Korean J. Math. **18** (2010), No. 3, pp. 229–241

CONVERGENCE THEOREMS FOR DENJOY-PETTIS INTEGRABLE FUZZY MAPPINGS

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Abstract. In this paper, we introduce the Denjoy-Pettis integral of fuzzy mappings in Banach spaces and obtain some properties of the Denjoy-Pettis integral of fuzzy mappings and the convergence theorems for Denjoy-Pettis integrable fuzzy mappings.

1. Introduction

Saks [11] introduced the Denjoy integral of real-valued functions which is a natural extension of the Lebesgue integral. Gordon [6] introduced the Denjoy-Pettis integral of Banach-valued functions in terms of the Denjoy integral which is the Denjoy extension of Pettis integral. Several types of integrals of set-valued mappings were introduced and studied by Aumann [1], Cascales and Rodriguez [2], Di Piazza and Musial [3,4], El Amri and Hess [5], Papageoriou [9] and others. In [10] we introduce the Denjoy-Pettis integral of set-valued mappings and investigate some properties of the integral and convergence theorems for set-valued Denjoy-Pettis integrable mappings. Another mathematicians also introduced the integrals of fuzzy mappings in Banach spaces in terms of the integrals of set-valued mappings. Kaleva [8] introduced the integral of fuzzy mappings in $\mathbb{R}^n$ in terms of the integral of set-valued mappings in $\mathbb{R}^n$. Xue, Ha and Ma [13], Xue, Wang and Wu [14] also introduced integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.

In this paper, we introduce the Denjoy-Pettis integral of fuzzy mappings in Banach spaces and obtain some properties of the Denjoy-Pettis

Received June 16, 2010. Revised August 6, 2010. Accepted August 19, 2010.
2000 Mathematics Subject Classification: 03E72, 26A39, 28B05, 28B20, 46G10, 54C60.
Key words and phrases: set-valued mapping, fuzzy mapping, Denjoy integral, Denjoy-Pettis integral.

integral of fuzzy mappings and the convergence theorems for Denjoy-Pettis integrable fuzzy mappings.

2. Preliminaries

Throughout this paper, $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $[a,b]$ and X a Banach space with dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. $CL(X)$ denotes the family of all nonempty closed subsets of X, $C(X)$ the family of all nonempty closed convex subsets of X, $CB(X)$ the family of all nonempty closed bounded convex subsets of X, $CWK(X)$ the family of all nonempty convex weakly compact subsets of X. Note that if X is reflexive then $CWK(X) = CB(X)$. For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A. For closed bounded subsets A, B of X, let $H(A,B)$ denote the Hausdorff metric of A and B defined by

$$H(A,B) = \max\left(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right),$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$ and $d(b,A) = \inf_{a\in A}\|a-b\|$. Especially,

$$H(A,B) = \sup_{\|x^*\|\le 1} |s(x^*,A) - s(x^*,B)|$$

whenever A, B are convex sets. The number $\|A\|$ is defined by $\|A\| = H(A,\{0\}) = \sup_{x\in A}\|x\|$. If $A \in CB(X)$ and $x_1^*, x_2^* \in X^*$, then $|s(x_1^*,A) - s(x_2^*,A)| \le \|x_1^* - x_2^*\|\|A\|$. Note that $(CWK(X), H)$ is a complete metric space.

Let $u : X \to [0,1]$. We denote $[u]^r = \{x \in X : u(x) \ge r\}$ for $r \in (0,1]$. u is called a *generalized fuzzy number* if for each $r \in (0,1]$, $[u]^r \in CWK(X)$ and $[u]^0 = cl\{x \in X : u(x) > 0\}$. Let $\mathcal{F}(X)$ denote the set of all generalized fuzzy numbers on X. The addition and scalar multiplication in $\mathcal{F}(X)$ are defined according to Zadeh's extension principle. For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, $[u+v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda[u]^r$ for each $r \in (0,1]$. Hence $u+v, \lambda u \in \mathcal{F}(X)$. For $u, v \in \mathcal{F}(X)$, we define $u \le v$ as follows:

$$u \le v \ \text{ if } \ u(x) \le v(x) \ \text{ for all } x \in X.$$

For $u, v \in \mathcal{F}(X)$, $u \leq v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0,1]$. Define $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]$ by the equation

$$D(u,v) = \sup_{r\in(0,1]} H([u]^r, [v]^r).$$

Then D is a metric on $\mathcal{F}(X)$. The norm $\|u\|$ of $u \in \mathcal{F}(X)$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{r\in(0,1]} H([u]^r, \{0\}) = \sup_{r\in(0,1]} \|[u]^r\|, \text{ where } \tilde{0} = \chi_{\{0\}}.$$

A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F(\cdot))$ is measurable. A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *measurable* if $F^{-1}(A) = \{t \in [a,b] : F(t) \cap A \neq \phi\} \in \mathcal{L}$ for every $A \in CL(X)$. If $F : [a,b] \to CL(X)$ is measurable then $F : [a,b] \to CL(X)$ is scalarly measurable. Let X be a separable Banach space. Then $F : [a,b] \to CWK(X)$ is measurable if and only if $F : [a,b] \to CWK(X)$ is scalarly measurable [5]. $f : [a,b] \to X$ is called a *selection* of $F : [a,b] \to CL(X)$ if $f(t) \in F(t)$ for all $t \in [a,b]$.

Definition 2.1[7,11]. Let $F : [a,b] \to X$ and let $t \in (a,b)$. A vector z in X is called the *approximate derivative* of F at t if there exists a measurable set $E \subseteq [a,b]$ that has t as a point of density such that $\lim\limits_{\substack{s\to t\\ s\in E}} \dfrac{F(s)-F(t)}{s-t} = z$. We will write $F'_{ap}(t) = z$. A function $f : [a,b] \to \mathbb{R}$ is said to be *Denjoy integrable* on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap} = f$ a.e. on $[a,b]$. In this case, we write $\int_a^b f(t)dt = F(b) - F(a)$. The function f is Denjoy integrable on a set $A \subseteq [a,b]$ if $f\chi_A$ is Denjoy integrable on $[a,b]$. In this case, we write $\int_A f(t)dt = \int_a^b f\chi_A(t)dt$.

Definition 2.2[6]. A function $f : [a,b] \to X$ is said to be *Denjoy-Pettis integrable* or simply *DP-integrable* on $[a,b]$ if for each $x^* \in X^*$ the function x^*f is Denjoy integrable on $[a,b]$ and if for every subinterval $[c,d]$ of $[a,b]$ there exists a vector $x_{[c,d]} \in X$ such that $x^*(x_{[c,d]}) = \int_c^d x^*f(t)dt$ for all $x^* \in X^*$. In this case, the vector $x_{[a,b]}$ is called *the*

Denjoy-Pettis integral of f on $[a,b]$ and is denoted by $(DP)\int_a^b f(t)dt$. The function $f : [a,b] \to X$ is Denjoy-Pettis integrable on a set $A \subseteq [a,b]$ if the function $f\chi_A$ is Denjoy-Pettis integrable on $[a,b]$. In this case, we write $(DP)\int_A f(t)dt = (DP)\int_a^b f\chi_A(t)dt$.

Definition 2.3[10]. A set-valued mapping $F : [a,b] \to CWK(X)$ is said to be *Denjoy-Pettis integrable* or simply *DP-integrable* on $[a,b]$ if for each $x^* \in X^*$, $s(x^*, F(\cdot))$ is Denjoy integrable on $[a,b]$ and for every subinterval $[c,d]$ of $[a,b]$ there exists $W_{[c,d]} \in CWK(X)$ such that

$$s(x^*, W_{[c,d]}) = \int_c^d s(x^*, F(t))dt$$

for each $x^* \in X^*$. We write $W_{[c,d]} = (DP)\int_c^d F(t)dt$.

3. Results

A mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is called a *fuzzy mapping* in a Banach space X. In this case, $\tilde{F}^r : [a,b] \to CWK(X)$ defined by $\tilde{F}^r(t) = [\tilde{F}(t)]^r$ is a set-valued mapping for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *measurable* (resp., *scalarly measurable*) if $\tilde{F}^r : [a,b] \to CWK(X)$ is measurable (resp., scalarly measurable) for each $r \in (0,1]$.

Definition 3.1. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *Denjoy-Pettis integrable* or simply *DP-integrable* on $[a,b]$ if for each subinterval $[c,d]$ of $[a,b]$ there exists $u_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = (DP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. In this case, $u_{[c,d]} = (DP)\int_c^d \tilde{F}(t)dt$ is called the *Denjoy-Pettis integral* of $\tilde{F}$ over $[c,d]$.

Theorem 3.2. *Let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ be DP-integrable on $[a,b]$ and $\lambda \geq 0$. Then*

(1) *$\tilde{F} + \tilde{G}$ is DP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(DP)\int_c^d \left\{\tilde{F}(t) + \tilde{G}(t)\right\} dt = (DP)\int_c^d \tilde{F}(t)dt + (DP)\int_c^d \tilde{G}(t)dt;$$

(2) *$\lambda\tilde{F}$ is DP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(DP)\int_c^d \lambda\tilde{F}(t)dt = \lambda(DP)\int_c^d \tilde{F}(t)dt.$$

Proof. The proof is straightforward.

□

THEOREM 3.3. *Let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ be DP-integrable fuzzy mappings. If $\tilde{F}(t) \le \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(DP)\int_c^d \tilde{F}(t)dt \le (DP)\int_c^d \tilde{G}(t)dt.$$

Furthermore, if $\tilde{F}(t) = \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$

$$(DP)\int_c^d \tilde{F}(t)dt = (DP)\int_c^d \tilde{G}(t)dt.$$

Proof. Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are DP-integrable on $[a,b]$, for each subinterval $[c,d]$ of $[a,b]$ there exist $u_{[c,d]}, v_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = (DP)\int_c^d \tilde{F}^r(t)dt$, $[v_{[c,d]}]^r = (DP)\int_c^d \tilde{G}^r(t)dt$ for each $r \in (0,1]$. If $\tilde{F}(t) \le \tilde{G}(t)$ a.e. on $[a,b]$, then by [12, Theorem 3.4], $[u_{[c,d]}]^r = (DP)\int_c^d \tilde{F}^r(t)dt \subseteq (DP)\int_c^d \tilde{G}^r(t)dt = [v_{[c,d]}]^r$ for each $r \in (0,1]$. Thus $(DP)\int_c^d \tilde{F}(t)dt = u_{[c,d]} \le v_{[c,d]} = (DP)\int_c^d \tilde{G}(t)dt$ for each subinterval $[c,d]$ of $[a,b]$.

□

Let X be a separable Banach space. If $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is DP-integrable on $[a,b]$, then $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is measurable on $[a,b]$.

A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *Denjoy integrably bounded* on $[a,b]$ if there exists a Denjoy integrable function h on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \le h(t)$ for all $x \in F(t)$. A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *Denjoy integrably bounded* on $[a,b]$ if there exists a Denjoy integrable function h on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \le h(t)$ for all $x \in \tilde{F}^0(t)$, where $\tilde{F}^0(t) = cl\left(\cup_{0<r\le 1}\tilde{F}^r(t)\right)$.

THEOREM 3.4. *Let X be a separable Banach space. If $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are Denjoy integrably bounded and DP-integrable on $[a,b]$, then $D(\tilde{F},\tilde{G})$ is Denjoy integrable on $[a,b]$ and*

$$D\left((DP)\int_a^b \tilde{F}(t)dt, (DP)\int_a^b \tilde{G}(t)dt\right) \le \int_a^b D(\tilde{F}(t),\tilde{G}(t))dt.$$

Proof. Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are measurable on $[a,b]$, there exist Castaing representations $\{f_n^r\}$ and $\{g_n^r\}$ for $\tilde{F}^r$ and $\tilde{G}^r$ for each $r \in (0,1]$. Since f_n^r and g_n^r are measurable on $[a,b]$ for all $n \in \mathbb{N}$,

$$H(\tilde{F}^r(t),\tilde{G}^r(t)) = \max\left(\sup_{n\ge 1}\inf_{k\ge 1}\|f_n^r(t) - g_k^r(t)\|, \sup_{n\ge 1}\inf_{k\ge 1}\|g_n^r(t) - f_k^r(t)\|\right)$$

is measurable on $[a,b]$ for each $r \in (0,1]$. Hence $D(\tilde{F}(t),\tilde{G}(t)) = \sup_{k\ge 1} H(\tilde{F}^{r_k}(t),\ \tilde{G}^{r_k}(t))$ is measurable on $[a,b]$, where $\{r_k : k \in \mathbb{N}\}$ is dense in $(0,1]$. Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are Denjoy integrably bounded on $[a,b]$, there exist Denjoy integrable functions h_1 and h_2 on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \le h_1(t)$ for all $x \in \tilde{F}^0(t)$ and $\|x\| \le h_2(t)$ for all $x \in \tilde{G}^0(t)$. Since h_1 and h_2 are nonnegative and Denjoy integrable on $[a,b]$, h_1 and h_2 are Lebesgue integrable on $[a,b]$. Hence we have

$$D(\tilde{F}(t),\tilde{G}(t)) \le D(\tilde{F}(t),\tilde{0}) + D(\tilde{G}(t),\tilde{0}) \le h_1(t) + h_2(t)$$

for each $t \in [a,b]$. Therefore $D(\tilde{F},\tilde{G})$ is Lebesgue integrable and so Denjoy integrable on $[a,b]$. By [10, Theorem 3.6], we have

$$H\left((DP)\int_a^b \tilde{F}^r(t)dt, (DP)\int_a^b \tilde{G}^r(t)dt\right) \le \int_a^b H(\tilde{F}^r(t),\tilde{G}^r(t))dt$$

for each $r \in (0,1]$. Hence we have

$$\begin{aligned}
&D\left((DP)\int_a^b \tilde{F}(t)dt, (DP)\int_a^b \tilde{G}(t)dt\right)\\
&= \sup_{r\in(0,1]} H\left(\left[(DP)\int_a^b \tilde{F}(t)dt\right]^r, \left[(DP)\int_a^b \tilde{G}(t)dt\right]^r\right)\\
&= \sup_{r\in(0,1]} H\left((DP)\int_a^b \tilde{F}^r(t)dt, (DP)\int_a^b \tilde{G}^r(t)dt\right)\\
&\leq \sup_{r\in(0,1]} \int_a^b H(\tilde{F}^r(t), \tilde{G}^r(t))dt\\
&\leq \int_a^b \sup_{r\in(0,1]} H(\tilde{F}^r(t), \tilde{G}^r(t))dt\\
&= \int_a^b D(\tilde{F}(t), \tilde{G}(t))dt.
\end{aligned}$$

□

The following theorem is the Dominated Convergence Theorem for the Denjoy-Pettis integral of fuzzy mappings.

THEOREM 3.5. *Let X be a reflexive and separable Banach space and let $\tilde{F}_n : [a,b] \to \mathcal{F}(X)$ be a DP-integrable fuzzy mapping for each $n \in \mathbb{N}$ and let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ be a fuzzy mapping such that $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$. If there exists a Denjoy integrable function h on $[a,b]$ such that $\|\tilde{F}_n^0(t)\| \leq h(t)$ on $[a,b]$ for each $n \in \mathbb{N}$, then $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is DP-integrable on $[a,b]$ and*

$$\lim_{n\to\infty} D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right) = 0.$$

Proof. Since $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$, for each $\epsilon > 0$ and $t \in [a,b]$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow D(\tilde{F}_n(t), \tilde{F}(t)) < \epsilon$.

For some $n \in \mathbb{N}$ with $n \geq N$,

$$\begin{aligned}\|\tilde{F}^0(t)\| &= D(\tilde{F}(t), \tilde{0}) \leq D(\tilde{F}(t), \tilde{F}_n(t)) + D(\tilde{F}_n(t), \tilde{0}) \\ &< \|\tilde{F}_n^0(t)\| + \epsilon \leq h(t) + \epsilon.\end{aligned}$$

for each $t \in [a,b]$. Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}^0(t)\| \leq h(t)$ on $[a,b]$. Thus $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is Denjoy integrably bounded on $[a,b]$. Since $\tilde{F}_n : [a,b] \to \mathcal{F}(X)$ is DP-integrable on $[a,b]$ for each $n \in \mathbb{N}$, there exists $u_n \in \mathcal{F}(X)$ such that $[u_n]^r = (DP)\int_a^b \tilde{F}_n^r(t)dt$ for each $r \in (0,1]$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$, $\lim_{n\to\infty} H(\tilde{F}_n^r(t), \tilde{F}^r(t)) = 0$ on $[a,b]$ for each $r \in (0,1]$. Since $\|\tilde{F}_n^0(t)\| \leq h(t)$ on $[a,b]$ for each $n \in \mathbb{N}$, $\|\tilde{F}_n^r(t)\| \leq h(t)$ on $[a,b]$ for each $r \in (0,1]$ and $n \in \mathbb{N}$. By [10, Theorem 3.7], $\tilde{F}^r : [a,b] \to CWK(X)$ is DP-integrable on $[a,b]$ for each $r \in (0,1]$. Let $[c,d]$ be any subinterval of $[a,b]$. Then there exists $M_r \in CWK(X)$ such that $M_r = (DP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(t) \supseteq \tilde{F}^{r_2}(t)$ for each $t \in [a,b]$. By [10, Theorem 3.4], $M_{r_1} = (DP)\int_c^d \tilde{F}^{r_1}(t)dt \supseteq (DP)\int_c^d \tilde{F}^{r_2}(t)dt = M_{r_2}$. Let $r \in (0,1]$ and $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(t) = \cap_{n=1}^{\infty} \tilde{F}^{r_n}(t)$ for each $t \in [a,b]$. By [13, Lemma 4.2], $\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(t)) = s(x^*, \tilde{F}^r(t))$ for each $t \in [a,b]$ and $x^* \in X^*$. For each $n \in \mathbb{N}$, $|s(x^*, \tilde{F}^{r_n}(t))| \leq \sup_{\|x^*\|\leq 1} |s(x^*, \tilde{F}^{r_n}(t))| = H(\tilde{F}^{r_n}(t), \{0\}) = \|\tilde{F}^{r_n}(t)\| \leq h(t)$ on $[a,b]$ for each $x^* \in B_{X^*}$. By the Dominated Convergence Theorem for the Denjoy integral, $s(x^*, \tilde{F}^r(t))$ is Denjoy integrable on $[c,d]$ and $\lim_{n\to\infty}\int_c^d s(x^*, \tilde{F}^{r_n}(t))dt = \int_c^d s(x^*, \tilde{F}^r(t))dt$ for each $x^* \in B_{X^*}$. Thus $\lim_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$ for each $x^* \in B_{X^*}$ and so $\lim_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$ for each $x^* \in X^*$. By [13, Lemma 4.2], $M_r = \cap_{n=1}^{\infty} M_{r_n}$. Let $M_0 = X$. By [13, Lemma 4.1], there exists $u_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = M_r = (DP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. Hence $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is DP-integrable on $[a,b]$. By Theorem 3.4 and the Dominated Convergence Theorem for the Denjoy integral,

$$D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right)$$
$$\leq \int_a^b D(\tilde{F}_n(t), \tilde{F}(t))dt \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\lim_{n\to\infty} D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right) = 0.$

□

Let $F_n : [a, b] \to CWK(X)$ be a set-valued mapping for each $n \in \mathbb{N}$. The sequence $\{F_n\}$ is said to be *monotone increasing* (resp., *monotone decreasing*) if for each $n \in \mathbb{N}$ $F_n(t) \subseteq F_{n+1}(t)$ (resp., $F_n(t) \supseteq F_{n+1}(t)$) for all $t \in [a, b]$. The sequence $\{F_n\}$ is said to be *monotone* if it is monotone increasing or monotone decreasing. Let $\tilde{F}_n : [a, b] \to \mathcal{F}(X)$ be a fuzzy mapping for each $n \in \mathbb{N}$. The sequence $\{\tilde{F}_n\}$ is said to be *monotone increasing* (resp., *monotone decreasing*) if the sequence $\{\tilde{F}_n^r\}$ is monotone increasing (resp., monotone decreasing) for each $r \in (0, 1]$. The sequence $\{\tilde{F}_n\}$ is said to be *monotone* if it is monotone increasing or monotone decreasing.

The following theorem is the Monotone Convergence Theorem for the Denjoy-Pettis integral of fuzzy mappings.

THEOREM 3.6. *Let X be a reflexive and separable Banach space and let $\{\tilde{F}_n\}$ be a monotone sequence of DP-integrable fuzzy mappings on $[a, b]$ and let $\tilde{F} : [a, b] \to \mathcal{F}(X)$ be a fuzzy mapping such that $D(\tilde{F}_1(t), \tilde{F}(t))$ is bounded and $\lim\limits_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a, b]$. If $\lim_{n\to\infty}(DP)\int_a^b \tilde{F}_n(t)dt \in \mathcal{F}(X)$, then $\tilde{F} : [a, b] \to \mathcal{F}(X)$ is DP-integrable on $[a, b]$ and*

$$\lim_{n\to\infty} D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right) = 0.$$

Proof. Since $\{\tilde{F}_n\}$ is a monotone sequence of fuzzy mappings on $[a, b]$, $\{\tilde{F}_n^r\}$ is a monotone sequence of $CWK(X)$-valued mappings on $[a, b]$ for each $r \in (0, 1]$. Since $\tilde{F}_n : [a, b] \to \mathcal{F}(X)$ is DP-integrable on

$[a,b]$ for each $n \in \mathbb{N}$, there exists $u_n \in \mathcal{F}(X)$ such that $\tilde{F}_n^r : [a,b] \to CWK(X)$ is DP-integrable on $[a,b]$ and $[u_n]^r = (DP)\int_a^b \tilde{F}_n^r(t)dt$ for each $r \in (0,1]$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$, $\lim_{n\to\infty} H(\tilde{F}_n^r(t), \tilde{F}^r(t)) = 0$ on $[a,b]$ for each $r \in (0,1]$. Since $D(\tilde{F}_1(t), \tilde{F}(t))$ is bounded on $[a,b]$, $H(\tilde{F}_1^r(t), \tilde{F}^r(t))$ is bounded on $[a,b]$ for each $r \in (0,1]$. Let $\lim_{n\to\infty}(DP)\int_a^b \tilde{F}_n(t)dt = u \in \mathcal{F}(X)$. Then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} D\left((DP)\int_a^b \tilde{F}_n(t)dt, u\right) &= D(u_n, u) \\ &= \sup_{r\in(0,1]} H([u_n]^r, [u]^r) < \epsilon. \end{aligned}$$

Hence for any $n \geq N$, we have

$$H\left((DP)\int_a^b \tilde{F}_n^r(t)dt, [u]^r\right) = H([u_n]^r, [u]^r) < \epsilon$$

for each $r \in (0,1]$. Thus $\lim_{n\to\infty}(DP)\int_a^b \tilde{F}_n^r(t)dt = [u]^r \in CWK(X)$ for each $r \in (0,1]$. By [10, Theorem 3.8], $\tilde{F}^r : [a,b] \to CWK(X)$ is DP-integrable on $[a,b]$ and

$$\lim_{n\to\infty} H\left((DP)\int_a^b \tilde{F}_n^r(t)dt, (DP)\int_a^b \tilde{F}^r(t)dt\right) = 0$$

for each $r \in (0,1]$. Let $[c,d]$ be any subinterval of $[a,b]$. Then there exists $M_r \in CWK(X)$ such that $M_r = (DP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(t) \supseteq \tilde{F}^{r_2}(t)$ for each $t \in [a,b]$. By [10, Theorem 3.4], $M_{r_1} = (DP)\int_c^d \tilde{F}^{r_1}(t)dt \supseteq (DP)\int_c^d \tilde{F}^{r_2}(t)dt = M_{r_2}$. Let $r \in (0,1]$ and $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(t) = \cap_{n=1}^\infty \tilde{F}^{r_n}(t)$ for each $t \in [a,b]$. By [13, Lemma 4.2], $\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(t)) = s(x^*, \tilde{F}^r(t))$ for each $t \in [a,b]$ and $x^* \in X^*$. $\{s(x^*, \tilde{F}^{r_n}(t)\}$ is a monotone decreasing sequence of

Denjoy integrable functions defined on $[a, b]$ for each $x^* \in X^*$. Now we have

$$\begin{aligned}
&\lim_{n\to\infty} H\left((DP)\int_c^d \tilde{F}_n^r(t)dt, (DP)\int_c^d \tilde{F}^r(t)dt\right)\\
&= \lim_{n\to\infty}\sup_{\|x^*\|\le 1}\left|s\left(x^*, (DP)\int_c^d \tilde{F}_n^r(t)dt\right) - s\left(x^*, (DP)\int_c^d \tilde{F}^r(t)dt\right)\right|\\
&= \lim_{n\to\infty}\sup_{\|x^*\|\le 1}\left|\int_c^d s(x^*, \tilde{F}_n^r(t))dt - \int_c^d s(x^*, \tilde{F}^r(t))dt\right| = 0.
\end{aligned}$$

Hence $\lim_{n\to\infty}\int_c^d s(x^*, \tilde{F}_n^r(t))dt = \int_c^d s(x^*, \tilde{F}^r(t))dt$ for each $x^* \in B_{X^*}$. Therefore $\lim\limits_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$ for each $x^* \in B_{X^*}$ and so $\lim\limits_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$ for each $x^* \in X^*$. By [13, Lemma 4.2], $M_r = \cap_{n=1}^{\infty} M_{r_n}$. Let $M_0 = X$. By [13, Lemma 4.1], there exists $u_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = M_r = (DP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0, 1]$. Hence $\tilde{F} : [a, b] \to \mathcal{F}(X)$ is DP-integrable on $[a, b]$. Since $\tilde{F}_n$ and $\tilde{F}$ are measurable on $[a, b]$, $D(\tilde{F}_n(t), \tilde{F}(t))$ is measurable on $[a, b]$ for each $n \in \mathbb{N}$. Since $\{\tilde{F}_n\}$ is monotone and $\lim\limits_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a, b]$, $D(\tilde{F}_n(t), \tilde{F}(t)) \ge D(\tilde{F}_{n+1}(t), \tilde{F}(t))$ on $[a, b]$ for each $n \in \mathbb{N}$. In particular, $D(\tilde{F}_n(t), \tilde{F}(t)) \le D(\tilde{F}_1(t), \tilde{F}(t))$ on $[a, b]$ for each $n \in \mathbb{N}$. Since $D(\tilde{F}_1(t), \tilde{F}(t))$ is bounded on $[a, b]$, $D(\tilde{F}_n(t), \tilde{F}(t))$ is Lebesgue integrable and so Denjoy integrable on $[a, b]$ for each $n \in \mathbb{N}$. Hence we have

$$\begin{aligned}
&D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right)\\
&= \sup_{r\in(0,1]} H\left(\left[(DP)\int_a^b \tilde{F}_n(t)dt\right]^r, \left[(DP)\int_a^b \tilde{F}(t)dt\right]^r\right)\\
&= \sup_{r\in(0,1]} H\left((DP)\int_a^b \tilde{F}_n^r(t)dt, (DP)\int_a^b \tilde{F}^r(t)dt\right)
\end{aligned}$$

$$= \sup_{r\in(0,1]} \sup_{\|x^*\|\le 1} \left| s\left(x^*, (DP)\int_a^b \tilde{F}_n^r(t)dt\right) - s\left(x^*, (DP)\int_a^b \tilde{F}^r(t)dt\right)\right|$$

$$= \sup_{r\in(0,1]} \sup_{\|x^*\|\le 1} \left| \int_a^b s(x^*, \tilde{F}_n^r(t))dt - \int_a^b s(x^*, \tilde{F}^r(t))dt\right|$$

$$\le \sup_{r\in(0,1]} \sup_{\|x^*\|\le 1} \int_a^b |s(x^*, \tilde{F}_n^r(t)) - s(x^*, \tilde{F}^r(t))|dt$$

$$\le \sup_{r\in(0,1]} \int_a^b \sup_{\|x^*\|\le 1} |s(x^*, \tilde{F}_n^r(t)) - s(x^*, \tilde{F}^r(t))|dt$$

$$= \sup_{r\in(0,1]} \int_a^b H(\tilde{F}_n^r(t), \tilde{F}^r(t))dt$$

$$\le \int_a^b \sup_{r\in(0,1]} H(\tilde{F}_n^r(t), \tilde{F}^r(t))dt$$

$$= \int_a^b D(\tilde{F}_n(t), \tilde{F}(t))dt$$

for each $n \in \mathbb{N}$. By the Monotone Convergence Theorem for the Denjoy integral we have

$$D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right)$$
$$\le \int_a^b D(\tilde{F}_n(t), \tilde{F}(t))dt \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\lim_{n\to\infty} D\left((DP)\int_a^b \tilde{F}_n(t)dt, (DP)\int_a^b \tilde{F}(t)dt\right) = 0.$

□

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