Korean J. Math. **18** (2010), No. 3, pp. 213–227

CONVERGENCE OF AN IMPLICIT ITERATIVE PROCESS FOR TWO FINITE FAMILIES OF NONEXPANSIVE MAPPINGS

S. Y. Cho, S. M. Kang, X. Qin and Y. C. Kwun*

Abstract. Convergence of an implicit iterative process is investigated for nonexpansive mappings. Strong and weak convergence theorems of common fixed points of two finite families of nonexpansive mappings are established in the framework of Banach spaces.

1. Introduction and preliminaries

Let E be a real Banach space. Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Let C be a nonempty closed and convex subset of a Banach space E. Let $T : C \to C$ be a mapping. Denote by $F(T)$ the fixed point set of T. Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We know that $F(T) \neq \emptyset$ if E is uniformly convex and C is bounded, see Browder [1].

Received May 18, 2010. Revised August 9, 2010. Accepted August 19, 2010.

2000 Mathematics Subject Classification: 47H09, 47H10, 47J25.

Key words and phrases: common fixed point, convergence, implicit iterative process, nonexpansive mapping.

This study was supported by research funds from Dong-A University.

*Corresponding author.

Recall that the normal Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$.

A celebrated weak convergence theorem was established by Reich [8], see also Falset et al. [5].

In 2001, Xu and Ori [12], in the framework of Hilbert spaces, introduced the following implicit Mann-like iterative process for a finite family of nonexpansive mappings $\{T_1, T_2, \cdots, T_N\}$ with $\{\alpha_n\}$ a real sequence in $(0,1)$ and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\cdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1},\\
&\cdots,
\end{aligned}$$

which can written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geq 1, \tag{1.2}$$

where $T_n = T_{n(modN)}$ (here the mod N takes values in $\{1, 2, \cdots, N\}$).

They obtained the following weak convergence theorem.

THEOREM 1.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by* (1.2). *If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^N$.*

Recently, Chidume and Shahzad [3] further considered the implicit Mann-like iterative process (1.2) in the framework of Banach spaces. To be more precise, they obtained the following.

Theorem 1.2. *Let E be a real uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in [0,1]$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i \in \{1, 2, \cdots, N\}$.*

Recall that a family $\{T_i\}_{i=1}^N : C \to C$ with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy *Condition* (B) on C if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in C$, $\max_{1\le i\le N}\{\|x - T_i x\|\} \ge f(d(x, F))$.

They also obtained the following strong convergence theorem with the help of Condition (B).

Theorem 1.3. *Let E be a real uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings satisfying Condition (B). Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in [0,1]$, then the sequence $\{x_n\}$ converges strongly to some point in F.*

Notice that, from the view of computation, the implicit iterative process (1.2) are often impractical. For each step, we must solve nonlinear operator equations exactly. Therefore, one of the interesting and important problems in the theory of implicit iterative algorithm is to consider the iterative algorithm with errors. That is an efficient iterative algorithm to compute approximately fixed point of nonlinear mappings.

In this work, motivated by the above results, we introduced the following generalized implicit iterative process for two finite families of nonexpansive mappings $\{S_1, S_2, \cdots, S_N\}$ and $\{T_1, T_2, \cdots, T_N\}$ as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1,\\
x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_1 + \gamma_2 T_2 x_2 + \delta_2 u_2,\\
&\cdots\\
x_N &= \alpha_N x_{N-1} + \beta_N S_N x_{N-1} + \gamma_N S_N x_N + \delta_N u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_1 x_N + \gamma_{N+1} T_1 x_{N+1} + \delta_{N+1} u_{N+1},\\
&\cdots,
\end{aligned}$$

where x_0 is the initial value, $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for each $n \geq 1$. The above table can be rewritten in the following compact form:

$$(1.3) \qquad x_n = \alpha_n x_{n-1} + \beta_n S_n x_{n-1} + \gamma_n T_n x_n + \delta_n u_n, \quad \forall n \geq 1,$$

where $S_n = S_{n(modN)}$ and $T_n = T_{n(modN)}$.

If $S_i = I$, where I denotes the identity, for each $i \in \{1, 2, \cdots, N\}$ and $\delta_n = 0$ for each $n \geq 1$, then (1.3) is reduced to Xu and Ori's implicit iterative process.

If $T_i = I$, where I denotes the identity, for each $i \in \{1, 2, \cdots, N\}$ and $\delta_n = 0$ for each $n \geq 1$, then (1.3) is reduced to the explicit Mann iterative process.

Note from [9] that the explicit Mann iterative process (1.1) and the implicit Mann iterative process (1.2) are independent. We remark that our implicit iterative process (1.3) is general which includes the explicit Mann iterative process (1.1) and the implicit Mann iterative process (1.2) as special cases.

If $S_i = T_i$ for each $i \in \{1, 2, \cdots, N\}$ and $\delta_n = 0$ for each $n \geq 1$, then (1.3) is reduced to

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \quad \forall n \geq 1,$$

which was considered by Thianwan and Suantai [11].

The purpose of this paper is to study the weak and strong convergence of the generalized implicit iteration process (1.3) for two finite families of nonexpansive mappings in a real uniformly convex Banach space.

In order to prove our main results, we also need the following conceptions and lemmas.

Recall that a Banach space E is said to satisfy *Opial's condition* [7] if, for each sequence $\{x_n\}$ in E, the convergence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \, (y \neq x).$$

It is known [7] that each l^p $(1 \leq p < \infty)$ enjoys this property. It is also known [4] that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition.

LEMMA 1.1. ([10]) *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

LEMMA 1.2. ([6]) *Let* E *be a uniformly convex Banach space, let* $s > 0$ *be a positive number and let* $B_s(0)$ *be a closed ball of* E. *There exits a continuous, strictly increasing and convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|ax + by + cz + dw\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|)$$

for all $x, y, z, w \in B_s(0) = \{x \in E : \|x\| \leq s\}$ *and* $a, b, c, d \in [0, 1]$ *such that* $a + b + c + d = 1$.

LEMMA 1.3. ([2]) *Let* E *be a real uniformly convex Banach space, let* C *be a nonempty closed and convex subset of* E *and let* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed at zero, that is,* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$ *imply that* $x = Tx$.

2. Main Results

Now, we are ready to give our main results.

THEOREM 2.1. *Let* E *be a real uniformly convex Banach space which satisfies Opial's condition, let* C *be a nonempty closed convex subset of* E *and let* $N \geq 1$ *be a positive integer. Let* $T_i : C \to C$ *be a nonexpansive mapping and let* $S_i : C \to C$ *be a nonexpansive mapping for each* $i \in D$, *where* $D = \{1, 2, \cdots, N\}$. *Assume that* $\mathcal{F} = \left(\bigcap_{i=1}^N F(T_i)\right) \bigcap \left(\bigcap_{i=1}^N F(S_i)\right) \neq \emptyset$. *Let* $\{u_n\}$ *a bounded sequence in* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.3). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b \in (0, 1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;

(b) *there exists a constant* $c \in (0, 1)$ *such that* $c \leq \beta_n, \gamma_n$, $\forall n \geq 1$;

(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then the sequence $\{x_n\}$ *converges weakly to some point in* $\mathcal{F}$.

Proof. It is easy to see that the sequence generated in (1.3) is well defined. Fixing $p \in \mathcal{F}$, we see that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + \beta_n\|S_n x_{n-1} - p\| + \gamma_n\|T_n x_n - p\| \\ &\quad + \delta_n\|u_n - p\| \\ &\le (\alpha_n + \beta_n)\|x_{n-1} - p\| + \gamma_n\|x_n - p\| + \delta_n\|u_n - p\|.\end{aligned}$$

It follows from the restrictions (a) and (b) that

$$\|x_n - p\| \le \|x_{n-1} - p\| + \frac{\delta_n}{a+c}\|u_n - p\|. \tag{2.1}$$

From Lemma 1.1, we see from the restriction (c) that $\lim_{n\to\infty}\|x_n - p\|$ exists. It follows that the sequence $\{x_n\}$ is bounded, so are $\{S_i x_n\}$ and $\{T_i x_n\}$, where $i \in D$. On the other hand, we see from Lemma 1.2 that

$$\begin{aligned}\|x_n - p\|^2 &\le \alpha_n\|x_{n-1} - p\|^2 + \beta_n\|S_n x_{n-1} - p\|^2 + \gamma_n\|T_n x_n - p\|^2 \\ &\quad + \delta_n\|u_n - p\|^2 - \alpha_n\beta_n g(\|S_n x_{n-1} - x_{n-1}\|) \\ &\le (\alpha_n + \beta_n)\|x_{n-1} - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|u_n - p\|^2 \\ &\quad - \alpha_n\beta_n g(\|S_n x_{n-1} - x_{n-1}\|).\end{aligned}$$

It follows that

$$\begin{aligned}\alpha_n\beta_n g(\|S_n x_{n-1} - x_{n-1}\|) &\le \|x_{n-1} - p\|^2 - \|x_{n-1} - p\|^2 \\ &\quad + \delta_n\|u_n - p\|^2.\end{aligned}$$

It follows from the restrictions (a), (b) and (c) that

$$\lim_{n\to\infty} g(\|S_n x_{n-1} - x_{n-1}\|) \le 0.$$

Since g is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, we obtain that

$$\lim_{n\to\infty}\|S_n x_{n-1} - x_{n-1}\| = 0. \tag{2.2}$$

In view of Lemma 1.2, we also see that

$$\begin{aligned}\|x_n - p\|^2 \le{}& \alpha_n\|x_{n-1} - p\|^2 + \beta_n\|S_n x_{n-1} - p\|^2 + \gamma_n\|T_n x_n - p\|^2 \\ &+ \delta_n\|u_n - p\|^2 - \alpha_n\gamma_n g(\|T_n x_n - x_{n-1}\|) \\ \le{}& (\alpha_n + \beta_n)\|x_{n-1} - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|u_n - p\|^2 \\ &- \alpha_n\gamma_n g(\|T_n x_n - x_{n-1}\|).\end{aligned}$$

It follows that

$$\alpha_n\gamma_n g(\|T_n x_n - x_{n-1}\|) \le \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + \delta_n\|u_n - p\|^2.$$

From the restrictions (a), (b) and (c), we obtain that

$$\lim_{n\to\infty} g(\|T_n x_n - x_{n-1}\|) = 0.$$

Since g is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, we obtain that

$$\lim_{n\to\infty} \|T_n x_n - x_{n-1}\| = 0. \tag{2.3}$$

Notice that

$$\|x_n - x_{n-1}\| \le \|S_n x_{n-1} - x_{n-1}\| + \|T_n x_n - x_{n-1}\| + \delta_n\|u_n - x_{n-1}\|.$$

In view of (2.2) and (2.3), we see from the restriction (c) that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0, \tag{2.4}$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0, \quad \forall i \in D. \tag{2.5}$$

Notice that

$$\|x_n - T_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|.$$

It follows from (2.3) and (2.4) that

$$\lim_{n\to\infty\infty} \|x_n - T_n x_n\| = 0. \tag{2.6}$$

Notice that

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \\ &\quad + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\|, \quad \forall i \in D.\end{aligned}$$

It follows from (2.5) and (2.6) that

$$\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0, \quad \forall i \in D.$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \forall i \in D. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}\|x_n - S_n x_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\| + \|S_n x_{n-1} - S_n x_n\| \\ &\le 2\|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\|.\end{aligned}$$

In view of (2.2) and (2.4), we see that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0.$$

In a similar way, we can obtain that

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0, \quad \forall i \in D. \tag{2.8}$$

Since $\{x_n\}$ is bounded and E is uniformly convex, we see that there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that x_{n_l} converges weakly to $\bar{x}$, where $\bar{x} \in C$. Note from Lemma 1.3 that $I - T_i$ and $I - S_i$ are demiclosed at zero for each $i \in D$. We see from (2.7) and (2.8) that $\bar{x} \in \mathcal{F}$.

Next we show $\{x_n\}$ converges weakly to $\bar{x}$. Suppose the contrary, then there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $x^* \in C$, where $x^* \neq \bar{x}$. Similarly, we can show $x^* \in \mathcal{F}$. Notice that we have proved that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$. Assume that $\lim_{n\to\infty} \|x_n - \bar{x}\| = d$ where d is a nonnegative number. By virtue of Opial's condition of H, we see that

$$\begin{aligned} d &= \liminf_{n_l\to\infty} \|x_{n_l} - \bar{x}\| < \liminf_{n_i\to\infty} \|x_{n_l} - x^*\| \\ &= \liminf_{n_j\to\infty} \|x_{n_j} - x^*\| < \liminf_{n_j\to\infty} \|x_{n_j} - \bar{x}\| = d. \end{aligned}$$

This is a contradiction. Hence $x^* = \bar{x}$. This completes the proof. □

If $S_i = I$, where I denote the identity, for each $i \in D$, then Theorem 2.1 is reduced the following.

Corollary 2.2. *Let E be a real uniformly convex Banach space enjoys Opial's condition, let C be a nonempty closed convex subset of E and let $N \geq 1$ be a positive integer. Let $T_i : C \to C$ be a nonexpansive mapping for each $i \in D$, where $D = \{1, 2, \cdots, N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in the following implicit iterative:*

$$(2.9) \qquad x_0 \in C, \quad x_n = (\alpha_n + \beta_n)x_{n-1} + \gamma_n T_n x_n + \delta_n u_n, \quad \forall n \geq 1.$$

Assume that the following restrictions are satisfied:

(a) *there exist constants $a, b \in (0,1)$ such that $a \leq \alpha_n \leq b$, $\forall n \geq 1$;*
(b) *there exist a constant $c \in (0,1)$ such that $c \leq \beta_n, \gamma_n$, $\forall n \geq 1$;*
(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

If $T_i = I$, where I denotes the identity, for each $i \in D$ and $\delta_n = 0$ for each $n \geq 1$, then Theorem 2.1 is reduced the following.

Corollary 2.3. *Let E be a real uniformly convex Banach space enjoys Opial's condition, let C be a nonempty closed convex subset of E and let $N \geq 1$ be a positive integer. Let $S_i : C \to C$ be a nonexpansive* *mapping for each* $i \in D$, *where* $D = \{1, 2, \cdots, N\}$. *Assume that* $\mathcal{F} = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in the following explicit iterative:*

$$x_0 \in C, \quad x_n = \frac{\alpha_n}{1-\gamma_n} x_{n-1} + \frac{\beta_n}{1-\gamma_n} S_n x_{n-1}, \quad \forall n \geq 1. \tag{2.10}$$

Assume that the following restrictions are satisfied:

(a) *there exist constants* $a, b \in (0, 1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;

(b) *there exist constants* $c \in (0, 1)$ *such that* $c \leq \beta_n$ $\forall n \geq 1$.

Then the sequence $\{x_n\}$ *converges weakly to some point in* $\mathcal{F}$.

In this paper, we introduce the following definition based on Condition (B).

Recall that two families $\{S_i\}_{i=1}^{N} : C \to C$ and $\{T_i\}_{i=1}^{N} : C \to C$ with $\mathcal{F} = \big(\bigcap_{i=1}^{N} F(S_i)\big) \bigcap \big(\bigcap_{i=1}^{N} F(T_i)\big) \neq \emptyset$ is said to satisfy *Condition* (B') if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - S_i x\|\} + \max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, F)).$$

Next, we give strong convergence theorems with the help of Condition (B').

THEOREM 2.4. *Let* E *be a real uniformly convex Banach space which satisfies Opial's condition, let* C *be a nonempty closed convex subset of* E *and let* $N \geq 1$ *be a positive integer. Let* $T_i : C \to C$ *be a nonexpansive mapping and let* $S_i : C \to C$ *be a nonexpansive mapping for each* $i \in D$, *where* $D = \{1, 2, \cdots, N\}$. *Assume that* $\mathcal{F} = \big(\bigcap_{i=1}^{N} F(T_i)\big) \bigcap \big(\bigcap_{i=1}^{N} F(S_i)\big) \neq \emptyset$. *Let* $\{u_n\}$ *a bounded sequence in* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.3). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b \in (0, 1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;

(b) *there exists a constant* $c \in (0, 1)$ *such that* $c \leq \beta_n, \gamma_n$, $\forall n \geq 1$;

(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \cdots, S_N\}$ *and* $\{T_1, T_2, \cdots, T_N\}$ *satisfies Condition* (B'), *then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

Proof. In view of (2.1), we see from Lemma 1.1 that $d(x_n, \mathcal{F})$ exists. In view of Condition (B'), we obtain from (2.7) and (2.8) that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Putting $\lambda_n = \frac{\delta_n}{a+c}\|u_n - p\|$, we see from (2.1) that

$$\|x_n - p\| \le \|x_{n-1} - p\| + \lambda_n.$$

For any positive integers m, n, where $m > n$, we see that

$$\|x_m - p\| \le \|x_n - p\| + \sum_{j=n+1}^{m} \lambda_j.$$

It follows that

$$\|x_n - x_m\| \le \|x_n - p\| + \|x_m - p\| \le 2\|x_n - p\| + \sum_{j=n+1}^{m} \lambda_j.$$

This implies that

$$\|x_n - x_m\| \le 2d(x_n, \mathcal{F}) + \sum_{j=n+1}^{m} \lambda_j.$$

It follows from the restriction (c) that $\{x_n\}$ is a Cauchy sequence in C and so $\{x_n\}$ converges strongly to some $q \in C$. Since $\mathcal{F} = \big(\bigcap_{i=1}^N F(S_i)\big) \bigcap \big(\bigcap_{i=1}^N F(T_i)\big)$ is closed, we obtain that $q \in \mathcal{F}$. This completes the proof. □

If $S_i = I$, where I denote the identity, for each $i \in D$, then Theorem 2.4 is reduced the following.

Corollary 2.5. *Let E be a real uniformly convex Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E and let $N \ge 1$ be a positive integer. Let $T_i : C \to C$ be a nonexpansive mapping for each $i \in D$, where $D = \{1, 2, \cdots, N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \ne \emptyset$. Let $\{u_n\}$ a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ such that*

$\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for each $n\geq 1$. Let $\{x_n\}$ be a sequence generated in (2.9)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants $a,b\in(0,1)$ such that $a\leq\alpha_n\leq b$, $\forall n\geq 1$;*
(b) *there exists a constant $c\in(0,1)$ such that $c\leq\beta_n,\gamma_n$, $\forall n\geq 1$;*
(c) $\sum_{n=1}^{\infty}\delta_n<\infty$.

If $\{T_1,T_2,\cdots,T_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.

If $T_i=I$, where I denotes the identity, for each $i\in D$ and $\delta_n=0$ for each $n\geq 1$, then Theorem 2.4 is reduced the following.

Corollary 2.6. *Let E be a real uniformly convex Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E and let $N\geq 1$ be a positive integer. Let $S_i:C\to C$ be a nonexpansive mapping for each $i\in D$, where $D=\{1,2,\cdots,N\}$. Assume that $\mathcal{F}=\bigcap_{i=1}^{N}F(S_i)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$ for each $n\geq 1$. Let $\{x_n\}$ be a sequence generated in* (2.10)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants $a,b\in(0,1)$ such that $a\leq\alpha_n\leq b$, $\forall n\geq 1$;*
(b) *there exists a constant $c\in(0,1)$ such that $c\leq\beta_n$, $\forall n\geq 1$;*
(c) $\sum_{n=1}^{\infty}\delta_n<\infty$.

If $\{S_1,S_2,\cdots,S_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.

Recall that a mapping $T:C\to C$ is *semicompact* if any sequence $\{x_n\}$ in C satisfying $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ has a convergent subsequence.

Finally, we give strong convergence theorems with the help of semicompact.

Theorem 2.7. *Let E be a real uniformly convex Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E and let $N\geq 1$ be a positive integer. Let $T_i:C\to C$ be a nonexpansive mapping and let $S_i:C\to C$ be a nonexpansive mapping for each $i\in D$, where $D=\{1,2,\cdots,N\}$. Assume that $\mathcal{F}=\big(\bigcap_{i=1}^{N}F(T_i)\big)\bigcap\big(\bigcap_{i=1}^{N}F(S_i)\big)\neq\emptyset$. Let $\{u_n\}$ a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0,1]$ such that*

$\alpha_n+\beta_n+\gamma_n+\delta_n=1$ *for each* $n\geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.3). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b \in (0,1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;
(b) *there exists a constant* $c \in (0,1)$ *such that* $c \leq \beta_n, \gamma_n$, $\forall n \geq 1$;
(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \cdots, S_N\}$ *and one of* $\{T_1, T_2, \cdots, T_N\}$ *are semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

Proof. We may, without loss of generality, assume that S_1 and T_1 are semicompact. From (2.7), we see that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0.$$

It follows from the semicompact of T_1 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in C$. Notice that

$$\|x^* - T_i x^*\| \leq \|x^* - x_{n_k}\| + \|x_{n_k} - T_i x_{n_k}\|.$$

This implies from (2.7) that $x^* \in F(T_i)$ for each $i \in D$. In the same way, we can show that $x^* \in F(S_i)$ for each $i \in D$. This is, $x^* \in \mathcal{F}$. For each $p \in \mathcal{F}$, we have that $\lim_{n\to\infty} \|x_n - p\|$ exits. Letting $p = x^*$, we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

If $S_i = I$, where I denote the identity, for each $i \in D$, then Theorem 2.7 is reduced the following.

Corollary 2.8. *Let* E *be a real uniformly convex Banach space which satisfies Opial's condition, let* C *be a nonempty closed convex subset of* E *and let* $N \geq 1$ *be a positive integer. Let* $T_i : C \to C$ *be a nonexpansive mapping for each* $i \in D$, *where* $D = \{1, 2, \cdots, N\}$. *Assume that* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{u_n\}$ *a bounded sequence in* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be sequences in* $[0,1]$ *such that* $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ *for each* $n\geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (2.9). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b \in (0,1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;
(b) *there exists a constant* $c \in (0,1)$ *such that* $c \leq \beta_n, \gamma_n$, $\forall n \geq 1$;
(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{T_1, T_2, \cdots, T_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

If $T_i = I$, where I denotes the identity, for each $i \in D$ and $\delta_n = 0$ for each $n \geq 1$, then Theorem 2.7 is reduced the following.

COROLLARY 2.9. *Let* E *be a real uniformly convex Banach space which satisfies Opial's condition, let* C *be a nonempty closed convex subset of* E *and let* $N \geq 1$ *be a positive integer. Let* $S_i : C \to C$ *be a nonexpansive mapping for each* $i \in D$, *where* $D = \{1, 2, \cdots, N\}$. *Assume that* $\mathcal{F} = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (2.10). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b \in (0, 1)$ *such that* $a \leq \alpha_n \leq b$, $\forall n \geq 1$;
(b) *there exists a constant* $c \in (0, 1)$ *such that* $c \leq \beta_n$, $\forall n \geq 1$;
(c) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \cdots, S_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

References

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. USA **54** (1965), 1041–1044.

[2] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[3] C. E. Chidume and N. Shahzad, *Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings*, Nonlinear Anal. **62** (2005), 1149–1156.

[4] D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mapping*, J. Lond. Math. Soc. **25** (1982), 139–144.

[5] J. G. Falset, W. Kaczor, T. Kuczumow and S. Reich, *Weak convergence theorems for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. **43** (2001), 377–401.

[6] Y. Hao, S. Y. Cho and X. Qin, *Some weak convergence theorems for a family of asymptotically nonexpansive nonself mappings*, Fixed Point Theory Appl. **2010** (2010), Article ID 218573.

[7] Z. Opial, *Weak convergence of the sequence of successive appproximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[8] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[9] B. E. Rhoades and S. M. Soltuz, *The convergence of mean value iteration for a family of maps*, Int. J. Math. Math. Sci. **2005** (2005), 3479–3485.

[10] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

[11] S. Thianwan and S. Suantai, *Weak and strong convergence of an implicity iteration process for a finite family of nonexpansive mappings*, Sci. Math. Jpn. **66** (2007), 221–229.

[12] H. K. Xu and M. G. Ori, *An implicit iterative process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767–773.

Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea
E-mail: ooly61@yahoo.co.kr

Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail: smkang@gnu.ac.kr

Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, China
E-mail: qxlxajh@163.com

Department of Mathematics
Dong-A University
Pusan 614-714, Korea
E-mail: yckwun@dau.ac.kr