \cdots + p_nx^n)$. So we have the following diagram by $\bar{s} : P[x] \to M_1$ and $\bar{t} : P[x] \to M_2$

$$\begin{array}{ccccccc} & & & (P[x] & \xrightarrow{id} & P[x]) & & & \\ & \swarrow^{\bar{s}} & & \downarrow k \;\; \swarrow^{\bar{t}} & & \downarrow h\circ k & & & \\ (M_1 & \xrightarrow{g} & M_2) & \longrightarrow (N_1 & \xrightarrow{h} & N_2) & \longrightarrow & (0 & \longrightarrow 0) \end{array}$$

as a commutative diagram. Hence, $P[x] \xrightarrow{id} P[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

□

COROLLARY 2.8. If P is a projective left R-module, then $P[[x]] \xrightarrow{id} P[[x]]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

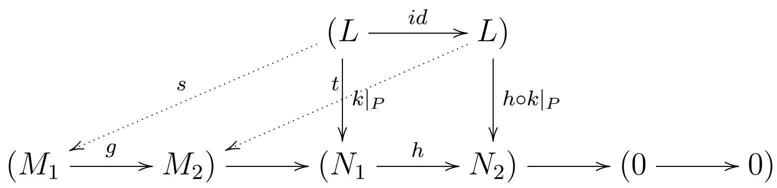
EXAMPLE 2.9. Let $R = Z_6$, then $P = Z_2$ is a projective Z_6-module. $Z_2[x] \xrightarrow{id} Z_2[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $Z_6[x]$-modules.

THEOREM 2.10. If $L \xrightarrow{id} L$ is a projective representation of $Q = \bullet \to \bullet$ as R-modules, then $L[x] \xrightarrow{id} L[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

Proof. Let M_1, M_2, N_1, N_2 be left $R[x]$-modules, and $\alpha : M_1 \to N_1$ and $\beta : M_2 \to N_2$ be onto $R[x]$-linear maps, and consider the following diagram

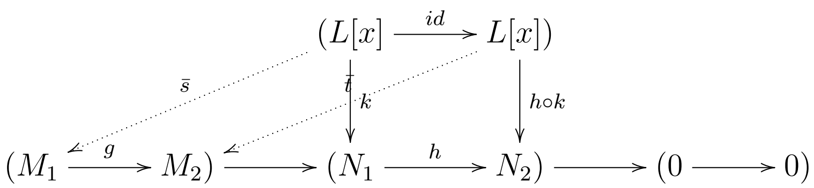
$$\begin{array}{ccccccc} & & & (L[x] & \xrightarrow{id} & L[x]) & & & \\ & & & \downarrow k & & \downarrow h\circ k & & & \\ (M_1 & \xrightarrow{g} & M_2) & \longrightarrow (N_1 & \xrightarrow{h} & N_2) & \longrightarrow & (0 & \longrightarrow 0). \end{array}$$

Since $L \to L$ is a projective representation, we can complete the following diagram



as a commutative diagram.

Define $\bar{s} : L[x] \to M_1$ by $\bar{s}(n_0 + n_1x + \cdots + n_ix^i) = s(n_0) + s(n_1)x + \cdots + s(n_i)x^i$ and $\bar{t} : L[x] \to M_2$ by $\bar{t}(n_0 + n_1x + \cdots + n_jx^j) = t(n_0) + t(n_1)x + \cdots + t(n_j)x^j$. Then we see that the following by $\bar{s} : L[x] \to M_1$ and $\bar{t} : L[x] \to M_2$



as a commutative diagram. Hence, $L[x] \to L[x]$ is a projective representation of a quiver $Q = \bullet \longrightarrow \bullet$ as $R[x]$-modules.

□

Corollary 2.11. If $L \xrightarrow{id} L$ is a projective representation of $Q = \bullet \to \bullet$ as R-module, then $L[[x]] \xrightarrow{id} L[[x]]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

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